



## Numerical method for evaluation triple integrals by using midpoint's rule

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### ABSTRACT

In this paper, we derive method to find the values of the triple integrals numerically its integrands continuous but have singularity in partial derivatives in the region of the integrals by using Midpoint's rule on the three dimensions  $x, y$  and  $z$  and how to find the general form of the errors (correction terms) and we will improve the results by using Romberg acceleration [3],[6] from correction terms that we found it when the number of subintervals( $l$ ) that divided interval integral on the exterior dimension  $z$  equal to twice the number of subintervals( $n$ ) on the interior dimension  $x$  and the number of subintervals ( $m$ ) on the middle dimension  $y$ , that is mean ( $h_3 = \frac{1}{2}h_1, h_1 = h_2$ ) when  $h_1$  means the distances between the ordinates on the  $x$ - axis,  $h_2$  means the distances between the ordinates on the  $y$ - axis and  $h_3$  means the distances between the ordinates on the  $z$ -axis and we denote to this method by  $Mid^3(h_1, h_2, h_3)$  and we can depend on it to calculate the triple integrals because it gave high accuracy in results by few subintervals .

### Keywords

Triple integral ; midpoint's rule ; Romberg Acceleration

### Academic Discipline And Sub-Disciplines

Applied Mathematics, Numerical Analysis



# Council for Innovative Research

Peer Review Research Publishing System

Journal: JOURNAL OF ADVANCES IN MATHEMATICS

Vol .11, No.1

[www.cirjam.com](http://www.cirjam.com) , [editorjam@gmail.com](mailto:editorjam@gmail.com)



### 1. INTRODUCTION

Many of scientists work in the single integrals like Fox.L, in 1967 [3] and Fox.L and Linda Hayes in 1970 [8] and shanks.J.A in 1972 [9] . In 1973 every of HansShjar and Jacobsen highlighted on the double integral its integrands continuous in the form  $f(x, y) = f_1(x)f_2(y)$  [4] and 1975 PhillipJ.Davis and Phillip Rabinowitz studied the integrals its integrands singularity but neglects the singularity[5] . in 2009 Dheyaa.A.M introduced four composition method its  $RM(RS), RM(RM), RS(RM), RS(RS)$  [1] .in2014Salman[ 7],introduced derive rule to find values of triple integrals, numerically its integrands have singular partial derivatives not on the end of the region of integration by using the midpoint rule with the three direction x,yand zwhere that divided interval integral on the interior dimension x equal to the number of the middle dimension y and to the exterior dimension z . In this paper we derive method to find values of the triple integrals and a special case where  $h_3 = \frac{1}{2}h_1, h_1 = h_2$  when  $h_1$  means the distances between the ordinates on the x-axis,  $h_2$  means the distances between the ordinates on the y- axis and  $h_3$  means the distances between the ordinates on the z- axis and we denote to this method by  $Mid^3(h_1, h_2, h_3)$

### 2. Triple Integrals With Continuous Integrands

Let  $I = \int\int\int_{e, c, a}^{g, d, b} f(x, y, z) dx dy dz$  such that  $f(x, y, z)$  continuous integrand in every point of region of integrals  $[a, b] \times [c, d] \times [e, g]$ .

In general we can write the integral  $I$  in the form:

$$I = \int\int\int_{e, c, a}^{g, d, b} f(x, y, z) dx dy dz = Mid^3(h_1, h_2, h_3) + E(h_1, h_2, h_3) = h_1 h_2 h_3 \sum_{k=1}^l \sum_{j=1}^m \sum_{i=1}^n f\left(x_0 + \frac{2i-1}{2}h_1 + y_0 + \frac{2j-1}{2}h_2, z_0 + \frac{2k-1}{2}h_3\right) + E(h_1, h_2, h_3) \tag{1}$$

such that  $Mid^3(h_1, h_2, h_3)$  is the numerical value of the integral by using the midpoint rule on the three dimensions  $x, y$  and  $z$  and  $E(h_1, h_2, h_3)$  is the series of correction terms that we can add to the value of  $Mid^3(h_1, h_2, h_3)$ ,  $h_1 = \frac{b-a}{n}, h_2 = \frac{d-c}{m}, h_3 = \frac{g-e}{l}$  by suppose  $l = 2n, n = m$  . In the Single Integrals the correction terms for continuous integrands by using midpoint method is given by :-

$$E(h) = A_1 h^2 + A_2 h^4 + \dots \tag{2}$$

Fox[3]. Such that  $A_1, A_2, \dots$  constants depend on the derivatives of the function  $f$  on the ends of interval integral.

In special case we take  $h_3 = \frac{1}{2}h_1, h_1 = h_2$  then we can write the correction terms as the following :-

$$E(h_1, h_2, h_3) = I - Mid^3(h_1, h_2, h_3) = D_1 h_1^2 + D_2 h_1^4 + D_3 h_1^6 + \dots \tag{3}$$

Where  $D_1, D_2, D_3, \dots$  are constants depend on the partial derivatives of (f) in the ends of the region of integrands  $[a, b] \times [c, d] \times [e, g]$ .

### 3. Triple Integrals For Continuous Integrands With Singular Partial Derivatives

$$let I = \int\int\int_{z_0, y_0, x_0}^{z_l, y_m, x_n} f(x, y, z) dx dy dz = Mid^3(h_1, h_2, h_3) + E(h_1, h_2, h_3)$$

**case (1):-**supposethat a function  $f(x, y, z)$  is defined in every point of region of integrals  $[x_0, x_n] \times [y_0, y_m] \times [z_0, z_l]$  but at least one of partial derivatives undefined inthe point  $(x_0, y_0, z_0)$



That is meaning Taylor's series for a function of triple variables( Sastry s.s [6 ]),exist in every point of region integral except the point  $(x_0, y_0, z_0)$  .

We can write triple integral  $I$  by the following :

$$\begin{aligned}
 I &= \int_{z_0}^{z_1} \int_{y_0}^{y_1} \int_{x_0}^{x_1} f(x, y, z) dx dy dz = \int_{z_0}^{z_1} \int_{y_0}^{y_1} \int_{x_0}^{x_1} f(x, y, z) dx dy dz + \int_{z_0}^{z_1} \int_{y_0}^{y_1} \sum_{r=1}^{l-1} \int_{x_r}^{x_{r+1}} f(x, y, z) dx dy dz \\
 &+ \int_{z_0}^{z_1} \sum_{s=1}^{m-1} \int_{y_s}^{y_{s+1}} \int_{x_0}^{x_1} f(x, y, z) dx dy dz + \int_{z_0}^{z_1} \sum_{s=1}^{m-1} \int_{y_s}^{y_{s+1}} \sum_{r=1}^{n-1} \int_{x_r}^{x_{r+1}} f(x, y, z) dx dy dz + \sum_{t=1}^{l-1} \int_{z_t}^{z_{t+1}} \int_{y_0}^{y_1} \int_{x_0}^{x_1} f(x, y, z) dx dy dz \\
 &+ \sum_{t=1}^{l-1} \int_{z_t}^{z_{t+1}} \sum_{r=1}^{n-1} \int_{x_r}^{x_{r+1}} \int_{y_0}^{y_1} f(x, y, z) dx dy dz + \sum_{t=1}^{l-1} \int_{z_t}^{z_{t+1}} \sum_{s=1}^{m-1} \int_{y_s}^{y_{s+1}} \int_{x_0}^{x_1} f(x, y, z) dx dy dz + \int_{z_1}^{z_1} \int_{y_1}^{y_1} \int_{x_1}^{x_1} f(x, y, z) dx dy dz \dots(4)
 \end{aligned}$$

[2].With respect to the first integral in partial region  $[x_0, x_1] \times [y_0, y_1] \times [z_0, z_1]$  we use Taylor's series of  $f(x, y, z)$  about the point  $(x_1, y_1, z_1)$  ,we obtain :-

$$\begin{aligned}
 f(x, y, z) &= [1 + (x - x_1)D_x + (y - y_1)D_y + (z - z_1)D_z + \frac{(x - x_1)^2}{2}D_x^2 + \frac{(y - y_1)^2}{2}D_y^2 + \frac{(z - z_1)^2}{2}D_z^2 + \\
 &(x - x_1)(y - y_1)D_x D_y + (x - x_1)(z - z_1)D_x D_z + (y - y_1)(z - z_1)D_y D_z + \frac{(x - x_1)^3}{3!}D_x^3 + \frac{(y - y_1)^3}{3!}D_y^3 \\
 &+ \frac{(z - z_1)^3}{3!}D_z^3 + \frac{(x - x_1)^2(y - y_1)}{2}D_x^2 D_y + \dots]f(x_1, y_1, z_1) \dots(5)
 \end{aligned}$$

Such that all partial derivatives of  $f(x, y, z)$  exist at the point  $(x_1, y_1, z_1)$  , by taking the triple integral for formula (4) in the region  $[x_0, x_1] \times [y_0, y_1] \times [z_0, z_1]$  we obtain :-

$$\begin{aligned}
 \int_{z_0}^{z_1} \int_{y_0}^{y_1} \int_{x_0}^{x_1} f(x, y, z) dx dy dz &= \left[ h_1 h_2 h_3 - \frac{h_1^2 h_2 h_3}{2} D_x - \frac{h_1 h_2^2 h_3}{2} D_y - \frac{h_1 h_2 h_3^2}{2} D_z + \frac{h_1^3 h_2 h_3}{6} D_x^2 + \frac{h_1 h_2^3 h_3}{6} D_y^2 \right. \\
 &+ \frac{h_1 h_2 h_3^3}{6} D_z^2 + \frac{h_1^2 h_2^2 h_3}{4} D_x D_y + \frac{h_1^2 h_2 h_3^2}{4} D_x D_z + \frac{h_1 h_2^2 h_3^2}{4} D_y D_z - \frac{h_1^4 h_2 h_3}{24} D_x^3 - \frac{h_1 h_2^4 h_3}{24} D_y^3 \\
 &\left. - \frac{h_1 h_2 h_3^4}{24} D_z^3 - \frac{h_1^3 h_2^2 h_3}{12} D_x^2 D_y + \dots \right] f(x_1, y_1, z_1) \dots(6)
 \end{aligned}$$

From (5) we obtain :-

$$\begin{aligned}
 f(x_0 + 0.5h_1, y_0 + 0.5h_2, z_0 + 0.5h_3) &= \left[ 1 - \frac{h_1}{2} D_x - \frac{h_2}{2} D_y - \frac{h_3}{2} D_z + \frac{h_1^2}{8} D_x^2 + \frac{h_2^2}{8} D_y^2 + \frac{h_3^2}{8} D_z^2 + \right. \\
 &\frac{h_1 h_2}{4} D_x D_y + \frac{h_1 h_3}{4} D_x D_z + \frac{h_2 h_3}{4} D_y D_z - \frac{h_1^3}{48} D_x^3 - \frac{h_2^3}{48} D_y^3 - \frac{h_3^3}{48} D_z^3 - \frac{h_1^2 h_2}{16} D_x^2 D_y \\
 &\left. + \dots \right] f(x_1, y_1, z_1) \dots(7)
 \end{aligned}$$

From the forms (6) and (7) we have :-

$$\begin{aligned}
 \int_{z_0}^{z_1} \int_{y_0}^{y_1} \int_{x_0}^{x_1} f(x, y, z) dx dy dz &= h_1 h_2 h_3 f \left( x_0 + \frac{1}{2} h_1, y_0 + \frac{1}{2} h_2, z_0 + \frac{1}{2} h_3 \right) + \left[ \frac{h_1^3 h_2 h_3}{24} D_x^2 + \frac{h_1 h_2^3 h_3}{24} D_y^2 \right. \\
 &\left. + \frac{h_1 h_2 h_3^3}{24} D_z^2 - \frac{h_1^4 h_2 h_3}{48} D_x^3 - \frac{h_1 h_2^4 h_3}{48} D_y^3 - \frac{h_1 h_2 h_3^4}{48} D_z^3 - \frac{h_1^3 h_2^2 h_3}{48} D_x^2 D_y + \dots \right] f(x_1, y_1, z_1) \dots(8)
 \end{aligned}$$



With respect to the another seven integrals we note that its function continuous in every point of its region integrals,so, by using mid point rule on all dimensions we obtain :-

$$\int_{z_0}^{z_1} \int_{y_0}^{y_1} \sum_{r=1}^{n-1} \int_{x_r}^{x_{r+1}} f(x, y, z) dx dy dz = h_1 h_2 h_3 \sum_{r=1}^{n-1} f(x_r + 0.5h_1, y_0 + 0.5h_2, z_0 + 0.5h_3) + a_1 h_1^2 + a_2 h_2^2 + a_3 h_3^2 + a_4 h_1^4 + a_5 h_2^4 + a_6 h_3^4 + \dots \dots (9)$$

$$\int_{z_0}^{z_1} \sum_{s=1}^{m-1} \int_{y_s}^{y_{s+1}} \int_{x_0}^{x_1} f(x, y, z) dx dy dz = h_1 h_2 h_3 \sum_{s=1}^{m-1} f(x_0 + 0.5h_1, y_s + 0.5h_2, z_0 + 0.5h_3) + b_1 h_1^2 + b_2 h_2^2 + b_3 h_3^2 + b_4 h_1^4 + b_5 h_2^4 + b_6 h_3^4 + \dots \dots (10)$$

$$\int_{z_0}^{z_1} \sum_{s=1}^{m-1} \int_{y_s}^{y_{s+1}} \sum_{r=1}^{n-1} \int_{x_r}^{x_{r+1}} f(x, y, z) dx dy dz = h_1 h_2 h_3 \sum_{s=1}^{m-1} \sum_{r=1}^{n-1} f(x_r + 0.5h_1, y_s + 0.5h_2, z_0 + 0.5h_3) + c_1 h_1^2 + c_2 h_2^2 + c_3 h_3^2 + c_4 h_1^4 + c_5 h_2^4 + c_6 h_3^4 + \dots \dots (11)$$

$$\sum_{t=1}^{l-1} \int_{z_t}^{z_{t+1}} \int_{y_0}^{y_1} \int_{x_0}^{x_1} f(x, y, z) dx dy dz = h_1 h_2 h_3 \sum_{t=1}^{l-1} f(x_0 + 0.5h_1, y_0 + 0.5h_2, z_t + 0.5h_3) + d_1 h_1^2 + d_2 h_2^2 + d_3 h_3^2 + d_4 h_1^4 + d_5 h_2^4 + d_6 h_3^4 + \dots \dots (12)$$

$$\sum_{t=1}^{l-1} \int_{z_t}^{z_{t+1}} \int_{y_0}^{y_1} \sum_{r=1}^{n-1} \int_{x_r}^{x_{r+1}} f(x, y, z) dx dy dz = h_1 h_2 h_3 \sum_{t=1}^{l-1} \sum_{r=1}^{n-1} f(x_r + 0.5h_1, y_0 + 0.5h_2, z_t + 0.5h_3) + e_1 h_1^2 + e_2 h_2^2 + e_3 h_3^2 + e_4 h_1^4 + e_5 h_2^4 + e_6 h_3^4 + \dots \dots (13)$$

$$\sum_{t=1}^{l-1} \int_{z_t}^{z_{t+1}} \sum_{s=1}^{m-1} \int_{y_s}^{y_{s+1}} \int_{x_0}^{x_1} f(x, y, z) dx dy dz = h_1 h_2 h_3 \sum_{t=1}^{l-1} \sum_{s=1}^{m-1} f(x_0 + 0.5h_1, y_s + 0.5h_2, z_t + 0.5h_3) + k_1 h_1^2 + k_2 h_2^2 + k_3 h_3^2 + k_4 h_1^4 + k_5 h_2^4 + k_6 h_3^4 + \dots \dots (14)$$

$$\int_{z_1}^{z_m} \int_{y_1}^{y_m} \int_{x_1}^{x_n} f(x, y, z) dx dy dz = h_1 h_2 h_3 \sum_{k=1}^n \sum_{j=1}^m \sum_{i=1}^l f\left(x_i + \frac{2i-1}{2} h_1, y_j + \frac{2j-1}{2} h_2, z_k + \frac{2k-1}{2} h_3\right) + w_1 h_1^2 + w_2 h_2^2 + w_3 h_3^2 + w_4 h_1^4 + w_5 h_2^4 + w_6 h_3^4 + \dots \dots (15)$$

Where as  $a_i, b_i, c_i, d_i, e_i, k_i, w_i$  are constants ,for all  $i = 1, 2, \dots$  .

By collect formulas (8),(9),(10),(11),(12),(13),(14),(15) we get

$$I = \int_{z_0}^{z_1} \int_{y_0}^{y_1} \int_{x_0}^{x_n} f(x, y, z) dx dy dz = h_1 h_2 h_3 \sum_{k=1}^l \sum_{j=1}^m \sum_{i=1}^n f\left(x_0 + \frac{2i-1}{2} h_1, y_0 + \frac{2j-1}{2} h_2, z_0 + \frac{2k-1}{2} h_3\right) + \left[ \frac{h_1^3 h_2 h_3}{24} D_x^2 + \frac{h_1 h_2^3 h_3}{24} D_y^2 + \frac{h_1 h_2 h_3^3}{24} D_z^2 - \frac{h_1^4 h_2 h_3}{48} D_x^3 - \frac{h_1^3 h_2^2 h_3}{48} D_x^2 D_y - \frac{h_1^3 h_2 h_3^2}{48} D_x^2 D_z - \frac{h_2^4 h_1 h_3}{48} D_y^3 - \frac{h_2^3 h_1^2 h_3}{48} D_y^2 D_x + \dots \right] f(x_1, y_1, z_1) + A_1^* h_1^2 + A_2^* h_2^2 + A_3^* h_3^2 + A_4^* h_1^4 + A_5^* h_2^4 + A_6^* h_3^4 + \dots \dots (16)$$

Where as  $A_i^*$  constants depend only on partial derivatives of  $f$  for all  $i = 1, 2, \dots$

In special case we take  $h_3 = \frac{1}{2} h_1$  ,  $h_1 = h_2$  then we can write (16) by the following :-



$$I = \int_{z_0}^{z_l} \int_{y_0}^{y_m} \int_{x_0}^{x_n} f(x, y, z) dx dy dz = 0.5h_1^3 \sum_{k=1}^{2n} \sum_{j=1}^n \sum_{i=1}^n f \left( x_0 + \frac{2i-1}{2} h_1, y_0 + \frac{2j-1}{2} h_1, z_0 + \frac{2k-1}{2} (0.5)h_1 \right) + \left[ \frac{0.5h_1^5}{24} D_x^2 + \frac{0.5h_1^5}{24} D_y^2 + \frac{(0.5)^3 h_1^5}{24} D_z^2 - \frac{0.5h_1^6}{48} D_x^3 - \frac{0.5h_1^6}{48} D_x^2 D_y - \frac{(0.5)^2 h_1^6}{48} D_x^2 D_z - \frac{0.5h_1^5}{48} D_y^3 - \frac{0.5h_1^6}{48} D_y^2 D_x + \dots \right] f(x_1, y_1, z_1) + B_1^* h_1^2 + B_2^* h_2^2 + B_3^* h_3^2 + B_4^* h_1^4 + B_5^* h_2^4 + B_6^* h_3^4 + \dots \quad \dots(17)$$

Where as  $A_i^*, B_i^*$  constants depend only on partial derivatives of  $f$  for all  $i = 1, 2, \dots$

**case (2):-**

Let  $f(x, y, z)$  defined in every point of region integral  $[x_0, x_n] \times [y_0, y_m] \times [z_0, z_l]$  and not have singularity but its partial derivatives undefined at the point  $(x_n, y_m, z_l)$  that is meaning Taylor's series for a function of triple variables exist in every point of points region integral except the point  $(x_n, y_m, z_l)$

We can write the integral  $I$  as follows :

$$I = \int_{z_0}^{z_l} \int_{y_0}^{y_m} \int_{x_0}^{x_n} f(x, y, z) dx dy dz = \int_{z_0}^{z_{l-1}} \int_{y_0}^{y_{m-1}} \int_{x_0}^{x_{n-1}} f(x, y, z) dx dy dz + \sum_{t=0}^{l-2} \int_{z_t}^{z_{t+1}} \sum_{s=0}^{m-2} \int_{y_s}^{y_{s+1}} \int_{x_0}^{x_n} f(x, y, z) dx dy dz + \sum_{t=0}^{l-2} \int_{z_t}^{z_{t+1}} \sum_{r=0}^{n-2} \int_{y_0}^{y_m} \int_{x_r}^{x_{r+1}} f(x, y, z) dx dy dz + \sum_{t=0}^{l-2} \int_{z_t}^{z_{t+1}} \int_{y_0}^{y_m} \int_{x_{n-1}}^{x_n} f(x, y, z) dx dy dz + \int_{z_{n-1}}^{z_l} \sum_{s=0}^{m-2} \int_{y_s}^{y_{s+1}} \sum_{r=0}^{n-2} \int_{x_r}^{x_{r+1}} f(x, y, z) dx dy dz + \int_{z_{l-1}}^{z_l} \sum_{s=0}^{m-2} \int_{y_s}^{y_{s+1}} \int_{x_0}^{x_n} f(x, y, z) dx dy dz + \int_{z_{n-1}}^{z_l} \sum_{r=0}^{n-2} \int_{y_0}^{y_m} \int_{x_r}^{x_{r+1}} f(x, y, z) dx dy dz + \int_{z_{n-1}}^{z_l} \int_{y_0}^{y_m} \int_{x_{n-1}}^{x_n} f(x, y, z) dx dy dz \quad \dots(18)$$

[2]. with respect to the eighth integral in the partial region  $[x_{n-1}, x_n] \times [y_{m-1}, y_m] \times [z_{l-1}, z_l]$  we use Taylor's series to  $f(x, y, z)$  about  $(x_n, y_m, z_l)$

$$f(x, y, z) = \left[ 1 + (x - x_{n-1})D_x + (y - y_{m-1})D_y + (z - z_{l-1})D_z + \frac{(x - x_{n-1})^2}{2} D_x^2 + \frac{(y - y_{m-1})^2}{2} D_y^2 + \frac{(z - z_{l-1})^2}{2} D_z^2 + (x - x_{n-1})(y - y_{m-1})D_x D_y + (x - x_{n-1})(z - z_{l-1})D_x D_z + \frac{(x - x_{n-1})^3}{6} D_x^3 + \frac{(y - y_{m-1})^3}{6} D_y^3 + \frac{(y - y_{m-1})^3}{6} D_z^3 + \frac{(x - x_{n-1})^2 (y - y_{m-1})}{2} D_x^2 D_y + \dots \right] f(x_n, y_m, z_l) \quad \dots(19)$$

By impose that all partial derivatives to  $f(x, y, z)$  exist at  $(x_{n-1}, y_{m-1}, z_{l-1})$  and by taking triple integral to the form (18) in the region  $[x_{n-1}, x_n] \times [y_{m-1}, y_m] \times [z_{l-1}, z_l]$  we get :-

$$\int_{z_{l-1}}^{z_l} \int_{y_{m-1}}^{y_m} \int_{x_{n-1}}^{x_n} f(x, y, z) dx dy dz = \left[ h_1 h_2 h_3 + \frac{h_1^2 h_2 h_3}{2} D_x + \frac{h_1 h_2^2 h_3}{2} D_y + \frac{h_1 h_2 h_3^2}{2} D_z + \frac{h_1^3 h_2 h_3}{6} D_x^2 + \frac{h_1 h_2^3 h_3}{6} D_y^2 + \frac{h_1 h_2 h_3^3}{6} D_z^2 + \frac{h_1^2 h_2^2 h_3}{4} D_x D_y + \frac{h_1^2 h_2 h_3^2}{4} D_x D_z + \frac{h_1 h_2^2 h_3^2}{4} D_y D_z + \frac{h_1^4 h_2 h_3}{24} D_x^3 + \frac{h_1 h_2^4 h_3}{24} D_y^3 + \frac{h_1 h_2 h_3^4}{24} D_z^3 + \frac{h_1^3 h_2^2 h_3}{12} D_x^2 D_y + \dots \right] f(x_{n-1}, y_{m-1}, z_{l-1}) \quad \dots(20) \text{ From (19)}$$

we get :-



$$\begin{aligned}
 f(x_n - 0.5h_1, y_m - 0.5h_2, z_l - 0.5h_3) = & \left[ 1 + \frac{h_1}{2} D_x + \frac{h_2}{2} D_y + \frac{h_3}{2} D_z + \frac{h_1 h_2}{4} D_x D_y + \frac{h_1 h_3}{4} D_x D_z + \right. \\
 & \frac{h_2 h_3}{4} D_y D_z + \frac{h_1^2}{8} D_x^2 + \frac{h_2^2}{8} D_y^2 + \frac{h_3^2}{8} D_z^2 + \frac{h_1 h_2}{16} D_x^2 D_y + \frac{h_2 h_3}{16} D_y^2 D_z + \frac{h_3 h_1}{16} D_z^2 D_x + \\
 & \left. \frac{h_1^2 h_3}{16} D_x^2 D_z + \frac{h_1^3}{48} D_x^3 + \frac{h_2^3}{48} D_y^3 + \frac{h_3^3}{48} D_z^3 + \dots \right] f(x_{n-1}, y_{m-1}, z_{l-1}) \quad \dots(21)
 \end{aligned}$$

From the formulas (20),(21) we get

$$\begin{aligned}
 \int_{z_{l-1}}^{z_l} \int_{y_{m-1}}^{y_m} \int_{x_{n-1}}^{x_n} f(x, y, z) dx dy dz = & h_1 h_2 h_3 f(x_n - 0.5h_1, y_m - 0.5h_2, z_l - 0.5h_3) + \left[ \frac{h_1^3 h_2 h_3}{24} D_x^2 + \frac{h_2^3 h_1 h_3}{24} D_y^2 + \right. \\
 & \frac{h_3^3 h_1 h_2}{24} D_z^2 + \frac{h_1^3 h_2^2 h_3}{48} D_x^2 D_y + \frac{h_1^3 h_3^2 h_2}{48} D_x^2 D_z + \frac{h_2^3 h_3^2 h_1}{48} D_y^2 D_z + \frac{h_1^4 h_2 h_3}{48} D_x^3 + \frac{h_2^4 h_1 h_3}{48} D_y^3 + \\
 & \left. \frac{h_3^4 h_1 h_2}{48} D_z^3 + \dots \right] f(x_{n-1}, y_{m-1}, z_{l-1}) \quad \dots(22)
 \end{aligned}$$

For the first seven integrals we note that the integrands continuous in every point of region integrals ,so, (according to midpoint rule ) we have:-

$$\begin{aligned}
 \int_{z_0}^{z_{l-1}} \int_{y_0}^{y_{m-1}} \int_{x_0}^{x_{n-1}} f(x, y, z) dx dy dz = & h_1 h_2 h_3 \sum_{k=1}^{l-1} \sum_{j=1}^{m-1} \sum_{i=1}^{n-1} f \left( x_0 + \frac{2i-1}{2} h_1, y_0 + \frac{2j-1}{2} h_2, z_0 + \frac{2k-1}{2} h_3 \right) \\
 & + A_1 h_1^2 + A_2 h_2^2 + A_3 h_3^2 + A_4 h_1^4 + A_5 h_2^4 + A_6 h_3^4 \quad \dots(23)
 \end{aligned}$$

$$\begin{aligned}
 \dots(24) \sum_{t=0}^{l-2} \int_{z_t}^{z_{t+1}} \sum_{s=0}^{m-2} \int_{y_s}^{y_{s+1}} \int_{x_{n-1}}^{x_n} f(x, y, z) dx dy dz = & h_1 h_2 h_3 \sum_{t=0}^{l-2} \sum_{s=0}^{m-2} f(x_{n-1} + 0.5h_1, y_s + 0.5h_2, z_t + 0.5h_3) + B_1 h_1^2 + \\
 & B_2 h_2^2 + B_3 h_3^2 + B_2 h_1^4 + B_2 h_2^4 + B_2 h_3^4 + \dots
 \end{aligned}$$

$$\begin{aligned}
 \dots(25) \sum_{t=0}^{l-2} \int_{z_t}^{z_{t+1}} \int_{y_{n-1}}^{y_m} \sum_{r=0}^{n-2} \int_{x_r}^{x_{r+1}} f(x, y, z) dx dy dz = & h_1 h_2 h_3 \sum_{t=0}^{n-2} \sum_{r=0}^{l-2} f(x_r + 0.5h_1, y_{n-1} + 0.5h_2, z_t + 0.5h_3) + \\
 & C_1 h_1^2 + C_2 h_2^2 + C_3 h_3^2 + C_4 h_1^4 + C_5 h_2^4 + C_6 h_3^4 + \dots
 \end{aligned}$$

$$\begin{aligned}
 \dots(26) \sum_{t=0}^{l-2} \int_{z_t}^{z_{t+1}} \int_{y_{n-1}}^{y_m} \int_{x_{n-1}}^{x_n} f(x, y, z) dx dy dz = & h_1 h_2 h_3 \sum_{t=0}^{l-2} f(x_{n-1} + 0.5h_1, y_{n-1} + 0.5h_2, z_t + 0.5h_3) + \\
 & D_1 h_3^2 + D_2 h_2^2 + D_3 h_3^2 + D_4 h_3^4 + D_5 h_3^4 + D_6 h_3^4 + \dots
 \end{aligned}$$

$$\begin{aligned}
 \dots(27) \int_{z_{l-1}}^{z_l} \sum_{s=0}^{m-2} \int_{y_s}^{y_{s+1}} \sum_{r=0}^{n-2} \int_{x_r}^{x_{r+1}} f(x, y, z) dx dy = & h_1 h_2 h_3 \sum_{s=0}^{m-2} \sum_{r=0}^{n-2} f(x_r + 0.5h_1, y_s + 0.5h_2, z_{n-1} + 0.5h_3) + \\
 & G_1 h_1^2 + G_1 h_2^2 + G_2 h_3^2 + G_4 h_1^4 + G_5 h_2^4 + G_6 h_3^4 + \dots
 \end{aligned}$$

$$\begin{aligned}
 \dots(28) \int_{z_{l-1}}^{z_l} \sum_{s=0}^{m-2} \int_{y_s}^{y_{s+1}} \int_{x_{n-1}}^{x_n} f(x, y, z) dx dy dz = & h_1 h_2 h_3 \sum_{s=0}^{m-2} f(x_{n-1} + 0.5h_1, y_s + 0.5h_2, z_{l-1} + 0.5h_3) + \\
 & H_1 h_2^2 + H_2 h_2^2 + H_3 h_3^2 + H_4 h_1^4 + H_5 h_2^4 + H_6 h_3^4 + \dots
 \end{aligned}$$



$$\dots(29) \int_{z_{l-1}}^{z_l} \int_{y_{m-1}}^{y_m} \sum_{r=0}^{n-2} \int_{x_r}^{x_{r+1}} f(x, y, z) dx dy dz = h_1 h_2 h_3 \sum_{r=0}^{n-2} f(x_r + 0.5h_1, y_{n-1} + 0.5h_2, z_{n-1} + 0.5h_3) + S_1 h_1^2 + S_2 h_2^2 + S_3 h_3^2 + S_4 h_2^4 + S_5 h_1^4 + S_6 h_2^4 + \dots$$

Where as  $A_i, B_i, C_i, D_i, G_i, H_i, S_i \dots$  constants depend on partial derivatives of function  $f$  for all  $i = 1, 2, \dots$

When we collect formulas (22),(23),(24),(25),(26),(27),(28),(29) we have :-

$$I = \int_{z_0}^{z_l} \int_{y_0}^{y_m} \int_{x_0}^{x_n} f(x, y, z) dx dy dz = h_1 h_2 h_3 \sum_{k=1}^l \sum_{j=1}^m \sum_{i=1}^n f \left( x_0 + \frac{2i-1}{2} h_1, y_0 + \frac{2j-1}{2} h_2, z_0 + \frac{2k-1}{2} h_3 \right) + \left[ \frac{h_1^3 h_2 h_3}{24} D_x^2 + \frac{h_2^3 h_1 h_3}{24} D_y^2 + \frac{h_3^3 h_1 h_2}{24} D_z^2 + \frac{h_1^3 h_2^2 h_3}{48} D_x^2 D_y + \frac{h_2^3 h_1^2 h_3}{48} D_y^2 D_x + \frac{h_1^3 h_2 h_3^2}{48} D_x^2 D_z + \frac{h_2^3 h_1^2 h_3^2}{48} D_y^2 D_z + \frac{h_3^3 h_1^2 h_2^2}{48} D_z^2 D_y + \frac{h_1^4 h_2 h_3}{48} D_x^3 + \frac{h_2^4 h_1 h_3}{48} D_y^3 + \frac{h_3^4 h_1 h_2}{48} D_z^3 + \dots \right] f(x_{n-1}, y_{m-1}, z_{l-1}) + p_1 h_1^2 + p_2 h_2^2 + p_3 h_3^2 + p_4 h_1^4 + p_5 h_2^4 + p_6 h_3^4 + \dots \dots(30)$$

Where as  $p_i$  constants depend on partial derivatives to the function  $f$  for all  $i = 1, 2, \dots$

In special case we take  $h_3 = \frac{1}{2} h_1, h_1 = h_2$  then we can write (31) by the following :-

$$I = \int_{z_0}^{z_l} \int_{y_0}^{y_m} \int_{x_0}^{x_n} f(x, y, z) dx dy dz = 0.5 h_1^3 \sum_{k=1}^{2n} \sum_{j=1}^n \sum_{i=1}^n f \left( x_0 + \frac{2i-1}{2} h_1, y_0 + \frac{2j-1}{2} h_1, z_0 + \frac{2k-1}{2} (0.5) h_1 \right) + \left[ \frac{0.5 h_1^5}{24} D_x^2 + \frac{0.5 h_1^5}{24} D_y^2 + \frac{(0.5)^3 h_1^5}{24} D_z^2 - \frac{0.5 h_1^6}{48} D_x^3 - \frac{0.5 h_1^6}{48} D_x^2 D_y - \frac{(0.5)^2 h_1^6}{48} D_x^2 D_z - \frac{0.5 h_1^5}{48} D_y^3 - \frac{0.5 h_1^6}{48} D_y^2 D_x + \dots \right] f(x_{n-1}, y_{m-1}, z_{l-1}) + q_1 h_1^2 + q_2 h_2^2 + q_3 h_3^2 + q_4 h_1^4 + q_5 h_2^4 + q_6 h_3^4 + \dots \dots(31)$$

Where as  $q_i$  constants depend on partial derivatives to the function for all  $i = 1, 2, \dots$

### 3.Examples :-

1-  $I = \int_1^2 \int_1^2 \int_1^2 \ln(x + y + z) dx dy dz$  its analytical value 1.4978022885754 close to thirteen decimal, the integrands is continuous for all  $(x, y, z) \in [1, 2] \times [1, 2] \times [1, 2]$  from the table (1) the values of integral by using  $Mid^3(h_1, h_2, h_3)$  be correct for eight decimal when  $l = 32, m = 16, n = 16$  and after using Romberg acceleration with we have correct value to thirteen decimal ( $2^{13}$  partial intervals).

n=m	l	$Mid^3((h_1, h_2, h_3))$	K=2	K=4	K=6	K=8
1	2	1.5025318004914				
2	4	1.4989997559488	1.4978224077680			
4	8	1.4981026388996	1.4978035998832	1.4978023460243		
8	16	1.4978774383089	1.4978023714453	1.4978022895494	1.4978022886530	
16	32	1.4978210799042	1.4978022937693	1.4978022885909	1.4978022885757	1.4978022885754

Table (1)



2-  $I = \int_0^{0.5} \int_0^{0.5} \int_0^{0.5} \sqrt{x + y + z} dx dy dz$  its analytic value 0.1065660178203 closed to thirteen decimal . the integrands is singular at the first partial derivative when  $(x, y, z) = (0, 0, 0)$  and the kind of singularity is root . From the table (2) the values of integral by using  $Mid^3(h_1, h_2, h_3)$  to five decimal and  $E_{Mid^3} = a_1 h_1^2 + a_2 h_1^{3.5} + a_3 h_1^4 + a_4 h_1^6 + \dots$  when  $m=n=32$  ,  $l=64$  and after using Romberg acceleration with  $Mid^3(h_1, h_2, h_3)$  we have a correct value to (12) decimal ( $2^{16}$  partial intervals) .

n=m	l	$Mid^3(h_1, h_2, h_3)$	k=2	K=3.5	K=4	K=6	K=8
1	2	0.1078739851085					
2	4	0.1069322306520	0.1066183124998				
4	8	0.1066624060560	0.1065724645241	0.1065680191807			
8	16	0.1065906376290	0.1065667148200	0.1065661573383	0.1065660332154		
16	32	0.1065722251397	0.1065660876432	0.1065660268332	0.1065660181329	0.1065660178935	
32	64	0.1065675746674	0.1065660245099	0.1065660183886	0.1065660178256	0.1065660178208	0.1065660178205

Table (2)



3-  $I = \int_{0.5}^1 \int_{0.5}^1 \int_{0.5}^1 zy \sqrt{1-xyz} dx dy dz$  its analytic value 0.0514008709909 closed to thirteen decimal , the integrand has singularity when  $(x, y, z) = (1, 1, 1)$  and the kind of singularity is in root . From the table (3) the values of integral by using  $Mid^3(h_1, h_2, h_3)$  correct to six decimal and





n=m	l	$(h_1, h_2, h_3)$	K=2	K=3.5	K=4	K=4.5	K=5.5	K=6	K=6.5
1	2	0.05281961 91616							
2	4	0.05179031 75234	0.0514472 169773						
4	8	0.05150248 54322	0.0514065 414018	0.0514025 975657					
8	16	0.05142674 16676	0.0514014 937461	0.0514010 043338	0.0514008 981184				
16	32	0.05140738 65152	0.0514009 347977	0.0514008 806030	0.0514008 723543	0.0514008 711630			
32	64	0.05140250 45567	0.0514008 772372	0.0514008 716562	0.0514008 710597	0.0514008 709999	0.0514008 709962		
64	128	0.05140127 98270	0.0514008 715838	0.0514008 710357	0.0514008 709943	0.0514008 709913	0.0514008 709911	0.0514008 709910	
128	256	0.05140097 32413	0.0514008 710460	0.0514008 709939	0.0514008 709911	0.0514008 709910	0.0514008 709909	0.0514008 709909	0.0514008 709909

Table (3)

$E_{Mid^3} = a_1 h_1^2 + a_2 h_1^{2.5} + a_3 h_1^4 + a_4 h_1^{4.5} + a_5 h_1^{5.5} + a_6 h_1^6 \dots$  when  $n=m=128$  ,  $l=256$  and after using Romberg acceleration with  $Mid^3(h_1, h_2, h_3)$  we have correct value to thirteen decimal ( $2^{22}$  partial intervals) .

#### 4. The discussion

We note from tables of this search when we evaluate extended value for Triple Integrals For Continuous Integrands With Singular Partial Derivatives in region integral by using midpoint rule on three dimension when the number of subintervals (l) that divided interval integral on the exterior dimension z equal to twice the number of subintervals(n) on the interior dimension x and the number of subintervals (m) on the middle dimension y and this rule give correct value for many decimal comparison with real values by use number of subintervals without use Romberg acceleration ,but when we use Romberg acceleration with this method gave best result in accuracy and speed of approaching in little subintervals comparison to the real values (Analytical values) then we can depend on this method in value of Triple Integrals For Continuous Integrands With Singular Partial Derivatives .

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