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On the Production of Nonseparable Solutions of Linear Partial Differential Equations

W. Robin
Engineering Mathematics Group Edinburgh Napier University
10 Colinton Road, Edinburgh, EH10 5DT, UK

ABSTRACT

A particular means of producing nonseparable solutions to a linear partial differential equation from given separable solutions to the same partial differential equation is exposed and extended. The method, involving symmetry operators, is straightforward, requiring only basic partial differentiation. Examples are presented for important standard second-order linear partial differential equations, with particular attention being given to Helmholtz equations, for which previously known *ad hoc* solutions are derived within the current methodology. Although the examples presented are restricted to second-order linear partial differential equations, the method is developed in sufficient generality to cover all linear partial differential equations and to make comparison with previous work meaningful. Indeed, comparison with previous work enables the development of certain novel results.

Keywords

Partial differential equations; symmetry operators; separable solutions; nonseparable solutions

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1. Introduction

One of the standard methods of solving linear partial differential equations (PDE) is the method of separation of variables (SOV) [4, 5]. Interestingly, it is possible to produce, in a particular and technically straightforward manner using *symmetry operators* [1, 2, 4], *nonseparable* solutions from separable solutions obtained from the method of SOV. As well as being relatively easy to produce, these nonseparable solutions have immediate application, for example, in the theory of elastic vibrations [6, 7], acoustical cavities [7] and electro-magnetic waveguides and cavities [3, 9-11]. In this paper, we expose and extend the basic ideas behind this nonseparable solution process, following [1] and [2], as well as providing examples of such nonseparable solutions to well-known second-order linear PDE.

We begin, in the next section, by examining and extending the elements of the 'nonseparable methodology' of references [1] and [2]. This is followed, in section 3, by two major examples of the 'nonseparable methodology' at work. Then, in section 4, we focus on the Helmholtz equation, for which the application of nonseparable solutions to various problems has been determined [3, 6, 7, 9-11] and develop various nonseparable solutions for this. Then, in section 5, we extend the method further and discuss, in particular, the related methodology of Overfelt [8], who produces nonseparable solutions to the Helmholtz equation from separable solutions, in an analogous manner to the present method, using other certain well-known symmetry operators [4, 8]. The paper finishes with a short conclusions section, section 6.

2. Generating Nonseparable Solutions: The Basic Ideas

We consider two particular approaches to the determination of nonseparable solutions to PDE from corresponding separable solutions that we will label the explicit approach [2] and the implicit approach [1]. As we will see, they both reduce to the same operational approach to the 'nonseparable' problem. In fact, following Chester [2], we consider a more general situation, in terms of linear operators, and then apply the more general results we obtain to the case of certain second-order linear PDE and their known variable separable solutions. So, generalizing Chester's method to an arbitrary number of variables/parameters, we begin by supposing that u is any solution of the linear PDE (later on we assume that u is a variable separable solution, at least to 'start' with)

$$L_1[u] = \lambda(a_1, a_2, \ldots a_n) L_2[u], \quad n = 2,2,3,\ldots \tag{2.1}$$

where L_1 and L_1 are arbitrary *linear differential operators* and the *parameter* $\lambda(a_1, a_2, \ldots a_n)$ depends only on the *other parameters* $a_1, a_2, \ldots a_n$. The relation (2.1) is a straightforward generalization of equation (1) of [2]. In equation (2.1) we have the explicit form of the problem, with the dependence on the parameters represented *explicitly* by $\lambda(a_1, a_2, \ldots a_n)$. We now find, extending Chester's work [2], that the functions

$$\mathrm{v}_{i,j} = \frac{\partial \lambda}{\partial a_j}\frac{\partial u}{\partial a_i} - \frac{\partial \lambda}{\partial a_i}\frac{\partial u}{\partial a_j}, \quad i \neq j = 1,2,3,\ldots,n \tag{2.2}$$

are also solutions of (2.1) when u is. That the solutions (2.2) of (2.1) are nonseparable if the solution u are separable should be clear, but is made more obvious in the examples presented below, in subsequent sections.

The argument leading to (2.2), extending that of [2], is elementary. First, differentiate (2.1) with respect to a_i and multiply by $\partial\lambda / \partial a_j$ to get

$$(\partial\lambda / \partial a_j) L_1[\partial u / \partial a_i] = (\partial\lambda / \partial a_j) L_2[\lambda(\partial u / \partial a_i) + (\partial\lambda / \partial a_i) u] \tag{2.3}$$

Next, swap-round a_i and a_j in (2.3), leaving

$$(\partial\lambda / \partial a_i) L_1[\partial u / \partial a_j] = (\partial\lambda / \partial a_i) L_2[\lambda(\partial u / \partial a_j) + (\partial\lambda / \partial a_j) u] \tag{2.4}$$

Now, on subtracting (2.4) form (2.3), we find that

$$L_1[(\partial\lambda / \partial a_j)(\partial u / \partial a_i) - (\partial\lambda / \partial a_i)(\partial u / \partial a_j)] = \lambda L_2[(\partial\lambda / \partial a_j)(\partial u / \partial a_i) - (\partial\lambda / \partial a_i)(\partial u / \partial a_j)] \tag{2.5}$$

Finally, comparing (2.5) with (2.1), the result (2.2) follows, Naturally, the result (2.2) depends on assuming that the partial differentiation process commutes with the linear operators L_1 and L_2; this will always be assumed in what follows.

It is apparent that we may identify an operator, to within a sign, which, assuming it commutes with L_1 and L_2, produces a new solution from the given solution u : from (2.5) we identify these (restricted) symmetry [4, 8] operators as

$$\mathrm{D}_{i,j} = \kappa_{i,j}\left(\frac{\partial\lambda}{\partial a_j}\frac{\partial}{\partial a_i} - \frac{\partial\lambda}{\partial a_i}\frac{\partial}{\partial a_j}\right), \quad i \neq j = 1,2,3,\ldots,n \tag{2.6}$$

with the $\kappa_{i,j}$ depending only on the parameters $a_1, a_2, \ldots a_n$. It is equally apparent that we may iterate the action of (2.6) and, indeed, form linear combinations of positive integral 'powers' of (2.6), leading to more general solutions of (2.1), than (2.2), through the application of these 'combination' operators to the given solution u. (In future, we will, mostly, take the word 'symmetry' as read.)

We note at this point, that the results (2.6) generalize those of Chakrabarti [1], who followed a more particular approach [1], with the appropriate choice of L_1 and L_2.

As presented above, the methodology appears quite general and may be applied to equations of any order. However, Chester warns that a certain amount of care is needed in its application, to avoid 'null' results [2]. On the other hand, the methodology determined through (2.1) to (2.6) is open to an iterative approach, leading to an, in principle, infinite sequence of alternative solutions being generated from a single original separable solution of (2.1) [7]. Further, as the operations involved are linear, it is possible to superimpose individual nonseparable solutions to obtain new nonseparable solutions from given ones. We will encounter these possibilities in the examples presented in the following sections.

We consider, now, the second or *implicit* approach to the problem of procuring nonseparable solutions to PDE from their corresponding separable solutions.

Suppose we have our relation, with L a linear operator, in the form

$$L[u] = 0 \tag{2.7}$$

so that u is any solution of the linear PDE (2.7) and we assume the problem depends *implicitly* on several parameters. That is, developing a variation of Chakrabarti's argument [1], we assume the existence of a parameter λ which is represented, now, through an implicit relation involving various other parameters, $a_1, a_2, \ldots a_n$. That is, we may write

$$\lambda(a_1, a_2, \ldots a_n) = 0 \tag{2.8}$$

and assume that we may, in principle, solve (2.8) for a_j in terms of a_i ($i \neq j$) or *vice versa*. We will now show that, as before where λ was given explicitly, if u is a solution of (2.7), then so are

$$\mathrm{v}_{i,j} = \frac{\partial\lambda}{\partial a_j}\frac{\partial u}{\partial a_i} - \frac{\partial\lambda}{\partial a_i}\frac{\partial u}{\partial a_j}, \quad i \neq j = 2,3,\ldots,n \tag{2.9}$$

First, we suppose that we may solve (2.8) for a_j in terms of a_i. If we differentiate (2.7) with respect to a_i with (2.8) in mind, then we find that

$$L\left[\frac{\partial u}{\partial a_i} + \frac{\partial u}{\partial a_j}\frac{da_j}{da_i}\right] = 0 \tag{2.10}$$

or

$$L\left[\left(\frac{\partial\lambda}{\partial a_j}\right)^{-1}\left(\frac{\partial\lambda}{\partial a_j}\frac{\partial}{\partial a_i} - \frac{\partial\lambda}{\partial a_i}\frac{\partial}{\partial a_j}\right)u\right] = 0 \tag{2.11}$$

Secondly, we suppose that we may solve (2.8) for a_i in terms of a_j. ($i \neq j$) Then, when we repeat the above argument with a replacing b and *vice versa*, we find that

$$L\left[-\left(\frac{\partial\lambda}{\partial a_i}\right)^{-1}\left(\frac{\partial\lambda}{\partial a_j}\frac{\partial}{\partial a_i}-\frac{\partial\lambda}{\partial a_i}\frac{\partial}{\partial a_j}\right)u\right]=0$$

(2.12)

Finally, multiplying (2.11) by $\partial\lambda/\partial a_j$ and (2.12) by $-\partial\lambda/\partial a_i$, we get, in both cases

$$L[(\partial\lambda/\partial a_j)(\partial u/\partial a_i)-(\partial\lambda/\partial a_i)(\partial u/\partial a_j)]=0 \qquad (2.13)$$

and the proposition (2.9) has been proven. As before, we may now define the operators (2.6) and we have recovered the previous (explicit) formalism.

However, there is more to this case. Indeed, if u is a solution of (2.7), then basic differentiation shows that the functions

$$\mathrm{v}_i=\frac{\partial u}{\partial a_i}, \quad i=1,2,3,\ldots,n \qquad (2.14)$$

are also solutions of (2.7), with corresponding (symmetry) operators $\partial/\partial a_i$. Of course, various combinations of the operators $\partial/\partial a_i$ are possible, as in the previous cases. With this, we leave the theoretical development of the method and move on to its application. We return to the consideration of further developments of the method in section 5, where we consider PDE with non-constant coefficients and the connection with other (corresponding) types of symmetry operators.

3. Examples of Nonseparable Solutions of Standard PDE

In the following, we restrict ourselves to a single application of one of the 'D' operators. Naturally, as mentioned already, in any example, we may *iterate* the use of any 'D' operator, or form various other combinations to produce new solutions. Also, we adopt a simpler more concrete, if more extensive, notation.

For our first example for this section, we consider the wave equation in three dimensions [5]

$$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 u}{\partial z^2}=\frac{1}{v^2}\frac{\partial^2 u}{\partial t^2} \qquad (3.1)$$

with separable solution (α, β, γ and δ are constants)

$$u(x,y,z,t)=\sin(ax+\alpha)\sin(by+\beta)\sin(cz+\gamma)\sin(dt+\delta) \qquad (3.2)$$

provided that

$$v^2=\frac{d^2}{a^2+b^2+c^2} \qquad (3.3)$$

In this case, the parameter in the PDE is $\lambda(a,b,c,d)=v^2$ and, given u is a solution of (3.1), six other solutions of (3.1) are

$$\mathrm{v}_{a,b}=\frac{\partial\lambda}{\partial b}\frac{\partial u}{\partial a}-\frac{\partial\lambda}{\partial a}\frac{\partial u}{\partial b},\quad \mathrm{v}_{a,c}=\frac{\partial\lambda}{\partial c}\frac{\partial u}{\partial a}-\frac{\partial\lambda}{\partial a}\frac{\partial u}{\partial c},\quad \mathrm{v}_{a,d}=\frac{\partial\lambda}{\partial d}\frac{\partial u}{\partial a}-\frac{\partial\lambda}{\partial a}\frac{\partial u}{\partial d},$$
$$\mathrm{v}_{b,c}=\frac{\partial\lambda}{\partial c}\frac{\partial u}{\partial b}-\frac{\partial\lambda}{\partial b}\frac{\partial u}{\partial c},\quad \mathrm{v}_{b,d}=\frac{\partial\lambda}{\partial d}\frac{\partial u}{\partial b}-\frac{\partial\lambda}{\partial b}\frac{\partial u}{\partial d},\quad \mathrm{v}_{c,d}=\frac{\partial\lambda}{\partial d}\frac{\partial u}{\partial c}-\frac{\partial\lambda}{\partial c}\frac{\partial u}{\partial d} \qquad (3.4)$$

with corresponding operators

$$\mathrm{D}_{a,b}=\kappa_{a,b}\left(\frac{\partial\lambda}{\partial b}\frac{\partial}{\partial a}-\frac{\partial\lambda}{\partial a}\frac{\partial}{\partial b}\right),\qquad \mathrm{D}_{a,c}=\kappa_{a,c}\left(\frac{\partial\lambda}{\partial c}\frac{\partial}{\partial a}-\frac{\partial\lambda}{\partial a}\frac{\partial}{\partial c}\right),$$

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$$\mathrm{D}_{a,d} = \kappa_{a,d}\left(\frac{\partial\lambda}{\partial d}\frac{\partial}{\partial a} - \frac{\partial\lambda}{\partial a}\frac{\partial}{\partial d}\right), \qquad \mathrm{D}_{b,c} = \kappa_{b,c}\left(\frac{\partial\lambda}{\partial c}\frac{\partial}{\partial b} - \frac{\partial\lambda}{\partial b}\frac{\partial}{\partial c}\right),$$

$$\mathrm{D}_{b,d} = \kappa_{b,d}\left(\frac{\partial\lambda}{\partial d}\frac{\partial}{\partial b} - \frac{\partial\lambda}{\partial b}\frac{\partial}{\partial d}\right), \qquad \mathrm{D}_{c,d} = \kappa_{c,d}\left(\frac{\partial\lambda}{\partial d}\frac{\partial}{\partial c} - \frac{\partial\lambda}{\partial c}\frac{\partial}{\partial d}\right) \tag{3.5}$$

and with $\kappa_{a,b},\ldots,\kappa_{c,d}$ depending only on the parameters $a,\ b,\ c$ or d.

So, for example, from (3.3) and (3.5), if (3.2) is a solution of (3.1), then so is

$$\mathrm{v}_{a,b}(x,y,z,t) = 2\kappa_{a,b}\left(b\frac{\partial u}{\partial a} - a\frac{\partial u}{\partial b}\right) \tag{3.6}$$

or, on setting $\kappa_{a,b} = 1/2,$ we get a nonseparable solution of (3.1) of the form

$$\mathrm{v}_{a,b}(x,y,z,t) = [bx\cos(ax+\alpha)\sin(by+\beta) - ay\sin(ax+\alpha)\cos(by+\beta)]\sin(cz+\gamma)\sin(dt+\delta) \tag{3.7}$$

which may be checked by differentiation and substitution.

We look, now, at an example in which we have an implicit relation among the parameters of the form of equations (2.7) and (2.8). So, as an example of the 'implicit situation', we examine Laplace's equation in three dimensions [5]

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = 0 \tag{3.8}$$

with separable solution

$$u(x,y) = e^{ax+by+\hat{i}cz} \tag{3.9}$$

provided that

$$\lambda(a,b,c) = a^2 + b^2 - c^2 = 0 \tag{3.10}$$

as $\hat{i}^2 = -1.$ From (2.9) and (3.10), we see that we may expect a nonseparable solution to (3.8) of the form, for example, of

$$\mathrm{v}_{a,b}(x,y,z) = 2\kappa_{a,b}\left(b\frac{\partial u}{\partial a} - a\frac{\partial u}{\partial b}\right) \tag{3.11}$$

On setting $\kappa_{a,b} = 1/2,$ we find that (3.11) yields a nonseparable solution to (3.8) of the form

$$\mathrm{v}_{a,b}(x,y) = (bx - ay)e^{ax+by+icz} \tag{3.12}$$

which may be checked, as usual, by differentiation and substitution.

4. Nonseparable Solutions of the Helmholtz Equation

In this section we are involved in somewhat further detail when we examine two forms of the Helmholtz equation, one in cartesian co-ordinates and one in polar co-ordinates. As well as providing further nonseparable solution examples, we also consider some unsolved problems left over from the original development of the method [1, 7] and show how they can be resolved. Note that various *ad hoc* expressions [6, 7] are here derived from the basic methodology

First, we suppose we are dealing with the Helmholtz equation in three variables [5, 7] and in cartesian co-ordinates, that is, we consider

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} + k^2 u = 0 \tag{4.1}$$

with separable solution [1, 7] ($\alpha,\ \beta$ and γ are constants)

$$u(x,y,z)=\sin(ax+\alpha)\sin(by+\beta)\sin(cz+\gamma)$$

(4.2)

provided that

$$k^2 = a^2 + b^2 + c^2 \tag{4.3}$$

In this case, the parameter in the PDE is $\lambda(a,b,c)=k^2$ and, from (2.9) and (4.3), if (4.2) is a solution of (4.1), then so are

$$\mathrm{v}_{a,b}=2\kappa_{a,b}\left(b\frac{\partial u}{\partial a}-a\frac{\partial u}{\partial b}\right),\ \mathrm{v}_{a,c}=2\kappa_{a,c}\left(c\frac{\partial u}{\partial a}-a\frac{\partial u}{\partial c}\right)\ \text{and}\ \mathrm{v}_{\mathrm{b,c}}=2\kappa_{b,c}\left(c\frac{\partial u}{\partial b}-b\frac{\partial u}{\partial c}\right) \tag{4.4}$$

or, setting $\kappa_{a,b}=\kappa_{a,c}=\kappa_{b,c}=1/2$

$$\mathrm{v}_{a,b}=bx\cos(ax+\alpha)\sin(by+\beta)\sin(cz+\gamma)-ay\sin(ax+\alpha)\cos(by+\beta)\sin(cz+\gamma) \tag{4.5a}$$

and

$$\mathrm{v}_{a,c}=cx\cos(ax+\alpha)\sin(by+\beta)\sin(cz+\gamma)-az\sin(ax+\alpha)\sin(by+\beta)\cos(cz+\gamma) \tag{4.5b}$$

and

$$\mathrm{v}_{b,c}=cy\sin(ax+\alpha)\cos(by+\beta)\sin(cz+\gamma)-bz\sin(ax+\alpha)\sin(by+\beta)\cos(cz+\gamma) \tag{4.5c}$$

We can obtain further nonseparable solutions to (4.1) in two ways. First, by linearly combining the operators of (4.4) and operating on (4.2) with the resultant operator combination and, secondly, by introducing other variable separable solutions of (4.1) and combining the results of operating on these other variable separable solutions of (4.1) with the operators (4.4).

Taking the first of these options, we set $\kappa_{a,b}=\kappa_{a,c}=1/2$ and $\kappa_{b,c}=-1/2$ in (4.4) and, assuming (4.2) and (4.3) again, we obtain the nonseparable solution

$$\mathrm{v}=\left(b\frac{\partial u}{\partial a}-a\frac{\partial u}{\partial b}\right)+\left(c\frac{\partial u}{\partial a}-a\frac{\partial u}{\partial c}\right)-\left(c\frac{\partial u}{\partial b}-b\frac{\partial u}{\partial c}\right) \tag{4.6}$$

or, from (4.5)

$$\begin{aligned}\mathrm{v}(x,y,z)=(b+c)x\cos(ax+\alpha)\sin(by+\beta)\sin(cz+\gamma)\\ -(a+c)y\sin(ax+\alpha)\cos(by+\beta)\sin(cz+\gamma)\\ -(a-b)z\sin(ax+\alpha)\sin(by+\beta)\cos(cz+\gamma)\end{aligned} \tag{4.7}$$

another result first presented by Moseley [7]. Of course, the operator implied by (4.6) was constructed to reproduce Moseley's result (4.7) (see also [1]). Written out explicitly and re-arranged, Moseley's operator (4.6) (again given *ad hoc* as equation (26) of [7]) producing (4.7) is

$$\mathrm{D_M}=(b+c)\frac{\partial}{\partial a}-(a+c)\frac{\partial}{\partial b}-(a-b)\frac{\partial}{\partial c} \tag{4.8}$$

As mentioned already, it is possible to iterate the above process by defining, for non-negative integers n, the operator

$$(\mathrm{D_M})^n=\left((b+c)\frac{\partial}{\partial a}-(a+c)\frac{\partial}{\partial b}-(a-b)\frac{\partial}{\partial c}\right)^n \tag{4.9}$$

and produce a sequence of nonseparable solution to (4.1) of the form [7]

$$\mathrm{v}_n(x,y,z)=(\mathrm{D_M})^n[\sin(ax+\alpha)\sin(by+\beta)\sin(cz+\gamma)]\quad n\ge 1 \tag{4.10}$$

with $\mathrm{v}_1(x,y,z)=\mathrm{v}(x,y,z)$ of (4.7). Naturally, further generalizations of these three-variable solutions may be obtained by invoking, again, the principle of linear superposition as in the two-variable case.

There are, of course, further variable separable solutions of (4.1); for example, we may consider [7]

$$u_1(x,y,z)=\sin(ax+\alpha)\cos(by+\beta)\cos(cz+\gamma) \tag{4.11a}$$

or

$$u_2(x,y,z)=\cos(ax+\alpha)\sin(by+\beta)\cos(cz+\gamma) \quad (4.11b)$$

or

$$u_3(x,y,z)=\cos(ax+\alpha)\cos(by+\beta)\sin(cz+\gamma) \quad (4.11c)$$

which are all particular solutions of (4.1), subject to condition (4.3).

If we examine, now, the particular construction from (4.4) and (4.11)

$$\mathrm{v}=2\kappa_{a,b}\left(b\frac{\partial u_3}{\partial a}-a\frac{\partial u_3}{\partial b}\right)+2\kappa_{a,c}\left(c\frac{\partial u_2}{\partial a}-a\frac{\partial u_2}{\partial c}\right)+2\kappa_{b,c}\left(c\frac{\partial u_1}{\partial b}-b\frac{\partial u_1}{\partial c}\right) \quad (4.12)$$

and set $\kappa_{a,b}=-1/2,$ $\kappa_{a,c}=1/2$ and $\kappa_{b,c}=-1/2,$ we find that (4.12) reduces to

$$\begin{aligned}\mathrm{v}=&\,x\sin(ax+\alpha)[b\cos(by+\beta)\sin(cz+\gamma)-c\sin(by+\beta)\cos(cz+\gamma)]\\&+y\sin(by+\beta)[c\sin(ax+\alpha)\cos(cz+\gamma)-a\cos(ax+\alpha)\sin(cz+\gamma)]\\&+z\sin(cz+\gamma)[a\cos(ax+\alpha)\sin(by+\beta)-b\sin(ax+\alpha)\cos(by+\beta)]\end{aligned} \quad (4.13)$$

which is just (the *ad hoc*) relation (31) of Moseley [7]. Chakrabarti [1] produces a different construction of (4.13).

Parenthetically, Moseley has produced a further nonseparable solution to (4.1) which he could not derive systematically, that is, equation (32) of [7]. In fact, if we take u as in (4.2) and form, using the operators implicit in (4.4) (with $\kappa_{a,b}=\kappa_{a,c}=\kappa_{b,c}=1$)

$$\mathrm{v}=-\frac{1}{2}\left[\left(b\frac{\partial}{\partial a}-a\frac{\partial}{\partial b}\right)^2+\left(c\frac{\partial}{\partial a}-a\frac{\partial}{\partial c}\right)^2+\left(c\frac{\partial}{\partial b}-b\frac{\partial}{\partial c}\right)^2\right][u] \quad (4.14)$$

then on setting $a=b=c=1,$ (4.14) reduces to equation (32) of [7].

The formalism of section 2 is not confined to cartesian co-ordinates. Consider, now, the Helmholtz equation in cylindrical co-ordinates with angular symmetry

$$\frac{\partial^2 u}{\partial r^2}+\frac{1}{r}\frac{\partial u}{\partial r}+\frac{\partial^2 u}{\partial z^2}+k^2u=0 \quad (4.15)$$

with separable solution [6, 7] (p and m are constants)

$$u(r,z)=J_0(pr)\sin(mz) \quad (4.16)$$

(with $J_0(pr)$ the parametric Bessel function of order zero) provided that

$$k^2=p^2+m^2 \quad (4.17)$$

In this case, the parameter in the PDE is $\lambda(p,m)=p^2+m^2$ and, from (2.6) and (4.17), if (4.16) is a solution of (4.15), then so is $\mathrm{D}_{p,m}[u(r,z)]$, where

$$\mathrm{D}_{p,m}=2\kappa_{p,m}\left(m\frac{\partial}{\partial p}-p\frac{\partial}{\partial m}\right) \quad (4.18)$$

Setting $\kappa_{p,m}=-1/2$ [6, 7], we get a nonseparable solution of (4.15) of the form (as $J_0'(pr)=J_1(pr)$)

$$\mathrm{v}_{p,m}(\mathrm{r,z})\left(p\frac{\partial}{\partial m}-m\frac{\partial}{\partial p}\right)[J_0(pr)\sin(mz)]=pzJ_0(pr)\cos(mz)+mrJ_1(pr)\sin(mz) \quad (4.19)$$

in agreement with Moseley's original, but *ad hoc*, result [6, 7].

The above example shows the superiority of the abstract approach pioneered by Chester [2], as the alternative approach of Chakrabarti is limited to PDE with constant coefficients and a particular (exponential) type of variable separable solution. Oddly, the above example appears to be the first application of Chester's methodology [2] to PDE with variable coefficients; of course, the ideas originate with Moseley [6, 7] (who did more work on this than mentioned here [6, 7]).

5. Discussion

In this section we will discuss two further topics. First, how the method developed here for producing nonseparable solutions to PDE from associated separable solutions is related to the similar method of Overfelt [7]. Then, following this, we show that the present methodology is sufficiently flexible to encompass further types of 'nonseparable operators' and their corresponding nonseparable solutions to PDE, leading to new results.

Overfelt [8] generates nonseparable solutions to the two-dimensional Helmholtz equation [1, 5, 6, 7]

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + k^2 u = 0 \tag{5.1}$$

using a Lie algebra of symmetry operators (Miller [4], chapter 1) different from those used here, but not unconnected, as we discuss in a moment. Overfelt discusses Moseley's work [6] and his original *ad hoc* operator

$$\mathrm{D} = b\frac{\partial}{\partial a} - a\frac{\partial}{\partial b} \tag{5.2}$$

for producing nonseparable solutions to (4.1) from the separable solution (α and β are constants)

$$u(x,y) = \sin(ax+\alpha)\sin(by+\beta) \tag{5.3}$$

provided that

$$k^2 = a^2 + b^2 \tag{5.4}$$

(Note that the operator (5.2) is easily extracted from (5.4) using the results of section 2, merely by realizing that $\lambda(a,b) = a^2 + b^2$.)

As Overfelt shows, (5.2) is insufficient to produce nonseparable solutions to (5.1) from the alternative starting separable solution

$$u(x,y) = \sin(ax)\sin(y/b) \tag{5.5}$$

with

$$k^2 = a^2 + 1/b^2 \tag{5.6}$$

However, this is not the case with the methodology discussed here. Indeed, the operators used here to derive nonseparable solutions from separable solutions are all also symmetry operators [4, 8] and will always lead to further solutions of the given equation, but subject to the *restriction* of λ having a particular functional form; also, the $'\mathrm{D}'$ operators do not lead to eigenvalue equations [4], which *restricts* their use. For example, in this instance the relevant operator is (2.6) and we get the (new) specific differential operator

$$\mathrm{D}_{a,b} = \left(\frac{\partial}{\partial a} + ab^3\frac{\partial}{\partial b}\right) \tag{5.7}$$

as $\lambda = a^2 + 1/b^2$ and $\kappa_{a,b} = -1/2b^3$, when the associated nonseparable solution of (5.1) is

$$\mathrm{v}_{a,b}(x,y) = \left(\frac{\partial}{\partial a} + ab^3\frac{\partial}{\partial b}\right)[\sin(ax)\sin(y/b)] = x\cos(ax)\sin(y/b) - aby\sin(ax)\cos(y/b) \tag{5.8}$$

Further, as we have seen already, the methodology generalizes to encompass the use of non-cartesian coordinates, in parallel with Overfelt's approach [8].

The two methods lead to different results, of course. In particular, if we consider the previous problem from Overfelt's point of view, we can obtain a nonseparable solution of (5.1) from (5.5) by applying the operator M of equation (3) of [8]. Thus, we get the nonseparable solution of (5.1) (with M defined implicitly)

$$\mathrm{v}_O(x,y) = M[\sin(ax)\sin(y/b)] = \left(y\frac{\partial}{\partial x} - x\frac{\partial}{\partial y} \right)[\sin(ax)\sin(y/b)]$$
$$= ay\cos(ax)\sin(y/b) - (x/b)\sin(ax)\cos(y/b) \tag{5.9}$$

The meeting point of the two approaches hinges on the fact that the separable solution (5.3) for (5.1) is *symmetric* to the interchange of a with x and b with y, so that the operator (5.2) becomes, after the above interchange, the operator M of equation (3) of [8]. Similar remarks apply to the separable solutions and their corresponding $'\mathrm{D}'$ operators in all of the examples presented here and elsewhere: if a separable solution of a PDE has a parameter-variable interchange symmetry, then new symmetry operators can be generated from the associated $'\mathrm{D}'$ operators, thus enabling the production of other classes of nonseparable solutions to said PDE.

All in all, the two methodologies appear complementary, although the method developed here can be a more 'elastic' approach to the problem. To show this, we will derive another of Moseley's *ad hoc* operators [7] from a more general symmetry operator.

Consider again equation (2.1): multiplying through (2.1) by $\partial\lambda/\partial a_j$, differentiating with respect to a_j and scaling the result by $(\partial\lambda/\partial a_i)/(\partial\lambda/\partial a_j)$, gives

$$\left(\frac{\partial\lambda/\partial a_i}{\partial\lambda/\partial a_j}\right) L_1\left[\frac{\partial}{\partial a_j}\left(\frac{\partial\lambda}{\partial a_j}u\right)\right] = \left(\frac{\partial\lambda/\partial a_i}{\partial\lambda/\partial a_j}\right) L_2\left[\frac{\partial\lambda}{\partial a_j}\left(\frac{\partial\lambda}{\partial a_j}u\right) + \lambda\frac{\partial}{a_j}\left(\frac{\partial\lambda}{\partial a_j}u\right)\right] \tag{5.10}$$

Now, if we subtract (5.6) from (2.2), we find that (after cancellation)

$$L_1\left[\frac{\partial\lambda}{\partial a_j}\frac{\partial u}{\partial a_i} - \left(\frac{\partial\lambda/\partial a_i}{\partial\lambda/\partial a_j}\right)\frac{\partial}{\partial a_j}\left(\frac{\partial\lambda}{\partial a_j}u\right)\right] = \lambda L_2\left[\frac{\partial\lambda}{\partial a_j}\frac{\partial u}{\partial a_i} - \left(\frac{\partial\lambda/\partial a_i}{\partial\lambda/\partial a_j}\right)\frac{\partial}{\partial a_j}\left(\frac{\partial\lambda}{\partial a_j}u\right)\right] \tag{5.11}$$

and we have another (nonseparable) solution of (2.1), with corresponding operator

$$\mathrm{D}'_{i,j} = \kappa'_{i,j}\left(\frac{\partial\lambda}{\partial a_j}\frac{\partial}{\partial a_i} - \left(\frac{\partial\lambda/\partial a_i}{\partial\lambda/\partial a_j}\right)\frac{\partial}{\partial a_j}\frac{\partial\lambda}{\partial a_j}\right) \tag{5.12}$$

To match this result (5.12) with Moseley's [7], we consider again equation (4.15), with separable solution (4.16) subject to the condition (4.17), when (5.12) takes the particular form, with $\kappa'_{i,j} = 1/2$, of Moseley's equation (A3) of [7], that is

$$\mathrm{D}'_{m,p} = \left(p\frac{\partial}{\partial m} - \frac{m}{p}\frac{\partial}{\partial p}p\right) \tag{5.13}$$

and we can continue by following Moseley's calculations [7] and produce further classes of nonseparable solutions to (4.15), should we so wish.

6. Conclusions

In conclusion, we have presented a systematic method for obtaining non-separable solutions to PDE when corresponding separable solutions are available. The method is a refinement and generalization of the work of Chakrabarti [1] and Chester [2], which was based on the original work of Moseley [6]. Examples involving important PDE of mathematical physics have been presented to support the basic theoretical ideas and the link with a similar approach [8] elucidated. Also, several unsolved problems from the literature [6, 7] have been resolved within the compass of the method presented here. Finally, although we have not discussed them here, it is apparent that boundary and initial conditions must be attended to when solving PDE. As Overfelt remarked [8], this is the nub of many practical problems, but with nonseparable solutions it may not be that easy and problems may have to be solved on a case-by-case basis [8].

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