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FEYNMAN INTEGRALS PERTAINING TO ALEPH FUNCTION AND TWO GENERAL CLASS OF POLYNOMIALS

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ABSTRACT

The object of the present paper is to derive certain integral properties of Aleph function and two general class of polynomials. During the course of finding, we obtain some particular cases, which are also new and of interest by themselves. The ℵ-function is a generalization of the familiar H-function and the I-function. The results derived are of general character.

Key Words and Phrases:

Aleph function (ℵ-function); Feynman integrals; General class of polynomials; Hermite polynomials; Laguerre polynomials.

SUBJECT CLASSIFICATION

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INTRODUCTION AND PRELIMINARIES

Feynman integrals are useful in the study and development of simple and multivariable hypergeometric series which in turn are useful in the statistical mechanics. The conventional formulation may fail pertaining to the domain of quantum cosmology but Feynman path integrals apply [10, 11]. Feynman path integrals reformulation of quantum mechanics is more fundamental than the conventional formulation in term of operators.

In the study of fractional driftless Fokker-Plank equations with power law diffusion coefficients, there arises naturally a special function, which is a special case of the ℵ-function i.e. Aleph function. The idea to introduced Aleph-function belongs to Südland et al. [3]. The complete definition is given in the following manner in terms of the Mellin-Barnes type integrals [4]:

$$\aleph[z]=\aleph^{M,N}_{u_i,v_i,\alpha_i;r}[z]=\aleph^{M,N}_{u_i,v_i,\alpha_i;r}\left[z\left|\begin{matrix}(a_j,A_j)_{1,N},\ldots[\alpha_i(a_{ji},A_{ji})]_{N+1,u_i}\\(b_j,B_j)_{1,M},\ldots[\alpha_i(b_{ji},A_{ji})]_{M+1,v_i}\end{matrix}\right.\right] \tag{1.1}$$

or

$$\aleph[z]=\aleph^{M,N}_{u_i,v_i,\alpha_i;r}[z]=\aleph^{M,N}_{u_i,v_i,\alpha_i;r}\left[z\left|\begin{matrix}(a_j,A_j)_{1,N},[\alpha_i(a_{ji},A_{ji})]_{N+1,u_i};r\\(b_j,B_j)_{1,M},[\alpha_i(b_{ji},B_{ji})]_{M+1,v_i};r\end{matrix}\right.\right]$$

$$=\frac{1}{2\pi\omega}\int_{\ell}\Lambda^{M,N}_{u_i,v_i,\alpha_i;r}(\eta)\,z^{-\eta}\,d\eta \tag{1.2}$$

For all z ≠ 0, where $\omega=\sqrt{-1}$ and

$$\Lambda^{M,N}_{u_i,v_i,\alpha_i;r}(\eta)=\frac{\prod_{j=1}^{M}\Gamma(b_j+B_j\eta)\prod_{j=1}^{N}\Gamma(1-a_j-A_j\eta)}{\sum_{i=1}^{r}\alpha_i\prod_{j=N+1}^{u_i}\Gamma(a_{ji}+a_{ji}\eta)\prod_{j=M+1}^{v_i}\Gamma(1-b_{ji}-B_{ji}\eta)} \tag{1.3}$$

The path $\ell=\ell_{i\gamma\infty}$ is a suitable contour which extends from $\gamma-i\infty$ to $\gamma+i\infty.\gamma\in R$, the integers M,N,$u_i$,$v_i$ satisfy the inequality $0\le N\le u_i, 1\le M\le v_i, \alpha_i>0, i=1,2,3,\ldots,r.$ The parameters A_j, B_j, A_{ji}, B_{ji} are positive real numbers and a_j, b_j, a_{ji}, b_{ji} are complex numbers, the poles suppose to be simple, such that the poles of $\Gamma(b_j+B_j\eta), j=1,2,3,\ldots,M$ separating from those of $\Gamma(1-a_j-A_j\eta), j=1,2,\ldots,N.$ All the poles of integrand (1.1) are supposed to easy and empty product are considered as unity. The existence conditions for the function (1.1) are given below:

$$\psi_c>0, |\arg(z)|<\frac{\pi}{2}\psi_c;\ c=1,2,\ldots,r \tag{1.4}$$

$$\psi_c\ge 0, |\arg(z)|<\frac{\pi}{2}\psi_c \text{ and } R\{\nu_c\}+1<0 \tag{1.5}$$

Where $\psi_c=\sum_{j=1}^{N}A_j+\sum_{j=1}^{M}B_j-\alpha_c\left(\sum_{j=N+1}^{u_c}A_{jc}+\sum_{j=M+1}^{v_c}B_{jc}\right)$

(1.6)

$$\nu_c=\sum_{j=1}^{M}b_j-\sum_{j=1}^{N}a_c+\alpha_c\left(\sum_{j=1}^{v_c}b_{jc}-\sum_{j=N+1}^{u_c}a_{jc}\right)+\frac{1}{2}(u_c-v_c) \tag{1.7}$$

The general class of polynomial introduced by Srivastava [6]

$$S_n^m[X]=\sum_{k=0}^{[n/m]}\frac{(-n)_{mk}}{k!}D_{n,k}X^k,\ n=0,1,2,\ldots \tag{1.8}$$

Where m is an arbitrary positive integer and the coefficient $D_{n,k}$ (n,k ≥ 0) are arbitrary constant, real or complex.

2. SOME IMPORTANT RESULTS

In this section we establish the following results

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(A) $$\int_0^1\int_0^1\left(\frac{1-x}{1-xy}y\right)^{\lambda}\left(\frac{1-y}{1-xy}\right)^{\mu}\frac{(1-xy)}{(1-x)(1-y)}S_{n'}^{m'}\left[\frac{(1-x)}{(1-xy)}wy\right]S_{n''}^{m''}\left[\left(\frac{1-x}{1-xy}wy\right)^2\right]\times\aleph_{u_i,v_i,\alpha_i;r}^{M,N}\left[\frac{(1-y)}{(1-xy)}w\right]dx\,dy$$

$$=\sum_{k'=0}^{[n'/m']}\frac{(-n')_{m'k'}}{k'!}D_{n',k'}\sum_{k''=0}^{[n''/m'']}\frac{(-n'')_{m''k''}}{k''!}D_{n'',k''}W^{k'+2k''}\Gamma(k'+2k''+\lambda)$$

$$\times\aleph_{u_i+1,v_i+1,\alpha_i;r}^{M,N+1}\left[W\left|\begin{matrix}(1-\mu;1),(a_j,A_j)_{1,N},[\alpha_i(a_{ji},A_{ji})]_{N+1,u_i;r}\\(b_j,B_j)_{1,M},[\alpha_i(b_{ji},B_{ji})]_{M+1,v_i;r},\alpha_i(1-k'-2k''-\lambda-\mu;1)\end{matrix}\right.\right] \qquad (2.1)$$

Provided that $R[\lambda+\mu+b_j/\lambda_j]>0,|\arg w|<\frac{1}{2}T\pi,m',m''$ are arbitrary positive integers and the coefficients $D_{n',k'},D_{n'',k''}$ where $(n',m'\geq 0)$ and $(n',m''\geq 0)$ are arbitrary constants, real or complex.

Proof. We have

$$S_{n'}^{m'}\left[\left(\frac{1-x}{1-xy}\right)wy\right]S_{n''}^{m''}\left[\left(\frac{1-x}{1-xy}wy\right)^2\right]\aleph_{u_i,v_i,\alpha_i;r}^{M,N}\left[\frac{1-y}{1-xy}w\right]$$

$$=\sum_{k'=0}^{[n'/m']}\frac{(-n')_{m',k'}}{k'!}D_{n',k'}\left\{\frac{1-x}{1-xy}wy\right\}^{k'}\frac{(-n'')_{m''k''}}{k''!}D_{n'',k''}\left\{\left(\frac{1-x}{1-xy}wy\right)^2\right\}^{k''}\times\frac{1}{2\pi\omega}\int_{\ell}\Lambda_{u_i,v_i,\alpha_i;r}^{M,N}(\eta)\left[\frac{(1-y)}{(1-xy)}w\right]^{-\eta}d\eta. \qquad (2.2)$$

Multiply both sides of eq.(2.2) by

$$\left[\frac{(1-x)}{(1-xy)}y\right]^{\lambda}\left[\frac{1-y}{1-xy}\right]^{\mu}\left[\frac{1-xy}{(1-x)(1-y)}\right]$$

and integrating it with respect to x and y between 0 and 1 for both the variables and using known result [8, page 145]. After simplify we get the required result (2.1).

(B) $$\int_0^{\infty}\int_0^{\infty}F(x+y)y^{\mu-1}x^{\lambda-1}S_{n'}^{m'}[x]S_{n''}^{m''}[x^2]\aleph_{u_i,v_i,\alpha_i;r}^{M,N}[y]dx\,dy$$

$$=\sum_{k'=0}^{[n'/m']}\frac{(-n')_{m'k'}}{k'!}D_{n',k'}\sum_{k''=0}^{[n''/m'']}\frac{(-n'')_{m''k''}}{k''!}D_{n'',k''}\Gamma\lambda+k'+2k''\int_0^{\infty}\phi(s)s^{(\mu+\lambda+k'+2k''-1)}ds$$

$$\times\aleph_{u_i+1,v_i+1,\alpha_i;r}^{M,N+1}\left[S\left|\begin{matrix}(1-\mu;1),(a_j,A_j)_{1,N},[\alpha_i(a_{ji},A_{ji})]_{N+1,u_i;r}\\(b_j,B_j)_{1,M},[\alpha_i(b_{ji},B_{ji})]_{M+1,v_i;r},\alpha_i(1-k'-2k''-\lambda-\mu;1)\end{matrix}\right.\right] \qquad (2.3)$$

Provided that $R(\mu+\lambda+b_j/\lambda_j)>0,m',m''$ are arbitrary positive integers and coefficients $D_{n',k'},D_{n'',k''}(n',n'',k',k''\geq 0)$ are arbitrary constants, real or complex.

Proof. We have

$$S_{n'}^{m'}[x]S_{n''}^{m''}[x^2]\aleph_{u_i,v_i,\alpha_i;r}^{M,N}[y]=\left[\sum_{k'=0}^{[n'/m']}\frac{(-n')_{m'k'}}{k'!}D_{n',k'}x^{k'}\right]\left[\sum_{k''=0}^{[n''/m'']}\frac{(-n'')_{m''k''}}{k''!}D_{n'',k''}x^{2k''}\right]\times\frac{1}{2\pi\omega}\int_{\ell}\Lambda_{u_i,v_i,\alpha_i;r}^{M,N}(\eta)\,y^{-\eta}d\eta \qquad (2.4)$$

Multiply both sides eq.(2.4) by $F(x+y)\,y^{\mu-1}x^{\lambda-1}$ and integrating with respect to x and y between 0 and ∞ both the variables and making a use of a known result [8, p.177]. After simplification, we get desired result(2.3).

(C) $$\int_0^1\int_0^1 F(xy)(1-x)^{\lambda-1}(1-y)^{\mu-1}\,y^{\lambda}\,S_{n'}^{m'}[y(1-x)]\,S_{n''}^{m''}[y^2(1-x)^2]\times\aleph_{u_i,v_i,\alpha_i;r}^{M,N}[1-y]dx\,dy$$

$$=\sum_{k'=0}^{[n'/m']}\frac{(-n')_{m'k'}}{k'!}D_{n',k'}\sum_{k''=0}^{[n''/m'']}\frac{(-n'')_{m''k''}}{k''!}D_{n'',k''}\Gamma(\lambda+k'+k'').\int_0^1 f(\xi)(1-\xi)^{\lambda+\mu+k'+k''-1}d\xi$$

$$\times \aleph^{M,N+1}_{u_i+1,v_i+1,\alpha_i;r}\left[(1-\xi)\left|\begin{matrix}(1-\mu;1),(a_j,A_j)_{1,N},[\alpha_i(a_{ji},A_{ji})]_{N+1},u_i;r\\(b_j,B_j)_{1,M},[\alpha_i(b_{ji},B_{ji})]_{M+1},v_i;r,\alpha_i(1-\lambda-\mu-k''-k';1)\end{matrix}\right.\right] \tag{2.5}$$

provided that $R(\lambda)>0, R(\mu)>0, m', m''$ are an arbitrary integer and coefficients $D_{n',k'}, D_{n'',k'}(n',n'',m',m''\geq 0)$ are arbitrary constants, real or complex.

Proof. We have

$$S^{m'}_{n'}[y(1-x)]S^{m''}_{n''}[\{y(1-x\}^2]\aleph^{M,N}_{u_i,v_i,\alpha_i;r}[1-y]$$

$$=\sum_{k'=0}^{[n'/m']}\frac{(-n')_{m'k'}}{k'!}D_{n',k'}\,y^{k'}(1-x)^{k'}\sum_{k''=0}^{[n''/m'']}\frac{(-n'')_{m'',k''}}{k''!}D_{n'',k''}[y(1-x)]^{2k'}\times\frac{1}{2\pi\omega}\int_{\ell}\Lambda^{M,N}_{u_i,v_i,\alpha_i;r}(\eta)(1-y)^{-\eta}d\eta \tag{2.6}$$

Multiplying both sides of eq.(2.6) by F(xy) $(1-x)^{\lambda-1}(1-y)^{\mu-1}y^{\lambda}$ and integrating with respect to x and y between 0 and 1 in view of result [8, p.243] and by further simplification, we get required result (2.5).

(D) $$\int_0^1\int_0^1\left[\frac{y(1-x)}{1-xy}\right]^{\lambda+\xi}\left[\frac{1-y}{1-xy}\right]^{\mu}\frac{1}{(1-x)}S^{m'}_{n'}\left[\frac{y(1-x)}{1-xy}\right]S^{m''}_{n'}\left[\frac{y^2(1-x)^2}{(1-xy)^2}\right]\times\aleph^{M,N}_{u_i,v_i,\alpha_i;r}\left[\left(\frac{1-x}{1-xy}\right)wy\right]dx\,dy$$

$$=\sum_{k'=0}^{[n'/m']}\frac{(-n')_{m'k'}}{k'!}D_{n',k'}\sum_{k''=0}^{[n''/m'']}\frac{(-n'')_{m''k''}}{k''!}D_{n'',k''}\Gamma(\mu+1)\times\aleph^{M,N+1}_{u_i+1,v_i+1,\alpha_i;r}\left[W\left|\begin{matrix}(1-\lambda-\xi-k'-k'';1),(a_j,A_j)_{1,N},[\alpha_i(a_{ji},A_{ji})]_{N+1},u_i;r\\(b_j,B_j)_{1,M},[\alpha_i(b_{ji},B_{ji})]_{M+1},v_i;r,\alpha_i(\lambda-\xi-k'-k''-\mu;1)\end{matrix}\right.\right] \tag{2.7}$$

provided that $R(\lambda+\mu+\xi+b_j/\mu_j)>0, |\arg w|<\frac{1}{2}T\pi, m', m''$ are an arbitrary integer and the coefficients $D_{n',k'}(n',k'\geq 0),(D_{n'',k''}(n'',k''>0)$ are arbitrary constants, real or complex.

Proof. We have

$$S^{m'}_{n'}\left[\frac{y(1-x)}{1-xy}\right]S^{m''}_{n''}\left[\frac{y^2(1-x)^2}{(1-xy)^2}\right]\aleph^{M,N}_{u_i,v_i,\alpha_i;r}\left[\left(\frac{1-x}{1-xy}\right)wy\right]$$

$$=\sum_{k'=0}^{[n'/m']}\frac{(-n')_{m'k'}}{k'!}D_{n',k'}\left\{\frac{y(1-x)}{1-xy}\right\}^{k'}\sum_{k''=0}^{[n''/m'']}\frac{(-n'')_{m'',k''}}{k''!}D_{n'',k''}\left\{\frac{y^2(1-x)^2}{(1-xy)^2}\right\}^{k''}\times\frac{1}{2\pi\omega}\int_{\ell}\Lambda^{M,N}_{u_i,v_i,\alpha_i;r}(\eta)\left[\frac{(1-x)}{(1-xy)}wy\right]^{-\eta}d\eta \tag{2.8}$$

Multiplying both sides of eq.(2.8) by $\left[\frac{y(1-x)}{1-xy}wy\right]^{\lambda+\xi}\left[\frac{1-y}{1-xy}\right]^{\mu}\frac{1}{(1-x)}$ and integrating with respect to x and y between 0 and 1 for both the variables, we get the result (2.8) after simplification.

3. SPECIAL CASES

As Aleph function is the most generalized special function, numerous special cases with useful transcendental functions as Bessel functions, hypergeometric function, I function, Fox H-function, generalized hypergeometric $_pF_q$ function and polynomials as Hermite polynomials, Laguerre polynomials and their special cases can be deduced by making suitable changes in the parameter.

(1) The case of Hermite polynomial [7] and [9] and the general class of polynomial introduced by Srivastava [6] and by setting

$$S^2_n(x)\to x^{n/2}H_n\left[\frac{1}{2\sqrt{x}}\right]$$

in which case $m=2, D_{n,k}=(-1)^k$, we have the following interesting consequences of the results (2.1), (2.3), (2.5) and(2.7).

(C.1) $$\int_0^1\int_0^1\left(\frac{1-x}{1-xy}y\right)^{\lambda}\left(\frac{1-y}{1-xy}\right)^{\mu}\frac{(1-xy)}{(1-x)(1-y)}\left[\frac{(1-x)}{(1-xy)}wy\right]^{n'/2}$$

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$$. \, H_{n'}\left[\frac{1}{2\sqrt{\frac{(1-x)}{(1-xy)}wy}}\right]\left[\frac{(1-x)^2}{(1-xy)^2}w^2y^2\right]^{n''/2} H_{n}''\left[\frac{1}{2\sqrt{\left(\frac{1-x}{1-xy}wy\right)^2}}\right] . \, \aleph^{M,N}_{u_i,v_i,\alpha_i;r}\left[\frac{1-y}{1-xy}w\right]dx\,dy$$

$$=\sum_{k'=0}^{[n'/2]}\frac{(-n')_{2k'}}{k'!}(-1)^{k'}\sum_{k''=0}^{[n''/2]}\frac{(-n'')_{2k''}}{k''!}(-1)^{k''}W^{k'+2k''}\Gamma(k'+2k''+\lambda)\times\aleph^{M,N+1}_{u_i+1,v_i+1,\alpha_i;r}\left[W\left|\begin{matrix}(1-\mu;1),(a_j,A_j)_{1,N},[\alpha_i(a_{ji},A_{ji})]_{N+1,u_i;r}\\(b_j,B_j)_{1,M},[\alpha_i(b_{ji},B_{ji})]_{M+1,v_i;r},\alpha_i(1-k''-k'-\lambda-\mu;1)\end{matrix}\right.\right]$$

the condition of validity are the same as stated in result (2.1) given in section 2.

(C.2) $$\int_0^\infty\int_0^\infty F(x+y)\,y^{\mu-1}\,x^{\lambda-1}\,x^{\lambda+\frac{n'}{2}+n''-1}\,H_{n'}\left[\frac{1}{2\sqrt{x}}\right]H_{n''}\left[\frac{1}{2\sqrt{x^2}}\right]\times\aleph^{M,N}_{u_i,v_i,\alpha_i;r}[y]\,dx\,dy$$

$$=\sum_{k'=0}^{[n'/2]}\frac{(-n')_{2k'}}{k'!}(-1)^{k'}\sum_{k''=0}^{[n''/2]}\frac{(-n'')_{2k''}}{k''!}(-1)^{k''}\Gamma(\lambda+k'+2k'')\int_0^\infty\phi(\xi)\xi^{\mu+\lambda+k'+2k''-1}d\xi$$

$$\times\left[\xi\left|\begin{matrix}(1-\mu;1),(a_j,A_j)_{1,N},[\alpha_i(a_{ji},A_{ji})]_{N+1,u_i;r}\\(b_j,B_j)_{1,M},[\alpha_i(b_{ji},B_{ji})]_{M+1,v_i;r},\alpha_i(1-k'-2k''-\lambda-\mu;1)\end{matrix}\right.\right]$$

the condition of validity are the same as stated in result (2.3) in section 2.

(C.3) $$\int_0^1\int_0^1 F(xy)(1-x)^{\lambda-1}(1-y)^{\mu-1}\,y^{\lambda+\frac{n'}{2}+n''}(1-x)^{n'/2}\,H_{n'}\left[\frac{1}{2\sqrt{y(1-x)}}\right](1-x)^{n''}H_{n''}\left[\frac{1}{2y(1-x)}\right]\aleph^{M,N}_{u_i,v_i,\alpha_i;r}[1-y]\,dx\,dy$$

$$=\sum_{k'=0}^{[n'/2]}\frac{(-n')_{2k'}}{k'!}(-1)^{k'}\sum_{k''=0}^{[n''/2]}(-1)^{k''}\frac{(-n'')_{2k''}}{k''!}\Gamma(\lambda+k'+2k''))\int_0^1 f(\xi)(1-\xi)^{\mu+\lambda+k'+k''-1}d\xi$$

$$\times\aleph^{M,N+1}_{u_i+1,v_i+1,\alpha_i;r}\left[1-\xi\left|\begin{matrix}(1-\mu;1),(a_j,A_j)_{1,N},[\alpha_i(a_{ji},A_{ji})]_{N+1,u_i;r}\\(b_j,B_j)_{1,M},[\alpha_i(b_{ji},B_{ji})]_{M+1,v_i;r},\alpha_i(1-\lambda-\mu-k'-k'';1)\end{matrix}\right.\right]$$

valid under the same condition as required for result (2.5) given in section 2.

(C.4) $$\int_0^1\int_0^1\left[\frac{y(1-x)}{(1-xy)}\right]^{\lambda+\xi}\left[\frac{1-x}{1-xy}\right]^{\mu}\frac{1}{(1-x)^{1-\frac{n'}{2}-n''}}\frac{y^{\frac{n'}{2}+n''}}{(1-xy)^{\frac{n'}{2}+n''}}$$

$$. \, H_{n'}\left[\frac{1}{2\sqrt{\left\{\frac{y(1-x}{1-xy}\right\}}}\right]H_{n''}\left[\frac{1}{2\left\{\frac{y(1-x}{1-xy}\right\}}\right]\aleph^{M,N}_{u_i,v_i,\alpha_i;r}\left[\frac{(1-x)}{(1-xy)}wy\right]dx\,dy$$

$$=\sum_{k'=0}^{[n'/2]}\frac{(-n')_{2k'}}{k'!}(-1)^{k'}\sum_{k''=0}^{[n''/2]}\frac{(-n'')_{2k''}}{k''!}(-1)^{k''}\Gamma(\mu+1)\times\left[W\left|\begin{matrix}(1-\lambda-\xi-k'-k'';1),(a_j,A_j)_{1,N},[\alpha_i(a_{ji},A_{ji})]_{N+1,u_i;r}\\(b_j,B_j)_{1,M},[\alpha_i(b_{ji},B_{ji})]_{M+1,v_i;r},\alpha_i(-\lambda-\xi-k'-k''-\mu;1)\end{matrix}\right.\right]$$

valid under the same conditions as required for (2.7) given in section 2.

(2) For the Laguerre polynomials ([7] and [9]) setting $S_n^{'}[x]\to L_n^{(\rho)}[x]$ in which case m = 1, $D_{n,k}=\binom{n+\rho}{n}\frac{1}{(\rho+1)^k}$, the results (2.1), (2.3), (2.5), (2.7) convert to the following formulae:

(D.1) $$\int_0^1\int_0^1\left(\frac{1-x}{1-xy}y\right)^{\lambda}\left(\frac{1-y}{1-xy}\right)^{\mu}\frac{(1-xy)}{(1-x)(1-y)}L_{n'}^{(\rho)}\left[\frac{1-x}{1-xy}wy\right] . \, L_{n''}^{(\rho)}\left[\left(\frac{1-x}{1-xy}wy\right)^2\right]\aleph^{M,N}_{u_i,v_i,\alpha_i;r}\left[\left(\frac{1-y}{1-xy}\right)w\right]dx\,dy$$

$$=\sum_{k'=0}^{[n']}\frac{(-n')_{k'}}{k'!}\binom{n'+\rho}{n'}\left(\frac{1}{(\rho+1)_{k'}}\right)\sum_{k''=0}^{[n'']}\frac{(-n'')_{k''}}{k''!}\binom{n''+\rho}{n''}\frac{1}{(\rho+1)_{k''}}W^{k'+k''}\Gamma(k'+k''+\lambda)$$

$$\times\aleph^{M,N+1}_{u_i+1,v_i+1,\alpha_i;\mu}\left[W\,\middle|\,\begin{matrix}(1-\mu;1),(a_j,A_j)_{1,N},[\alpha_i(a_{ji},A_{ji})]_{N+1,u_i;r}\\(b_j,B_j)_{1,M},[\alpha_i(b_{ji},B_{ji})]_{M+1,v_i;r},\alpha_i(1-k'-2k''-\lambda-\mu;1)\end{matrix}\right]$$

the condition of the validity are the same as stated in (2.1) given in section 2.

(D.2) $$\int_0^\infty\int_0^\infty F(x+y)\,y^{\mu-1}\,x^{\lambda-1}L^{(\rho)}_{n'}(x)\,L^{(\rho)}_{n''}(x^2)\aleph^{M,N}_{u_i,v_i,\alpha_i;r}[y]\,dx\,dy$$

$$=\sum_{k'=0}^{[n']}\frac{(-n')_{k'}}{k'!}\binom{n'+\rho}{n'}\frac{1}{(\rho+1)_{k'}}\sum_{k''=0}^{[n'']}\frac{(-n'')_{k''}}{k''!}\binom{n''+\rho}{n''}\frac{1}{(\rho+1)_{k''}}\Gamma(\lambda+k'+k'')$$

$$\times\int_0^\infty\phi(\xi)\xi^{\mu+\lambda+k'+2k''-1}d\xi\aleph^{M,N+1}_{u_i+1,v_i+1,\alpha_i;\mu}\left[\xi\,\middle|\,\begin{matrix}(1-\mu;1),(a_j,A_j)_{1,N},[\alpha_i(a_{ji},A_{ji})]_{N+1,u_i;r}\\(b_j,B_j)_{1,M},[\alpha_i(b_{ji},B_{ji})]_{M+1,v_i;r},\alpha_i(1-k'-k''-\lambda-\mu;1)\end{matrix}\right]$$

the condition of validity are the same as stated in (2.3),given in section 2.

(D.3) $$\int_0^1\int_0^1 F(xy)(1-x)^{\lambda-1}(1-y)^{\mu-1}\,y^{\lambda}\,L^{(\rho)}_{n'}[y(1-y)]L^{(\rho)}_{n''}[y^2(1-y)^2]\times\aleph^{M,N}_{u_i,v_i,\alpha_i;r}[1-y]\,dx\,dy$$

$$=\sum_{k'=0}^{[n']}\frac{(-n')_{k'}}{k'!}\binom{n'+\rho}{n'}\frac{1}{(\rho+1)_{k'}}\sum_{k''=0}^{[n'']}\frac{(-n'')_{k''}}{k''!}\binom{n''+\rho}{n''}\times\frac{1}{(\rho+1)_{k''}}\Gamma(\lambda+k'+k'')\int_0^1 f(\xi)(1-\xi)^{\mu+\lambda+k'+k''-1}d$$

$$\times\aleph^{M,N+1}_{u_i+1,v_i+1,\alpha_i;r}\left[1-\xi\,\middle|\,\begin{matrix}(1-\mu;1),(a_j,A_j)_{1,N},[\alpha_i(a_{ji},A_{ji})]_{N+1,u_i;r}\\(b_j,B_j)_{1,M},[\alpha_i(b_{ji},B_{ji})]_{M+1,v_i;r},\alpha_i(1-\lambda-\mu-k'-k'';1)\end{matrix}\right]$$

valid under the same conditions as required for result(2.5) given in section 2.

(D.4) $$\int_0^1\int_0^1\left[\frac{y(1-x)}{(1-xy)}\right]^{\lambda+\xi}\left[\frac{1-y}{1-xy}\right]^{\mu}\frac{1}{(1-x)}L^{(\rho)}_{n'}\left[\frac{y(1-x)}{(1-xy)}\right].\ L^{(\rho)}_{n''}\left[\left\{\frac{y(1-x)}{(1-xy)}\right\}^2\right]\aleph^{M,N}_{u_i,v_i,\alpha_i;r}\left[\frac{(1-x)}{(1-xy)}wy\right]dx\,dy$$

$$=\sum_{k'=0}^{[n']}\frac{(-n')_{k'}}{k'!}\binom{n'+\rho}{n'}\frac{1}{(\rho+1)_{k'}}\sum_{k''=0}^{[n'']}\frac{(-n'')_{2k''}}{k''!}\binom{n''+\rho}{n''}\frac{1}{(\rho+1)_{k''}}\Gamma(\mu+1)\times\left[W\,\middle|\,\begin{matrix}(1-\lambda-\xi-k'-k'';1),(a_j,A_j)_{1,N},[\alpha_i(a_{ji},A_{ji})]_{N+1,u_i;r}\\(b_j,B_j)_{1,M},[\alpha_i(b_{ji},B_{ji})]_{M+1,v_i;r},\alpha_i(-\lambda-\xi-k'-k''-\mu;1)\end{matrix}\right]$$

valid under the same conditions as required for (2.7) given in section 2.

Remark 1. For $\alpha_1=\alpha_2=\ldots=\alpha_r=1$ in (1) then it reduces to the I-function [13] as

$$I[z]=\aleph^{M,N}_{u_i,v_i,1;r}[z]=\aleph^{M,N}_{u_i,v_i;1,r}\left[z\,\middle|\,\begin{matrix}(a_j,A_j)_{1,N},\ldots(a_j,A_j)_{N+1,u_i}\\(b_j,B_j)_{1,M},\ldots(b_j,B_j)_{M+1,v_i}\end{matrix}\right]=\frac{1}{2\pi\omega}\int_{\ell}\Lambda^{M,N}_{u_i,v_i,1;r}(\eta)z^{-\eta}\,d\eta \tag{3.1}$$

where $\Lambda^{M,N}_{u_i,v_i,1;r}(\eta)$ is defined in known result (3.1). The existence conditions for the integral in eq.(3.1) are the same as given in (1.4) – (1.7) with $\alpha_i=1, i=1,2,\ldots,r$.

Remark 2. If we set $r=1,\alpha_i=1, i=1,2,\ldots,r,$ then eq.(3.1) reduces to the H-function as

$$H^{M,N}_{u,v}[z]=\aleph^{M,N}_{u_i,v_i,1;1}[z]=\aleph^{M,N}_{u_i,v_i,1;1}\left[z\,\middle|\,\begin{matrix}(a_u,A_u)\\(b_v,A_v)\end{matrix}\right]=\frac{1}{2\pi\omega}\int_{\ell}\Lambda^{M,N}_{u_i,v_i,1;1}(\eta)z^{-\eta}\,d\eta \tag{3.2}$$

4.CONCLUSION

The Aleph-function, presented in this paper, is quite basic in nature. The results discussed here are unified in nature are likely to find useful application in several fields such as probability, electrical networks and statistical mechanics. Therefore, on specializing the parameters of the function, we may obtain various other special functions such as Fox's H-function, Meijer's G-function, Wright's generalized Bessel function, Wright's generalized hypergeometric function, generalized hypergeometric function, exponential function, binomial function etc.

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