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Existence of solution for a coupled system ofVolterra type integro - differential equations with nonlocal conditions

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Abstract

In this paper we study the existence of a unique solution for a boundary value problemof a coupled system of Volterra type integro-differential equations under nonlocal conditions.

Keywords: Nonlocal boundary value problems, integro - differential equation;coupled system; Lipschitz condition; Banach fixed point theorem.

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1 INTRODUCTION

The study of value problem with nonlocal conditions is of significance, since they have application in problems in physics, engineering, economics and other areas of applied Mathematics. This feature allows the study of several types of integral equations such as:Fredholm, Volterra, Hammerstein, Urysohn, for different classes of functionals [see(6),(7)]. The main object of this paper to study the existence of solution$x, y \in C[0,1]$ and$x, y \in AC[0,1]$ for the coupled system of Volterra integro- differential equations

$$\frac{dx}{dt} = \int_0^t \mathrm{f}_1\left(t, s, \frac{dy}{ds}\right) ds, \quad t \in (0,1)$$

(1)

$$\frac{dy}{dt} = \int_0^t \mathrm{f}_2\left(t, s, \frac{dx}{ds}\right) ds, \quad t \in (0,1)$$

with the nonlocal boundary conditions

$$x(\tau) = \alpha x(\xi), \quad \tau \in [0,1), \quad \xi \in (0,1], \quad \alpha \neq 1 \quad (2)$$

and

$$y(\tau) = \beta y(\xi), \quad \tau \in [0,1), \quad \xi \in (0,1], \quad \beta \neq 1 \quad (3)$$

Let

$\frac{dx}{dt} = u$, and $\frac{dy}{dt} = v$ in (1), we obtain

$$u(t) = \int_0^t \mathrm{f}_1(t, s, v(s)) ds, \quad t \in (0,1)$$

(4)

$$v(t) = \int_0^t \mathrm{f}_2(t, s, u(s)) ds, \quad t \in (0,1)$$

where

$$x(t) = x(0) + \int_0^t u(s)\, ds \quad (5)$$

$$y(t) = y(0) + \int_0^t v(s)\, ds \quad (6)$$

Using the nonlocal boundary condition (2), we obtain

$$x(\tau) = x(0) + \int_0^\tau u(s)\, ds,$$

and

$$x(\xi) = x(0) + \int_0^\xi u(s)\, ds,$$

then

$$x(0) = \frac{\alpha}{1-\alpha}\int_0^\xi u(s)\, ds - \frac{1}{1-\alpha}\int_0^\tau u(s)\, ds$$

Substituting in (5), we obtain

$$x(t) = \frac{\alpha}{1-\alpha}\int_0^\xi u(s)\, ds - \frac{1}{1-\alpha}\int_0^\tau u(s)\, ds + \int_0^t u(s)\, ds. \quad (7)$$

And using the nonlocal boundary condition (3), we obtain

$$y(\tau) = y(0) + \int_0^\tau v(s)\, ds,$$

and

$$y(\xi) = y(0) + \int_0^{\xi} v(s)\, ds,$$

then

$$y(0) = \frac{\beta}{1-\beta}\int_0^{\xi} v(s)\, ds - \frac{1}{1-\beta}\int_0^{\tau} v(s)\, ds.$$

Substituting in (6), we obtain

$$y(t) = \frac{\beta}{1-\beta}\int_0^{\xi} v(s)\, ds - \frac{1}{1-\beta}\int_0^{\tau} v(s)\, ds + \int_0^{t} v(s)\, ds. \quad (8)$$

3. Existence of a unique continuous solution

Here, we study the existence of a unique continuous solution of the coupled system of integral equations (4) , under the following assumptions:

(1) $f_i : [0,1] \times [0,1]\ R_+ \to R$ are continuous, and satisfy the Lipschitz condition

$$|f_i(t,s,x) - f_i(t,s,y)| \le k_i(t,s)|x - y|, \quad i = 1,2.$$

where

$$k_i : [0,1] \times [0,1] \to R_+ \text{are integral in } (t,s).$$

(2) $sup_t \int_0^1 k_i\ (t,s)ds \le M_i, \quad t \in [0,1], \quad i = 1,2.$

Let $X = \{U = (u,v): u,v \in C[0,1]\}$, and its norm defined as

$$\|(u,v)\| = \|u\| + \|v\| = sup_t |u(t)| + sup_t |v(t)|, \qquad t \in [0,1].$$

Now, for the existence of a unique continuous solution for the coupled system of the integral equations (4) , we have the following theorem.

Theorem 1. Let the assumption (1)-(2) be satisfied. $If M_i < 1,\ i = 1,2$, then the coupled system of integral equations (4) has a unique solution in X.

Proof. Define the operator F associated with the coupled system of integral equations (4) by

$$F(u,v) = (F_1 v, F_2 u)$$

Where

$$F_1 v = \int_0^t f_1(t,s,v(s))ds$$

$$F_2 u = \int_0^t f_2(t,s,u(s))ds,$$

Firstly prove that $F : X \to X$.

Let

$u,v \in C[0,1], t_1,t_2 \in [0,1],\ t_1 < t_2$, and $|t_2 - t_1| \le \delta$, now to prove

$F_1 v(t): C[0,1] \to C[0,1]$, then

$$|F_1v(t_2) - F_1v(t_1)| = |\int_0^{t_2} f_1(t_2,s,v(s))\,ds - \int_0^{t_1} f_1(t_1,s,v(s))ds|$$

$$= |\int_0^{t_1} f_1(t_2,s,v(s))\,ds + \int_{t_1}^{t_2} f_1(t_2,s,v(s))\,ds$$

$$-\int_0^{t_1} f_1(t_1,s,v(s))ds|$$

$$\le |\int_0^{t_1} f_1(t_2,s,v(s))\,ds - \int_0^{t_1} f_1(t_1,s,v(s))ds|$$

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$+\left|\int_{t_1}^{t_2} f_1(t_2, s, v(s))\, ds\right|$

$\leq \int_0^{t_1} \left| f_1(t_2, s, v(s)) - f_1(t_1, s, v(s))\right| ds$

$+\int_{t_1}^{t_2} \left| f_1(t_2, s, v(s))\right| ds.$

This prove that $F_1 v(t) : C[0,1] \to C[0,1],\ \forall v(t) \in C[0,1]$.

As done before, we obtain

$$F_2 u(t) : C[0,1] \to C[0,1], \qquad \forall u(t) \in C[0,1].$$

Now since $F(u,v) = (F_1 v, F_2 u)$

$$F(u,v)(t_2) - F(u,v)(t_1) = F(u(t_2), v(t_2)) - F(u(t_1), v(t_1))$$

$$= (F_1 v(t_2) - F_1 v(t_1), F_2 u(t_2) - F_2 u(t_1)).$$

Then

$$\|F(u,v)(t_2) - F(u,v)(t_1)\| = \|(F_1 v(t_2) - F_1 v(t_1)\| + \|F_2 u(t_2)\ F_2 u(t_1))\|.$$

Hence

$F : X \to X.$

Secondly to prove that F is a contraction, we have following.

Let

$Z = (u,v) \in X$ and $Z_1 = (u_1, v_1) \in X,$ we have

$$F(u,v) = (F_1 v(t), F_2 u(t))$$

and

$$F(u_1,v_1) = (F_1 v_1(t), F_2 u_1(t)),$$

then

$$|F_1 v(t) - F_1 v_1(t)| = \left|\int_0^t f_1(t,s,v(s))\, ds - \int_0^t f_1(t,s,v_1(s))\, ds\right|$$

$$\leq \int_0^t \left|f_1(t,s,v(s)) - f_1(t,s,v_1(s))\right| ds$$

$$\leq \int_0^t k_1(t,s)\, |v(s) - v_1(s)|\, ds$$

$$\leq \int_0^t k_1(t,s) \sup_t |v(s) - v_1(s)|\, ds$$

$$\leq \|v(s) - v_1(s)\| \int_0^t k_1(t,s)\, ds.$$

Then

$$\|F_1 v(t) - F_1 v_1(t)\| \leq M_1 \|v - v_1\| .$$

Since $M_1 < 1$, then F_1 is a contraction.

As done before, we obtain

$$|F_2 u(t) - F_2 u_1(t)| \leq M_2 \|u - u_1\|.$$

Since $M_2 < 1$, then F_2 is a contraction.

Then

$$\|F(u,v) - F(u_1,v_1)\| = \|(F_1 v, F_2 u) - (F_1 v_1, F_2 u_1)\|$$

$$= \|F_1 v - F_1 v_1, F_2 u - F_2 u_1\|$$

$$= \|F_1 v - F_1 v_1\| + \|F_2 u - F_2 u_1\|$$

$$\leq max\{M_1, M_2\}\|(u, v) - (u_1, v_1)\|.$$

$Let\ M = max\{M_1, M_2\}$

$$\|F(u,v) - F(u_1,v_1)\| \leq M\|(u,v) - (u_1,v_1)\|.$$

Since$M < 1$, then F is a contraction, by using Banach fixed point Theorem[(5)], then there exists a unique solution in X for the coupled system of the integral equations (4).

4 Solution of the problem (1)-(3)

Consider now the problem (1)-(3).

Theorem 2. Let the assumption of the theorem 1 be satisfied, then there exists a unique solution $x, y \in C[0,1]$ of the problem (1)-(3) .

Proof. The solution of the problem (1) and (3) is given by

$$x(t) = \frac{\alpha}{1-\alpha}\int_0^{\xi} u(s)\,ds - \frac{1}{1-\alpha}\int_0^{\tau} u(s)\,ds + \int_0^{t} u(s)\,ds \quad \in C[0,1],$$

and

$$y(t) = \frac{\beta}{1-\beta}\int_0^{\xi} v(s)\,ds - \frac{1}{1-\beta}\int_0^{\tau} v(s)\,ds + \int_0^{t} v(s)\,ds \quad \in C[0,1].$$

Where

$$u(t) = \int_0^t f_1(t,s,v(s))ds, \quad \in C[0,1].$$

$$v(t) = \int_0^t f_2(t,s,u(s))ds, \quad \in C[0,1].$$

Then from Theorem 1 we can deduce that there exists a unique continuous solution of the problem (1)-(3).

5. Existence of a unique L^1 –solution

Here, we study the existence of integrable solution of the coupled system of integral equations (4) under the following assumptions:

(i) $f_i : [0,1] \times [0,1] \times R_+ \to R$ are measurable in (t,s), and satisfy the Lipschitz condition

$$|f_i(t,s,x) - f_i(t,s,y)| \leq k_i |x - y|, \quad i = 1,2.$$

(ii) $f_i(t,s\ 0) \in L^1[0,1]$, and

$$\int_0^1 |f_i(t\ s\ 0)|\ dt \leq Mi, \quad t \in [0,1], \quad i = 1,2.$$

Let $Y = \{U = (u,v) : u, v \in L^1[0,1]\}$, and its norm defined as

$$\|(u,v)\| = \|u\| + \|v\| = \int_0^1 |u(t)|dt + \int_0^1 |v(t)|dt$$

Now, for the existence of integrable solution for the coupled system of the integral equations(4), we have the following theorem.

Theorem 3. Let the assumption (i)-(ii) be satisfied. $If\ k_i < 1, i = 1,2,$ then the coupledsystem of the integral equations (4) has a unique solution in Y .

Proof. Define the operator G associated with the coupled system of integralequations (4)by

$$G(u,v) = (G_1 v, G_2 u).$$

Where

$$G_1 v = \int_0^t f_1(t,s,v(s))ds$$

$$G_2 u = \int_0^t f_2(t,s,u(s))ds.$$

Firstly to prove that $G : Y \to Y$,

now to prove $G_1 v : L^1[0,1] \to L^1[0,1]$, then

$$|f_1(t,s,v)| - |f_1(t,s,0)| \le |f_1(t,s,v) - f_1(t,s,0)| \le k_1 |v|$$

and

$$|f_1(t,s,v)| \le k_1 |v| + |f_1(t,s,0)|.$$

Hence

$$|G_1 v(t)| = \int_0^t |f_1(t,s,v(s))ds| \le \int_0^t k_1 |v(s)| ds + |f_1(t,s,0)|.$$

Integrating both sides with respect to t, we obtain

$$\int_0^1 |\int_0^t f_1(t,s,v(s))ds| dt \le \int_0^1 \int_0^t k_1 |v(s)| ds\, dt + \int_0^1 |f_1(t,s,0)| dt$$

$$\le k_1 \int_0^1 |v(s)| dt + \int_0^1 |f_1(t,s,0)| dt$$

$$\le k_1 \|v\|_{L^1} + M_1.$$

Then $\|G_1 v\|_{L^1} \le k_1 \|v\|_{L^1} + M_1$.

This proves that $G_1 v : L^1[0,1] \to L^1[0,1]$.

As done before, we obtain

$$\|G_2 u\|_{L^1} \le k_2 \|u\|_{L^1} + M_2.$$

This proves that $G_2 u : L^1[0,1] \to L^1[0,1]$.

Hence

$$\|G(u,v)\| = \|G_1 v, G_2 u\|$$

$$= \|G_1 v\| + \|G_2 u\|$$

$$= k_1 \|v\|_{L^1} + M_1 + k_2 \|v\|_{L^1} + M_2.$$

This proves $G: Y \to Y$.

Secondly prove that G is a contraction.

Let $Z = (u,v) \in Y$ and $Z_1 = (u_1, v_1) \in Y$.

Then $G(u,v) = (G_1 v, G_2 u)$ and $G(u_1,v_1) = (G_1 v_1, G_2 u_1)$

$$|G_1 v - G_1 v_1| = |\int_0^t f_1(t,s,v(s))ds - \int_0^t f_1(t,s,v_1(s))ds|.$$

Integrating both sides with respect to t, we

$$\int_0^1 |G_1 v - G_1 v_1| dt \le \int_0^1 |\int_0^t [f_1(t,s,v(s)) - f_1(t,s,v_1(s))] ds| dt$$

$$\le \int_0^1 \int_0^t k_1 | v(s) - v_1(s)| ds\, dt$$

$$\leq k_1 \int_0^1 |v(s) - v_1(s)|\, dt$$

$$\leq k_1 \| v - v_1 \|_{L^1}.$$

Then

$$\| G_1 v - G_1 v_1 \|_{L^1} \leq k_1 \| v - v_1 \|_{L^1}.$$

Since $k_1 < 1$, then G_1 is a contraction.

As done before, we obtain

$$\| G_2 u - G_2 u_1 \|_{L^1} \leq k_2 \| u - u_1 \|_{L^1}.$$

Since $k_2 < 1$, then G_2 is a contraction.

Hence

$$\| G(u,v) - G(u_1,v_1) \| = \| (G_1 v, G_2 u) - (G_1 v_1, G_2 u_1) \|$$

$$= \| G_1 v - G_1 v_1, G_2 u - G_2 u_1 \|$$

$$\leq max\{k_1, k_2\} \| (u,v) - (u_1,v_1) \|_{L^1}.$$

Let $k = max\{k_1, k_2\}$.

Then

$$\| G(u,v) - G(u_1,v_1) \|_{L^1} \leq k \| (u,v) - (u_1,v_1) \|_{L^1}.$$

Since $k < 1$, then G is a contraction, by using Banach fixed point Theorem [(5)], then there exists of solution in Y for the coupled system of the integral equations (4).

6. Solution of the problem (1)-(3)

Consider now the problem (1)-(3).

Theorem 4. Let the assumption of the theorem 3 be satisfied, then there exists a uniquesolution $x, y \in AC[0,1]$ of the boundary value problem (1)-(3).

Proof. The solution of the problem (1) - (3) is given by

$$x(t) = \frac{\alpha}{1-\alpha} \int_0^{\xi} u(s)\, ds - \frac{1}{1-\alpha} \int_0^{\tau} u(s)\, ds + \int_0^{t} u(s)\, ds \in AC[0,1],$$

and

$$y(t) = \frac{\beta}{1-\beta} \int_0^{\xi} v(s)\, ds - \frac{1}{1-\beta} \int_0^{\tau} v(s)\, ds + \int_0^{t} v(s)\, ds \in AC[0,1].$$

Where

$$u(t) = \int_0^1 f_1(t, s, v(s))\, ds \quad \in L^1[0,1]$$

$$v(t) = \int_0^1 f_2(t, s, u(s))\, ds \quad \in L^1[0,1].$$

Then from Theorem 3 we can deduce that there exists a unique solution of the problem (1) - (3).

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