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Real Characterization on Order Banach Algebra

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Abstract

In this paper we will study the real character on order Banach algebra with identity, and study character onorder Banach algebrawithout identity. We proved some properties on real character it. We define set of all character and prove it with some condition convex set and used it to prove all character is extreme point, we introduce dual cone in order Banach space .Also we show any order Banach algebra satisfying certain conditions is isomorphic to the space valued continuous function $C_0(x)$ forsuitable a locally compact Housdorf space.

Keywords: order Banach algebra; algebra cone; dual cone; character.

Introduction

The study of Banach algebra entails combining the use of algebraic and analytic (or topological) methods. It has become an important field modern operator theory having many practical and theoretical applications.

We will show that Banach algebra techniques combine with order structures yield new insights. We prove our results in structure will called order Banach algebra A , the ordering which is induced by a subset C of A with certain special properties, called an algebra cone which is compatible with the algebraic structure of A, and this the useful references for the material on Banach space are [J.B.Conway,1990].References for Banach algebra are [D,Holland,2015], [J.M.Erdman,2011] and references for order Banach Algebra are[D.Robimsond,1983],[H.Rouben,1996], and [R.Dejong,2010].

1-order Banach algebra

Definition 1.1[1] :- Let A be a Banach with respect to real numbers. $A$is called Branch algebra , if there exist operation from $A \times A$ to A such that $(x, y) \to xy$. for all x, y and z belong to A and $\alpha \in \mathbb{R}$with the following properties

1 − $(xy)z = x(yz)$
2 − $(\lambda x + y)z = \lambda xz + yzandz(\lambda x + y) = \lambda zx + zy$ (commutive)
3-$\|xy\| \leq \|x\|\|y\|$ (Sub multiplicative)

Definition 1.2[1]:-We called a Banach algebra$A$ is until, if it has a unite element e such that $e.x = x.e = x \ \forall \ x \in A$. If A has unite then ||e||=1

Definition (W*-topology) 1.3[5]:-Let A be a Banach space and A* be the dual space of A ,the weak topology is the weakest topology such the map T: A→ R is continuous .A weak*topology is the weakest topology making all functional on $A^*$are continuous.

Theorem (Hahn Banach) 1.4[6]:- Let X be a linear space in a real numbers$\mathbb{R}$ and let q be a sub linear functional on X, if M be a linear subset of X and if $f: M \to \mathbb{R}$ be a linear functional with f(x)≤q(x) for all $x \epsilon M$.Then there exist a linear functional$K : X \to \mathbb{R}$ with $K(x) \leq q(x)$ for all $x \epsilon X$.

Proposition 1.5[6]:-If X is a normal space and x in X.Then $\|x\| = \sup\{|f(x)| : f \in X*: \|f\| \leq 1\}$.

Definition1.6 [6]:-The closed unit ball in X which denoted by $CL(x)$ when X is normed space is define by $CL(x) = \{x \epsilon X: \|x\| \leq 1\}$.

Theorem(Alaoglu's theorem) 1.7[4]:-$CL(x)$ is a W*-compact if X is normed space.

Definition(Extreme point) 1.8[6]:-If X is a linear space and K ⊆X ,such K is convex and xϵK ,then x is extreme point if there is no proper open line segment in K , i.e if$x = ta + (1-t)b$,such that a, bϵ K and $0 < t < 1$ then $a = b$. And we denote to the set of extreme point by $e(K)$

Theorem (The KrienMilman theorem) 1.9[4]:-Let X be a locally convex space andK bea non-empty compact convex subset of X .Then $e(K)$≠ϕ and is equal to theclosed convex hull of $e(K)$.

Definition 1.10[3]:- Let A be a real Banach algebra with unit e and C non-empty subset of A .We call C a cone if it satisfies the following

1. a +b ϵ C for all a ,b ϵ C ,
2. λaϵ C for all a ϵ C and $\lambda \geq 0$.

In addition if C satisfies C ∩ −C= {0}, then C will be called a proper cone induced an ordering (≤) on A by a ≤b if and only if b-a ϵ C for all a, bϵC.We say that C is algebra cone if it is satisfies the following:

1. a· b ϵC for all a, b ϵC ,
2. e ϵ C.

Definition (Ordered Banach Algebras) 1.11[3]:- Let A be a real Banach algebra with unit e is called order Banach algebra (OBA) when A is ordered by a relation (≽) such that for every a,b,cϵC and $\lambda \geq 0$

1. a ,b≽ 0 ⟹a+b≽0
2. a≽ 0 ,$\lambda \geq 0$⟹ λa ≽ 0
3. a ,b ≽ 0 ⟹a· b ≽ 0 .
4. e≽0

So if A ordered by an algebra cone C, we will obtain (A, C) is an order Banach algebra.

Definition 1.12[6]:- Let A be OBA and C be algebra cone so

- We will say that C is *normal* if there is $\beta > 0$,and for any a , b ϵ A , 0≤ $a \leq b \Longrightarrow$ ‖a‖≤ β‖b‖ .
- We will say that C is α *−normal* if there is $\alpha > 0$,and for any a , b ,c ϵ A , b≤ $a \leq c$ ⟹ ‖a‖≤ α max(‖b‖,‖c‖) .

If α=1 we say that C is 1$-max$ −normal.

Proposition 1.13[6]:-Every normal algebra cone C is a proper algebra cone.

Definition 1.14:- Let A be aorder Banachalgebra, we called the function $f$is*multiplicative* if satisfies:$f(ab) = f(a)f(b)$.A functional f from order Banach A into the set of real numbers R is called real character if f is linear, multiplicative, and f(e)=1

. Notation:-When A is ordered by an algebra cone C .If we denote $\mathrm{M_A}$ to the set of all real characters, and $\mathrm{M_A}^+$ to the subset of $\mathrm{M_A}$such that $\mathrm{f(x)} \geq 0$,for all x in C.

Lemma 1.15: -M_A and $M_A{}^+$ are convex sets.

Proof: - Let $f_1, f_2 \in M_A$ and $0 \leq \beta \leq 1$ $, a, b \in A$

$$1-(\beta f_1 + (1-\beta)f_2)(a+b) = \beta f_1(a+b) + (1-\beta)f_2(a+b)$$
$$= \beta f_1(a) + \beta f_1(b) + (1-\beta)f_2(a) + (1-\beta)f_2(b)$$
$$= \beta f_1(a) + (1-\beta)f_2(a) + \beta f_1(b) + +(1-\beta)f_2(b)$$
$$= (\beta f_1 + (1-\beta)f_2)(a) + (\beta f_1 + (1-\beta)f_2)(b)$$

$$2-(\beta f_1 + (1-\beta)f_2)(\alpha a) = \beta f_1(\alpha a) + (1-\beta)f_2(\alpha a)$$

Since $f_1, f_2 \in M_A$ then we obtain

$= \alpha\beta f_1(a) + \alpha(1-\beta)f_2(a) = \alpha[\beta f_1(a) + (1-\beta)f_2(a)] \qquad = \alpha[\beta f_1 + (1-\beta)f_2](a)$

3-To prove that $[\beta f_1 + (1-\beta)f_2](ab) = (\beta f_1 + (1-\beta)f_2)(a).(\beta f_1 + (1-\beta)f_2)(b)$

Let$0\leq \beta \leq 1$, $\varepsilon > 0$ and $f_{1,} f_2 \epsilon M_A$

Let$t=\frac{i}{2^n}$for some$i, n\epsilon\mathbb{N}$,such that $|\beta - t| < \varepsilon$

And t$f_1 + (1-t)f_2 \epsilon$$M_A$.

Let$\{f_\lambda\}_{\lambda\epsilon\Lambda}$be net in M_A satisfy $f_\lambda \to \beta$,$\lambda\epsilon\Lambda$ and

$f_\lambda f_1(ab) + (1-f_\lambda)f_2(ab) = f_\lambda f_1 + (1-f_\lambda)f_2$)(a).($f_\lambda f_1$ +(1-f_λ)f_2)(b) for every a,bϵA.

Hence (βf_1(ab) + (1- β) f_2(ab)) =βf_1+(1-β)f_2)(a). (βf_1+(1-β)f_2)(b), for every a, b ϵA .Then from (1) ,(2) ,(3) we obtain $\beta f_1 + (1-\beta)f_2 \epsilon \mathrm{MA}$.

$M_A{}^+$Will be convex by follows the covexity of M_A .□

Proposition 1.16:- Let A be anOBA without identity and f be linear, multiplicative on A .Then we can extended f to real character$\overline{f}$ on OBA with identity A_e such that Ais sub algebra ofA_e and $\bar{f}\,/_A = f$.

Proof:- Let A be a OBA ,$and\ let\ A_e = A \times \mathbb{R}$

To prove thatA_e is an OBA with identity asfollows :

a- *Algebra*

We, define operations on,A_e as follows:

$1-(a,\alpha)+(b,\mu) = (a+b, \alpha+\mu) \quad for\ all\ a, b\epsilon A\ , \alpha, \mu\epsilon\ \mathbb{R}$

2- β (a,μ)=(βa,βμ) for all βϵR,aϵ A

$3-(a,\alpha).(b,\mu) = (ab+\alpha b+\mu a, \alpha\mu)$.

It clear that A_e is an algebra with identity (0,1) and A is sub algebra of A_e.

b- *Normed algebra*

If $\|(a,\alpha)\|$=‖a‖=$|\alpha|$ for all aϵA , $\alpha\epsilon\mathbb{R}$ .Then A_e be a normed algebra.

c- *Banach algebra*

To prove that $A_e$be a complete.

Let $\{x_n, \alpha_n\}$ be a Cauchy sequence in A_e

for all $\varepsilon > 0, there\ exist\ k\epsilon\ \mathbb{Z}^+\ such\ that \|(x_n, \alpha_n) - (x_m, \alpha_m)\| < \varepsilon$∀n>k

$\|(x_n - x_m, \alpha_n - \alpha_m)\| < \varepsilon$∀n>k

$\|x_n - x_m\| + |\alpha_n - \alpha_m| < \varepsilon$

$\|x_n - x_m\| < \varepsilon$and$|\alpha_n - \alpha_m| < \varepsilon$

Since $\{x_n\}$ be a Cauchy sequence in A ,$\{\alpha_n\}$ be a Cauchy sequence in $\mathbb{R}$. since A is Banach algebra ,there exist $x\epsilon A$ such that $x_n \to x\ as\ n \to \infty$

i.e $\|x_n - x\|$<$\varepsilon for\ all\ n \geq k$

Since$\mathbb{R}$ $is\ complete\ so\ there\ exist\ \alpha\epsilon\mathbb{R}\ such\ that\ |\alpha_n - \alpha| < \varepsilon$

Obtain $\|x_n - x\| + |\alpha_n - \alpha| < \varepsilon\varepsilon for\ all\ n \geq k$

Implies that$\|(x_n, \alpha_n) - (x_m, \alpha_m)\| < \varepsilon\varepsilon for\ all\ n \geq k$

Since $\{x_n\}$ in A and, $\{\alpha_n\}$ in $\mathbb{R}$.

Then $\{(x_n,\alpha_n)\}$ in A_e so A_e is complete .

d- Order Banach algebra

Let a,b$\epsilon A, a \succcurlyeq 0 , b \succcurlyeq 0$ since A is OBA then a+b$\succcurlyeq$ 0 and if $\alpha \geq 0 , \mu \geq 0$ $then$ $\alpha + \mu \geq o$.

Now to show that $C_e = C \times \mathbb{R}$ be an algebra cone, let $a, b\epsilon C$ and $\alpha, \mu\epsilon\mathbb{R}$ such that$x = (a, \alpha), y = (b, \mu)\ \epsilon C_e$.

1) $x + y = (a, \alpha) + (b, \mu) = (a + b, \alpha + \mu)\ \epsilon C_e \quad for\ all\ x, y\epsilon C_e$,
2) $\mu x = \mu(a, \alpha) = (\mu a, \mu\alpha)\ \epsilon C_e$,
3) $x.y = (a, \alpha).(b, \mu) = (ab + \alpha b + \mu a, \alpha\mu)$.
4) $(0,1)\epsilon C_e$

A cone C_e on A_e induced an ordering relation $\succcurlyeq$ on A_e by the way $for\ all\ (a, \alpha), (b, \mu)\epsilon C_e\ if\ a \succcurlyeq b , \alpha \geq \mu$ .This ordering satisfies:

1-[reflaxive]

for all $(a, \alpha)\epsilon A_e\ since\ a \succcurlyeq a\ and\ \alpha \geq \alpha\ .Then\ (a, \alpha) \succcurlyeq (a, \alpha)$

2-[transitive]

For all$(a, \alpha), (b, \mu), (c, \lambda)\epsilon A_e$ since $a \succcurlyeq b , b \succcurlyeq c\ obtain\ a \succcurlyeq c$, and since $\alpha \geq \mu , \mu \geq \lambda$

$obtain\ \alpha \geq \lambda$so if $(a, \alpha) \succcurlyeq (b, \mu)\ and\ (b, \mu) \succcurlyeq (c, \lambda)\ then\ (a, \alpha) \succcurlyeq (c, \lambda)$

Now to satisfy conditions of OBAfor all$(a, \alpha), (b, \mu)\epsilon A_e$,$(a, \alpha) \succcurlyeq (b, \mu)$such that a$\succcurlyeq$ $b\ \alpha \geq \mu$

$$1 - (a, \alpha) \succcurlyeq 0\ ,(b, \mu) \succcurlyeq o \Longrightarrow (a + b, \alpha + \mu) \succcurlyeq 0$$

$$2 - (a, \alpha) \succcurlyeq 0\ ,\lambda \geq 0 \qquad \Longrightarrow \lambda(a, \alpha) = (\lambda a, \lambda\alpha) \succcurlyeq 0$$

$$3 - (a, \alpha) \succcurlyeq 0, (b, \mu) \succcurlyeq 0 \Longrightarrow (a, \alpha).(b, \mu) = (ab + \alpha b + \mu a, \alpha\mu) \succcurlyeq 0,$$

$$4 - (0,1) \succcurlyeq 0.$$

e- $\bar{f}$ character

We define$\bar{f}$:$A_e \rightarrow \mathbb{R}$ asfollows:$\bar{f}\big((a, \alpha)\big) = f(a) + \alpha$ forall$a\epsilon A$,$\alpha\epsilon\mathbb{R}$
To prove that $\bar{f}$ is character

$$\bar{f}((a, \beta) + (b, \alpha)) = f(a + b, \beta + \alpha\ ) = f(a) + f(b) + \alpha + \beta = \ \bar{f}(a, \alpha) +\ \bar{f}(b, \alpha)$$

$$\bar{f}((\ \lambda a, \lambda\alpha) = f(\lambda a) + \lambda\alpha = \lambda\bar{f}(a, \alpha)$$

$$\bar{f}\big((a, \alpha)(b, \mu)\big) = \bar{f}(ab + \alpha b + \mu a.\alpha\mu)$$

$$= f(ab + \alpha b + \mu\alpha) + \alpha\mu$$
$$= f(ab) + f(\alpha b) + f(\mu\alpha) + \alpha\mu$$
$$= [f(a) + \alpha].[f(b) + \mu]$$
$= \bar{f}\big((a, \alpha)\big).\bar{f}\big((b, \mu)\big)$for all $\mu\epsilon\mathbb{R}$

Hence we obtain that $\bar{f}$ is realcharacter. □

Definition1.19:-A_e is called order adjunction with identity of an order Banach algebra without identity(A,C).

Proposition1.20 [1]:- Let Abe a real Banach algebra with identity and a$\epsilon A$is an invertible element,suppose b ϵA with$\|b - a\| < \frac{1}{\|a^{-1}\|}$, then$b$ is invertible and $\|(a - b)^{-1}\| \leq \frac{\|a^{-1}\|^2\|b-a\|}{1-\|a^{-1}\|\|b-a\|}$.

Lemma1.21:- let A be a real Banach algebra without identity. Then for every character in M_A is bounded.

Proof: - Since A without identity then we extend f to the character $\bar{f}$ of $A_e = A \times \mathbb{R}$ in this way we have $\bar{f}oi$ such that $i: A \to A_e$ defined by $(a) = (a.o)$,which is inclusion map.Toprove that$\bar{f}$is bounded i.e.to prove$|f(a)| \leq \|a\|$

Assume that $a \in A_e$with$|f(a)| > \|a\|$.

Take $\beta = f(b)$and $a_1 = \beta^{-1}b$ which has a norm$\|a_1\| < 1$ by proposition (1.20)$a = 1$ and $b = 1 - a_1$, which follows that $1 - a_1 = 1 - \beta^{-1}b$ is invertible,

since $\beta \neq 0$,so we have $c = \beta(1 - a_1) = \beta 1 - b$is invertible ,

weobtain$f(c) = \beta f(1) - f(b) = \beta - f(b) = 0$,

but

$1 = f(1) = f(cc^{-1}) = f(c^{-1})f(c) = 0$ which is imposible ,so $|f(a)| \leq \|a\|$. □

2.Dual cone in Order Banach algebra

Definition 2.1[6]:- Let A^*be a dual of order Banach space with algebra cone C ,a *dual cone* is the set of all continuous (bounded) linear functional and it is non-negative on C which isdenoted by C^* i.e $C^* = \{f \epsilon A^*: f(x) \geq 0\ , for\ all\ x \epsilon C\}$

Lemma 2.2[6]:- Let S be a closed and convex set of aBanach space A with respect to real numbers and a$\notin S$.Then there is a continuous (bounded) linear functional $f\ such\ that\ f(a) < f(b)$for all b$\epsilon C$.

Lemma 2.3:- Let C be an algebra cone in A.Then $a \epsilon C$if and only if $f(a) \geq 0\ \ for\ all\ f \epsilon C^*$and $C \neq A$ implies $C^* \neq \{0\}$

Proof:- Suppose that $a \notin C$, since C is closed and convex cone so from above lemmawith$f(a) < f(b) forall\ b \epsilon C$, $thus\ f(a) < 0 = f(0)$,since b ϵCthen we obtain $\beta b\ \epsilon C\ and\ \beta > 0$.
Then$f(b) \geq 0, Hence\ f \epsilon C^*$.□

Definition2.4]:- Let A^* be a dual of order Banach space and C^* be a dual cone in A^* which is called $-generated$, if for each $f \epsilon A^*\ and\ g_1, g_2 \epsilon C^* such\ that\ f = g_{1-}g_2\ satisfies\ that\ \alpha\|f\| \geq \|g_1\| + \|g_2\|$,If $f = g_{1-}g_2\ for\ each\ g_1, g_2 \epsilon C^* and \|f\| = \|g_1\| + \|g_2\|$.

we will call that C^* by *1-generated* .

Proposition 2.5[2]:- Let (A, C) be an order Banach algebra .Then C is$\alpha - \mathrm{normal}$ if and only if C^* is $\alpha - \mathrm{generated}$ .In particular Cis $1 - \mathrm{normal}$if and only if each$f \epsilon A^*$ satisfies $f = g_{1-}g_2$ for each $g_1, g_2 \epsilon C^*$.

Definition 2.8:- A closed ball of A^* which is denoted by$\Im_A = \{f \epsilon A^* : \|f\| \leq 1\}$.

Lemma2.9:- M_A and ${M_A}^+$ and $\Im_A^+ = \Im_A + C^*$are W^*- compact

Proof: -Since $\|f\| \leq 1$ and $M_A \subset \Im_A$ then we obtain $\Im_A$ is W^*-compact

sinceA^* is normed space. Then$\Im_A = CL(A^*)$ by [Alaoglu'stheorem].

To show that M_A is W^*-compact we have to prove M_A is W^*-closed in $\Im_A$
Let take a net$(f_\lambda)_{\lambda \in \wedge}\ in\ M_A$ and $f \in \Im_A$such that $f = \lim_{\lambda \in \wedge} f_\lambda$ to show that $f \in M_A$

$$f(ab) = \lim_{\lambda \in \wedge} f_\lambda\,(ab), \forall\, a, b \in A$$

It is clear from that f is liner and Multiplicative and f is not trivial since $f(e) = lim_{\lambda \in \wedge} f_\lambda\,(e) = 1$
Hence $f \in M_A$and M_A is W^*- closed in $\Im_A$.

To show that ${M_A}^+$is W^*- compact we have to prove that ${M_A}^+$is W^*-closed
Let $(f_\lambda)_{\lambda \in \wedge}$ be a net in ${M_A}^+$ and let $f \in {M_A}^+$
Then for $a \in C$ we get $(f_\lambda)(a) \geq 0$ so $f(a) \geq 0\ \ for\ all\ a \epsilon\ C$
Thus $f \in {M_A}^+$ and ${M_A}^+$ is W^*-closed

To prove that $\mathfrak{I}_A^+$is W^*-closed

Let $(f_\lambda)_{\lambda\in\wedge}$ be a net in $\mathfrak{I}_A^+$ and let$\|f\| \leq 1$, such that $f \in M_A$and $f = lim_{\lambda\in\wedge} f_\lambda$.

Hence therefor $a \in C$ we have$\| f_\lambda(a) \| \geq 0$ and $\|f(a)\| \geq 0$

Thus $f \in \mathfrak{I}_A^+$and we obtain $\mathfrak{I}_A^+$is W^*-closed. □

3. Main results

Proposition3.1:- every character in M_A is extreme point.

Proof: - Let f be a character and let $f = tf_1 + (1-t)f_2$ such that $f_1, f_2 \in M_A$

To prove that $f_1 = f_2$

Assume that$f_1 \neq f_2$,there exist aϵA such that$f_1(a) \neq f_2(b)$.Then $tf_1(a) \neq tf_2(b) and (1-t)f_1(a) \neq (1-t)f_2(b)$,implies that $tf_1(a).f_1(b) + (1-t)f_1(a)f_1(b) \neq tf_2(a).f_2(b) + (1-t)f_2(a).f_2(b)$ for someb$\neq 0$.

$tf_1(a).f_1(b) + (1-t)f_2(a).f_2(b) \neq \big(t\,f_1(a) + (1-t)f_2(b)\big).\big(tf_1(b) + (1-t)f_2(b)\big)$.

Then$f(ab) \neq f(a)f(b)$ this impossible because f character. □

- Now we consider the following conditions

1- $\bar{f}(y\,ab) \geq 0$ for every $f \in {M_A}^+$ and $a, b, y \in A_e$

2- $(ya) \in C_e$for every $a \in C_e$, and $y \in A_e$.

Lemma 3.2:- let A be real order Banach algebraand $\{f_\lambda\}_{\lambda\epsilon\wedge}$ be a netA_e satisfying that $f_\lambda a \to a$ for all $a \in A_e$,let $y \in A_e$.then A_e satisfies the conditions (1) and (2).

Proof: - Let $\{f_\lambda\}_{\lambda\epsilon\wedge}$ a net and f_λa→a for all aϵA_e, let $y\epsilon A_e$,then we have$yf_\lambda \in C_e$ and $(abf_\lambda) \in C_e$ so $(yf_\lambda)(abf_\lambda) \in C_e$

Then $(yf_\lambda)(abf_\lambda)$ is positive, thus $(yf_\lambda)(abf_\lambda) \geq 0$

Let$y', a', b' \in A$ and $\alpha, \beta \in \mathbb{R}$ such that $y = (y', \beta)$, $ab = \big((a'b'), \alpha\big)$

Without products$(yf_\lambda)(abf_\lambda)$and yab , since f is continuous and liner map we obtain $f\big((yf_\lambda)(abf_\lambda) - \alpha\beta f_\lambda\big)$

Convergent to $\bar{f}\big((f_\lambda)(abf_\lambda) - \alpha\beta\big)$, since $\|f\| \leq 1$

We have $f(\beta\alpha f_\lambda) \leq \beta\alpha$

0$\leq f(yf_\lambda)(abf_\lambda)$=f(y$f_\lambda)(ab) - \beta\alpha f_\lambda) + f(\beta\alpha f_\lambda) \leq f(yf_\lambda)(abf_\lambda) - \beta\alpha f_\lambda) + \beta\alpha$ Convergent to$\bar{f}(y\,ab - \beta\alpha) + \beta\alpha = \bar{f}(y\,ab)$

so condition (1) is satisfied and condition (2)directly by a $\lim_{\lambda\epsilon\wedge}(y\,f_\lambda)a$.□

Proposition3.3:-let X be a locally compact Hausdorff spaceand $C_0(X)$= {f:f: X→$\mathbb{R}$, $continuos$},let $C_0^+(X) = \{f\epsilon C_0(X): f(x) \geq 0, \forall x\epsilon X$,then$C0+X$be an algebra cone and make$C0X$by an OBA by $C0+X$.

Proof:-To show that $C_0^+(X)$ be an algebra cone ,let $f_1, f_2\epsilon C_0^+(X)$,$\lambda \geq 0$ and $x\epsilon X$ such that$f_1(x) \geq 0$, $f_2(x) \geq 0$

1- $f_1(x) + f_2(x) = (f_1 + f_2)(x) \geq 0$

2- $\lambda f_1(x) \geq 0 \quad for\ all \lambda \geq 0$

3- $f_1 f_2(x) = f_1(x) f_2(x) \geq 0$,

4- $f_1(1) = 1 \geq 0$.

It can show for every $f_1, f_2, f_3 \epsilon C_0^+(X)$ this satisfies that ordering is

a-[reflexive] i.e$f_1 \geqslant f_1$ if $f_1(x) \geq f_1(x)$.

b-[transitive] i.e$f_1 \geqslant f_2\ and\ f_1 \geqslant f_2 then f_1 \geqslant f_3$ if$f_1(x) \geq f_2(x) and f_2(x) \geq f_3(x)\ then f_1(x) \geq f_3(x)$.

So we obtain that $C_0^+(X) = \{f\epsilon C_0(X): f(x) \geq 0 , \forall x\epsilon X\}$ be an algebra cone which is make$C_0(X)$ an order Banach algebra(OBA).□

Theorem3.4:- Let (A, C) be an order Banach algebra andC be a closed and$\alpha-normal$ cone. Then A isomorphic to $C_0(X)$,if and only if for a in A there is a net$\{f_\lambda\}_{\lambda\epsilon\wedge}\, such\ that f_\lambda$a→a.

Proof:- *let F* be a set of extreme points of M_A .
By (Urysohn'sLemma) that there is a net $\{f_\lambda\}_{\lambda\epsilon\wedge}\, such\ that f_\lambda$a→a satisfying for all a ϵA.

Now let $X = F/\{0\}$with $w^* - topology$.

Suppose that C=A .then for all $a, b\epsilon A$have $\leq a \leq b$.

Since C is $\alpha - normal$ this a contradiction,

Thus$C \neq A$and$C^* \neq \{0\}$,

$so\ by\ lemma 2.3\ M_A^+ \neq \{0\}, thus\ X \neq \emptyset$,

So $X = F/\{0\}$ is a locally compact Hausdorff space.

Let $\gamma: C(X) \to A\ defined\ by\ \gamma(f) = f(a)\ \ for\ all\ f\ \epsilon X$.

Since$f$ is character, soγ is an algebra homomorphism.

Andγ $will\ be\ isomorphism$ to show that , let $a, b\epsilon A$such that $a \leq b$,so$\gamma(f)(b-a) = f(b-a) \geq 0 for\ all\ f\epsilon X\ and\ \gamma(f)(b) - \gamma(f)(a)\epsilon C^+(X, \mathbb{R})$,

$$thus\ \gamma(f)(b) \geq \gamma(f)(a) .$$

Conversely, l$et\ a, b\epsilon A$and $\gamma(f)(b) \geq \gamma(f)(a)$.

Then $f(b-a) \geq 0, for\ all\ f\ \epsilon C(X)$,

so by(Klein- Mailman theorem) and since $M_A^+ = F\ is\ the\ w^* - closed$ convex hull and M_A^+ generates C^*,

so$f(b-a) \geq 0$ for all $f\epsilon C^*$ and $b - a\ \epsilon C$ that by lemma 2.3□

Lemma3.5:- Let (A, C) be a real order Banach algebra with closed andnormal C . Then A is an isomrphisim to$C_0(X)$ if and only if the following conditions are satisfies:-

1. Fora, bϵAand $\|a\| = \|b\| = \alpha$, there existc ϵAwith$\|c\| = \alpha$such that $c \geq a, b$ and $c \geq 0$

2.For $0 \leq$a, b≤ 1,$\alpha \geq 0$ and $\|a\| = \|b\| = \alpha$Then ab≤a and ab≤b

3.For every $x\epsilon C$ there is two nets$\{f_{\lambda 1}\}_{\lambda\epsilon\wedge}, \{f_{\lambda 2}\}_{\lambda\epsilon\wedge}$ of psitive elements and$\|f_{\lambda 1}\| \leq 1$,$\|f_{\lambda 2}\| \leq 1$ such that $\lim_n ff_{\lambda 1} = f = \lim_n f_{\lambda 2} f$, let C be an algebra cone .Then there exist a net $\{f_\lambda\}_{\lambda\epsilon\wedge}$such that $f_\lambda a \longrightarrow a$ satisfies for all a ϵA .

Proof:- To prove that $C_0(X)$ satisfies condition (1),(2) and (3)

let$f_1, f_2\epsilon C_0(X)$ and $\|f_1\| = \|f_2\| = \alpha$,let $f_3 = \max\{|f_1|\,|f_2|\}$

then $f_3\epsilon C_0(X)$ and $\|f_3\| = \alpha$ with $f_3 \geq f_1, f_2\ and f_3 \geq 0$,so condition (1) satisfies.

If $f_1, f_2\epsilon C_0^+(X)$ i.e $f_1(x) \geq 0\ and\ f_2(x) \geq 0\ for\ all\ x\epsilon X$ and $\|f_1\| = \|f_2\| = \alpha$.

Then $(f_1 - f_1 f_2)(x) = (f_1(x) - f_1(x) f_2(x)) \geq 0$ for all $x\epsilon X$.

Then$f_1 f_2 \leq f_1$. In the same way we obtain $f_1 f_2 \leq f_2$.

To show that condition (3) satisfies .

Let k$f_\lambda.(1 + kf_\lambda)^{-1}\epsilon C_0^+(X)$ and$\|$ k$f_\lambda.(1 + kf_\lambda)^{-1}\| \leq 1$,

$0 \leq$k$f_\lambda(x).(1 + kf_\lambda(x))^{-1} \leq \frac{1}{k}$ for each ,$k\epsilon\mathbb{N}$.

Therefor

$\|f - f_\lambda(\mathrm{k}f_\lambda.(1+kf_\lambda)^{-1})\| = \|\frac{f_\lambda(1+kf_\lambda)-kf_\lambda^2}{1+kf_\lambda}\| = \|\frac{f_\lambda+kf_\lambda^2-kf_\lambda^2}{1+kf_\lambda}\| = \|\frac{f_\lambda}{1+kf_\lambda}\|$

$= \|f_\lambda(1+kf_\lambda)^{-1}\| = \sup\{f_\lambda(x)(1+kf_\lambda(x))^{-1} : x\epsilon X\} \leq \frac{1}{k}$.

It follows that $\lim_n f_\lambda(\mathrm{k}f_\lambda.(1+kf_\lambda)^{-1} = \lim_n(\mathrm{k}f_\lambda.(1+kf_\lambda)^{-1}.f_\lambda$.

there is a net $\{f_\lambda\}_{\lambda\epsilon\Lambda}$, *such that* f_λa→a satisfying for all aϵA.

Now to prove the sufficient part

i.e. Proved thatf_λa→a satisfying for all aϵA.

σbe the normality constant and Λ be the set of all positive elementsλ with $\|\lambda\|$=α and let $f_\lambda = \lambda$ such that $\{f_\lambda\}_{\lambda\epsilon\Lambda}$ be a net we first show that f_λa→a for all a$\epsilon A$and $a \geq 0$ let $a \geq 0$ .If $a = 0$ *the equation directly* , suppose $a \neq 0$.We will assume that $\|a\|$=α ,let $0 \leq \varepsilon \leq 1$ from (3) there is $b \geq 0$ *and* $\|b\| \leq 1$ *such that* $\|a - ba\| < \epsilon$ let $\lambda_1 = \|b\|^{-1}b$ and suppose that $\lambda\epsilon\Lambda$ such that $\lambda \geq \lambda_1$ obtain $\lambda \geq \|b\|^{-1}b \geq b$ so $(\lambda - b)a\epsilon C$ and thus $a - f_\lambda a = a - \lambda a \leq a - ba$.

From (2) obtain f_λa $\leq a$, so $a - f_\lambda$a ≥ 0

this shown $\|a - f_\lambda$a $\| \leq \sigma\|\mathrm{a} - \mathrm{b}a\| < \sigma\varepsilon$ forall $\lambda\epsilon\Lambda$, $\lambda \geq \lambda_1$.

then with arbitrary ε obtain f_λa→a for all aϵA with $a \geq 0$.

Now when aϵA .If $a = 0$ *the equation directly*. Suppose $a \neq 0$, from (1) there exist $b \geq 0$ such that $b \geq -a\|a\|^{-1}, a\|a\|^{-1}$.

Let $a = a_1 - a_2$ such that $a_1 = \frac{1}{2}(b\|a\| + \mathrm{a})$ and $a_2 = \frac{1}{2}(b\|a\| - \mathrm{a})$.therefore $a_1, a_2 \geq 0$ .Since f_λ a→a for all aϵA with $a \geq 0$.

Then $\lim_{\lambda\epsilon\Lambda} f_\lambda a_1 = a_1$ and$\lim_{\lambda\epsilon\Lambda} f_\lambda a_2 = a_2$.

Thus $\lim_{\lambda\epsilon\Lambda} f_\lambda a = \lim_{\lambda\epsilon\Lambda} f_\lambda(a_1 - a_2) = \lim_{\lambda\epsilon\Lambda} f_\lambda a_1 - \lim_{\lambda\epsilon\Lambda} f_\lambda a_2 = a_1 - a_2 = a$.

Then we obtain f_λa→a for all aϵA □

References

[1] D.Holland "Banach Algebra Notes", OregonState University .American, February 2015.

[2] D.W. Robinson and S. Yammer, "Addition of an identity to an ordered Banach Space", J. Austral. Math. Soc. 35 (1983), 200-210.

[3] H.Raubenheimer and S. Rode "Cone in Banach Algebra", M.Sc.thesis, University Of the Orange Free State, Indag.Mathem .N.S 7(4).489.502, 1996.

[4] J.B. Conway," A course in Functional Analysis", NewYork, 1990.

[5] J.M. Erdman ", Lecture Notes on Operator Algebra", Portland state University, Ph.D. thesis,Portland State University,2011.

[6] R. de Jong," Ordered Banach Algebras" Ms. Thesis, Leiden University, 2010.