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REGIONAL EXPONENTIAL REDUCED OBSERVABILITY IN DISTRIBUTED PARAMETER SYSTEMS

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ABSTRACT

The regional exponential reduced observability concept in the presence for linear dynamical systems is addressed for a class of distributed parameter systems governed by strongly continuous semi group in Hilbert space. Thus, theexistence of necessary and sufficient conditions is established for regional exponential reduced estimator in parabolic infinite dimensional systems. More precisely, the introduced approach is developed by using the decomposed system and reduced system in connection with various new concepts of (stability, detectability, estimator, observability and strategic sensors).Finally, we alsoshow that there exists a dynamical systemfor two-phase exchange system described by the coupled parabolic equations is not exponentially reduced observable in usual sense, but it may be regionally exponentially reduced observable.

Keywords: $\omega_{E\mathcal{R}}$-observability; ω_E-detectability; $\omega_{E\mathcal{R}}$-strategic sensor; $\omega_{E\mathcal{R}}$- detectability; exchange system.

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1. NTRODUCTION

One of the most important concepts in infinite dimensional systems analysis is observability concept. Many researches of these concept included the notion of exponential observer(estimator), where Luenberger introduced this notion for finite dimensional systems [22], and has been generalized to infinite dimensional systems described by strongly continuous linear semi-group operators by Gressang and Lamont [20]. The purpose of an exponential estimator is to provide an exponential state estimation for the considered system state [16]. New concept of regional analysis for a class of distributed parameter systems was extended by Al-Saphory and El Jai *et al*. as in ref.s [1-7, 18, 16, 25-29]. Various asymptotic characterizations have been established and explored in connection with sensors structures [1, 6]. In this paper, we introduce and study the notion of exponential regional reduced state observability in a given region ω of the domain Ω. Thus the developed approach is an extension of previous works to the regional case as in [2]. Moreover the relationship between this notion, regional detectability and strategic sensors are studied and discussed. The main reason behind the study of this notion (reduced observability), there exist some problem in the real world cannot observe the system state in the whole domain, but in a part of this domain. The scenario described by (Figure 1) below, one is interested in estimating the state in the green zone rather than in the entire space [12]. This problem falls into a class of so-called regional observation and estimation problem introduced by Al-Saphory and El-Jai and their workers as in [1-7, 25-29].

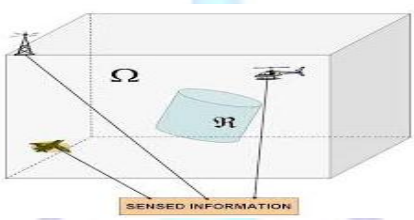

Fig. 1: Zone control $\mathcal{R}$ with fixed and mobile sensors

This paper is organized as follows. Section 2 is devoted to the introduction of regional exponential detectability and considered system with ω_E-detectability and ω-observability. We study the links of this notion with the regional exponential observability and strategic sensors. In Section 3, we study a regional exponential observability through the relations between ω_E-estimator reconstruction method and ω_E- observability. In section 4 we introduce regional exponential reduced observability notion for a distributed parameter system in terms of regional exponential reduced detectability and reduced strategic sensors. In the last section, we illustrate applications with different domains and circular strategic sensors of two-phase exchange systems.

2. REGIONAL EXPONENTIAL DETECTABILITY

The detectability is in some sense a dual notion of stabilizability [15]. This notion was considered and studied in the whole domainΩ.

2.1Considered Systems

Let Ω be a bounded and open subset ofR^n, with boundary $\partial\Omega$. Let$[0,T], T>0$ a time measurement interval and ω be a non-empty given subregion ofΩ. We denote $Q=\Omega\times(0,\infty)$and $\Theta=\partial\Omega\times(0,\infty)$. Let $X, U,$ and $\mathcal{O}$be separable Hilbert spaces, where X is the state space,U the control space and $\mathcal{O}$ the observation space. We consider $X=L^2(\Omega), U=L^2(0,\infty,R^p)$ and$\mathcal{O}=L^2(0,\infty,R^q)$ where$p$ and$q$ hold for the number of actuators and sensors [17]. The considered distributed parameter systems are described by the following parabolic equations

$$\begin{cases}\frac{\partial x}{\partial t}(\xi,t)=Ax(\xi,t)+Bu(t) & Q\\ x(\eta,t)=0 & \Theta\\ x(\xi,0)=x_0(\xi) & \Omega\end{cases} \quad (1)$$

augmented with the output function

$$y(.,t)=Cx(.,t) \quad (2)$$

where$A$ is a second-order linear differential operator, which generates a strongly continuous semigroup $(S_A(t))_{t\geq 0}$ on the Hilbert space $X=L^2(\Omega)$, and is self-adjoint with compact resolvent. The operators $B\in\mathcal{L}(R^P,X)$ and $C\in\mathcal{L}(X,R^q)$ depend on the structures of actuators and sensors [17] see (Figure 2).

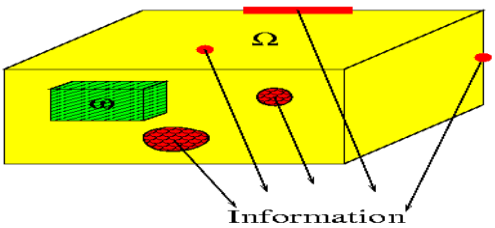

Fig. 2: The domain of Ω, the sub-region ω, various sensors locations.

That means, in the case of pointwise (internal or boundary) and boundary zone sensors (actuators), we have $B \notin \mathcal{L}(R^P, X)$ and $C \notin \mathcal{L}(X, R^q)$ [12, 22]. Thus, the system (1) has a unique solution given by

$$x(\xi,t) = S_A(t)x_0(\xi) + \int_0^t S_A(t-\tau)Bu(\tau)d\tau. \qquad (3)$$

The problem is that how to give an approach which enable to estimate the system state in a sub-region ω. The regional exponential reduced estimator is defined when the output give a part of the state vector in this region.

2.2 Definitions and Characterizations

We extend some definitions and characterizations in the Hilbert space$L^2(\Omega)$ as ref.s [15, 19].

Definition 2.1: The semi-group $(S_A(t))_{t\geq 0}$ is said to be exponential stable on Ω or (Ω_E-stable) if there exist two positive constants $M$and α such that

$$\|S_A(t)\|_{L^2(\Omega)} \leq Me^{-\alpha t}\ ;\ \ t \geq 0 \qquad (4)$$

If $(S_A(t))_{t\geq 0}$ is an Ω_E-stable semi-group, then for all$x_0(.) \in X$, the solution of the associated autonomous system satisfies

$$\|x(.,t)\|_{L^2(\Omega)} = \|S_A(t)x_0(.)\|_{L^2(\Omega)} \leq Me^{-\alpha t}\|x_0(.)\|_{L^2(\Omega)}$$

and therefore

$$\lim_{t\to\infty}\|x(.,t)\|_{L^2(\Omega)} = \lim_{t\to\infty}\|S_A(t)x_0(.)\|_{L^2(\Omega)} = 0.$$

we shall consider the following usual definition of stability.

Definition 2.2: The system (1) is said to be Ω_E-stable if the operator A generates a semi-group which is Ω_E-stable.

Definition 2.3: The system (1) together with the output (2) is said to be detectable on Ω if there exists an operator $H : R^q \to L^2(\Omega)$ such that $(A - HC)$ generates a strongly continuous semi-group $(S_A(t))_{t\geq 0}$ which is Ω_E-stable.

Thus, if a system is (Ω_E-detectable), then it is possible to construct an exponential Ω-estimator for the system state [9].

Remark 2.4: In this paper, we only need the relation (4) to be true on a given subdomain $\omega \subset \Omega$, i.e., if we consider a subdomain ω of the domain Ω and let χ_ω be the function defined by

$$\chi_\omega : L^2(\Omega) \to L^2(\omega) \qquad (5)$$

$$x \to \chi_\omega x = x|_\omega$$

where$x|_\omega$ is the restriction of x to ω. Thus

$$\|\chi_\omega S_A(t)\|_{\mathcal{L}(L^2(\omega),\ L^2(\Omega))} \leq Me^{-\alpha t}\ ;\ \ t \geq 0. (6)$$

and then

$$\lim_{t\to\infty}\|x(.,t)\|_{L^2(\omega)} = 0.$$

We may refer to this as regional exponential stability (or ω_E-stability), which is the equivalent for the considered class of systems to the exponential stability.

Definition 2.5: The system (1) is said to be regionally ω_E-stable if the operator A generates a semi-group which is regional exponential stable (or ω_E-stable).

In this section, we shall extend the definition of detectability by using equation (5) to the regional case by considering ω as subregion of Ω.

Definition 2.6: The system (1)-(2) is said to be ω_E-detectable if there exists an operator $H_\omega : R^q \to L^2(\omega)$such that$(A - H_\omega C)$generates a strongly continuous semi-group$(S_{H_\omega}(t))_{t\geq 0}$, which is ω_E-stable.

The main reason for introducing the concept of ω_E-detectability is the possibility of constructing an ω_E-estimator for the state of system (1).

2.3 ω_E-Detectability andω-Observability

It has been shown that a system which is exactly observable is detectable [16]. For linear systems, we recall the ω_E-observability [2]. Now consider the autonomous system of (1) by the following form

$$\begin{cases} \frac{\partial x}{\partial t}(\xi,t) = Ax(\xi,t) & \mathcal{Q} \\ x(\eta,t) = 0 & \Theta \\ x(\xi,0) = x_0(\xi) & \Omega \end{cases} \quad (7)$$

where$x(.,0)$ is supposed to be unknown. The knowledge of $x(.,0)$ allows one to observe the state $x(t,0)$ at any time t. Measurements are obtained by the output function (2). The solution of the system (6) is given by:

$$x(.,t) = S_A(t)x(.,0). \quad (8)$$

Now define the operator:

$$K : x \in X \to Kx = CS_A(t)x \in \mathcal{O}, \quad (9)$$

then$y(.,t) = K(t)x_0(.,0)$. We denote by $K^*: \mathcal{O} \to X$ the adjoint of K, and then, it is given by

$$K^*y^* = \int_0^t S^*{}_A(s)C^*y^*ds \quad (10)$$

- The system (6)-(2) is said to be exactly ω-observable if

$$\mathrm{Im}\chi_\omega K^* = L^2(\omega)$$

- The system (6)-(2) is said to be weakly ω-observable if

$$\overline{\mathrm{Im}\ \chi_\omega K^*} = L^2(\omega).$$

- If the system (6)-(2) is weakly ω-observable, then $x(.,0)$ is given by

$$x_0 = (K^*K)^{-1}K^*y = K^\dagger y,$$

where$K^\dagger$ is the pseudo-inverse of the operator K [15, 25]. These definitions have been extended to regional boundary case for parabolic, hyperbolicas in [26-28] linear, semi-linear and nonlinear [10-11, 29]. However, we can introduce the following important result.

Corollary 2.7: If the system (1)-(2) is exactly ω-observable, then it is ω_E-detectable. This result allows

$\exists\gamma > 0$ such that $\|\chi_\omega x\|_{L^2(\omega)} \leq \gamma\|CS(.)x_0\|_{L^2(0,\infty,\mathcal{O})}, \forall x \in L^2(\omega)$.(11)

Proof: We conclude the proof of this corollary from the results on observability considering $\chi_\omega K^*$[14]. We have the following forms:

(a) $Im\ F \subset Im\ G$

(b) There exist $\gamma > 0$ such that $\|F^*x^*\|_{P^*} \leq \gamma\|G^*x^*\|_{U^*}, \forall x^* \in V^*$.

From the right hand said of this relation $\exists\ M, \alpha > 0$ with $\gamma < M$ such that

$$\gamma\|G^*x^*\|_{U^*} \leq Me^{-\alpha t}\|x^*\|_{U^*}$$

whereP, U and V be Banach reflexive space and $F \in L(P,V), G \in L(U,V)$.

Now, Let $P = V = L^2(\omega)$, $U = 0$, $F = I$ to $L^2(\omega)$ and $G = S_A^*(.)\,\chi_\omega^* C^*$ where $S_A(.)$ is a strongly continuous semi group generates by A, witch is ω_E-stable then, it is ω_E-detectable■.

As in El Jai and Pritchard [17], we will develop a characterization result that links the ω_E-detectability in terms of sensors structures. So, we recall some definitions related to sensors.

- A sensor is defined by any couple (D, f) where D a non-empty closed subset of Ω, is the spatial support of the sensor, and $f \in L^2(D)$ defines the spatial distribution of the sensing measurements on D.

In the case of a pointwise sensor, D is reduced to a point $\{b\}$ and $f = \delta(.-b)$, where D is the Dirac mass concentrated in b. Depending on the choice of the parameters D and f we have various types of sensors, the output function (2) may be written in the form

$y(t) = \int_D x(\xi,t) f(\xi) d\xi$ (zone case)

$y(t) = \int_\Omega x(\xi,t) \delta(\xi - b) d\xi = x(b,t)$ (pointwise case)

In the case of boundary measurements (pointwise or zone) the support of sensors D is subset of $\partial\Omega$. Then, the output function (2) given by

$y(t) = \int_\Omega \frac{\partial x}{\partial v}(\xi,t) \delta(\xi - b) d\xi$ (Boundary pointwise case) (12)

Now in the case where the zone measurements, with $D = \Gamma \subset \partial\Omega$ and $f \in L^2(\Gamma)$. Then, the output function (2) given by

$y(t) = \int_\Gamma \frac{\partial x}{\partial v}(\xi,t) f(\xi) d\xi$ (boundary zone case) (13)

- The sensors (zone or pointwise) $(D_i, f_i)_{1\le i\le q}$ are said to be ω -strategic sensors if the system (1)-(2) is weakly ω-observable.

Let us consider the set (φ_{nj}) of orthonormal functions in $L^2(\omega)$ associated with the eigenvalues λ_n of multiplicity r_n [15] and suppose that the system (1) has J unstable modes. We have the following characterization of ω_E-detectability in the terms of the structure sensors.

Proposition 2.8: Suppose that there are q zone sensors$(D_i, f_i)_{1\le i\le q}$. If

(1) $q \ge r$.

(2) $Rank\ G_n = r_n, \forall n,\ n = 1, \dots, J$

with$G = (G_n)_{ij} = \left(\langle \varphi_{jk}, f_i \rangle_{L^2(D_i)}\right)$ where$\sup_n r_n = r < \infty$and$j = 1, \dots, r_n$.

Then the system (1)-(2) is ω_E-detectable.

Proof:by the result on observability considering $\chi_\omega K^*$[14], we can proof this theorem. We see that if the system is satisfy the condition (2) above. Since $Rank\ G_n = r_n$, therefore, the sensor of the system (1)-(2) is strategic sensor, and this system (1)-(2) is weakly ω-observable, then it's exactly ω-observable, finally we have the system (1)-(2) is ω_E-detectable.

3. REGIONAL EXPONENTIAL OBSERVABILITY

In this section, we give an approach which allows constructing an ω-estimator of$\hat{T}x(\xi,t)$. This method avoids the calculation of the inverse operators, and the consideration of the initial state [18]. It enables to observe the current state in ω without needing the effect of the initial state of the original system.

3.1ω_E-Estimator Reconstruction Method

We consider the system and the output specified by the following form:

$$\begin{cases} \frac{\partial x}{\partial t}(\xi,t) = Ax(\xi,t) + Bu(t) & Q \\ x(\eta,t) = 0 & \Theta \\ x(\xi,0) = x_0(\xi) & \Omega \\ \qquad y(.,t) = Cx(.,t)Q \end{cases} \quad (14)$$

Let $\omega \subset \Omega$ be a given subdomain (region) of Ω and assume that for $T \in \mathcal{L}(L^2(\Omega))$, and $\hat{T} = \chi_\omega T$ (where χ_ω is defined in (5)) there exists a system with state $z(.,t)$ such that

$z(\xi,t) = \hat{T}x(\xi,t).$ (15)

Thus, if we can build a system which is an exponential estimator for $z(\xi,t)$, then it will also be an exponential estimator for $\hat{T}x(.,t)$, that is to say an exponential estimator to the restriction of $Tx(.,t)$ to the region ω. The equations (2)-(15) give

$\begin{bmatrix} y \\ z \end{bmatrix} = \begin{bmatrix} C \\ \hat{T} \end{bmatrix} x.$ (16)

If we assume that there exist two linear bounded operators R and S, where $R: \mathbb{R} \to L^2(\omega)$ and $S: L^2(\omega) \to L^2(\omega)$, such that $RC + S\hat{T} = I$, then by deriving $z(\xi,t)$ we have

$$\frac{\partial z}{\partial t}(\xi,t) = \hat{T}\frac{\partial x}{\partial t}(\xi,t) = \hat{T}Ax(\xi,t) + \hat{T}Bu(t)$$
$$= \hat{T}ASz(\xi,t) + \hat{T}ARy(.,t) + \hat{T}Bu(t).$$

Consider now the system (which is destined to be the maximal ω_E-estimator for z)

$$\begin{cases}\frac{\partial \hat{z}}{\partial t}(\xi,t)=F_\omega \hat{z}(\xi,t)+G_\omega u(t)+H_\omega y(.,t) & Q\\ \hat{z}(\eta,t)=0 & \Theta\\ \hat{z}(\xi,0)=\hat{z}_0(\xi) & \Omega\end{cases} \quad (17)$$

whereF_ω generates a strongly continuous semi-group $(S_{F_\omega}(t))_{t\geq 0}$, which is regionally exponentially stable on $X=L^2(\omega)$, i.e.,$\exists M_{F_\omega}, \alpha F_\omega>0$, such that

$\|\chi_\omega S_{F_\omega}(.)\|_{L^2(\omega)} \leq M_{F_\omega} e^{-\alpha_{F_\omega} t}, \forall t\geq 0$.(18)

and$G_\omega \in \mathcal{L}(R^p, L^2(\omega))$ and $H_\omega \in \mathcal{L}(R^q, L^2(\omega))$. The solution of (17) is given by

$\hat{z}(.,t)=S_{F_\omega}(t)\hat{z}_0(.)+\int_0^t S_{F_\omega}(t-\tau)[G_\omega u(\tau)+H_\omega y(.,\tau)]d\tau$ (19)

3.2ω_E- Observability

In this case, we consider$\hat{T}=I$, and $X=Z$, so the operator equation $\hat{T}A-F_\omega\hat{T}=H_\omega C$ of the ω-observable reduces to $F_\omega=A-H_\omega C$, where A and C are known. Thus, the operator H_ω must be determined such that the operator F_ω is ω_E-stable. For the system (14), consider the dynamic system

$$\begin{cases}\frac{\partial \hat{z}}{\partial t}(\xi,t)=A\hat{z}(\xi,t)+Bu(t)+H_\omega(y(.,t)-C\hat{z}(\xi,t)) & Q\\ \hat{z}(\eta,t)=0 & \Theta\\ \hat{z}(\xi,0)=0 & \Omega\end{cases} \quad (20)$$

Thus, a sufficient condition for existence of ω_E-estimator is formulated in the following proposition.

Proposition 3.1: Suppose that the system (1)-(2) is ω_E-detectable,and then the dynamical system (20) achieve the ω_E-observability for the system (1)-(2), i.e.

$$\lim_{t\to\infty}\|x(\xi,t)-\hat{z}(\xi,t)\|_{L^2(\omega)}=0.$$

Proof: By the same way with minor modifications as in ref. R. Al-Saphory [2] we can prove the proposition 3.1 in different case of sensors (zone, pointwise) internal or boundary.

4. REGIONAL REDUCED EXPONENTIAL OBSERVABILITY

In this section we need some of additional assumptions, concerning the semigroup, its infinitesimal generator, and the observation space, under which condition can be given a regional reduced estimator for the state system (1)-(2).

4.1 General Decomposed System

Now, under the assumption of strongly continuous semigroup we have the system (1)-(2) is reduced as in the additional assumptions allow a decomposition of (1) to a form of the stabilizing operator H. These assumptions are as follows.

(1) A has a pure point spectrum, denoted by $\sigma(A)$.

(2) $S_A(t)$ is a compact operator for some $t>0$

(3) For $\delta>0$, $\sigma(A)$ the spectrum of A contained in the closed half plane $\{\lambda: Re\,\lambda\geq -\delta\}$.

(4) The subspace associated with each finite dimensional point of $\sigma(A)$ in the half plane $\{\lambda: Re\,\lambda\geq -\delta\}$.

(5) $\mathcal{O}$is finite dimensional.

These five assumptions are strong. The Hille-Yosida theorem implies that the set of spectral point of A lying in the half plane $\{\lambda: Re\,\lambda\geq -\delta\}$ forms a bounded spectral set. Denote this spectral set by $\sigma(A_1)$. Using the spectral set $\sigma(A_1)$, a reduced form of (1) can be derived. Denote $\sigma(A)-\sigma(A_1)$ by $\sigma(A_2)$. As A is a closed operator with nonempty resolvent set, operational calculus can be used to completely reduce the operator A in terms of the spectral sets $\sigma(A_1)$ and $\sigma(A_2)$ [20]. $\sigma(A_1)$and $\sigma(A_2)$ determine subspaces X_1 and X_2,

$X=X_1\oplus X_2$, (21)

and projections $E_1: X\to X_1, E_2: X\to X_2,$ such that

$$E_1Ax=AE_1x$$

$$E_2Ax=AE_2x$$

Defining $A_1x=AE_1x,\ D(A_1)=D(A)\cap X_1$

and$A_2x=AE_2x,\ D(A_2)=D(A)\cap X_2$

the operator A can be represented by

$Ax=\begin{bmatrix}A_1 & 0\\ 0 & A_2\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix}andB=\begin{bmatrix}B_1\\ B_2\end{bmatrix}$ (22)

Where $x = x_1 + x_2$,

$x \in D(A), x_1 \in D(A_1), x_2 \in D(A_2), B_1 \in \mathcal{L}(R^P, X_1)andB_2 \in \mathcal{L}(R^P, X_2)$

as$D(A)$ is dense in $X, D(A_1)$ is dense in X_1, and $D(A_2)$ is dense in X_2.

A_1andA_2 are closed operators as A is closed operator. If A is the infinitesimal generator of a strongly continuous semigroup, then the Hille-Yosida theorem shows that both A_1 and A_2 are infinitesimal generators. Using the decomposition of X and A given by (21)-(22), and then (1)-(2) can be rewritten in the following forms [20]

$$\begin{cases} \frac{\partial x_1}{\partial t}(\xi,t) = A_1 x_1(\xi,t) + E_1 B_1 u(t) & Q \\ x_1(\eta,t) = 0 & \Theta \\ x_1(\xi,0) = x_{1_0}(\xi) & \Omega \end{cases} \quad (23)$$

and

$$\begin{cases} \frac{\partial x_2}{\partial t}(\xi,t) = A_2 x_2(\xi,t) + E_2 B_2 u(t) & Q \\ x_2(\eta,t) = 0 & \Theta \\ x_2(\xi,0) = x_{2_0}(\xi) & \Omega \end{cases} \quad (24)$$

Augmented with the output function

$y(.,t) = Cx_1(\xi,t)$(25)

Equations (24)-(25) are called the reduced form of (1)-(2).

Since A_1 is the restriction of A to X_1, and $D(A_1) = D(A) \cap X_1$, the spectrum of A_1 is $\sigma(A_1)$ [21]. As the points of $\sigma(A_1)$ are isolated, each point by itself is a spectral set, and the spectral sets so formed are pairwise disjoint. Thus a projection E_{ij} and subspace X_{ij} can be associated with each point $\lambda_j \in \sigma(A_1)$, and the subspace X_1 completely reduced to

$$X_1 = X_{11} \oplus X_{12} \oplus \dots \oplus X_{in}$$

where$n$ is the number of points in $\sigma(A_1)$. Each X_{ij} is finite dimensional by assumption, hence X_1 is finite dimensional, and A_1 is a bounded operator. Then choosing bases for X_1 and $\mathcal{O}$, (24) can be represented as a linear constant coefficient ordinary differential equation, and C restricted to X_1 can be expressed as a matrix.

In terms of the finite dimensional bases for X_1 and $\mathcal{O}$, the homogeneous equations corresponding to (24)-(25) are

$$\begin{cases} \frac{\partial x_1}{\partial t}(\xi,t) = A_1 x_1(\xi,t) \\ x_1(\xi,0) = x_{1_0}(\xi) \end{cases} \quad (26)$$

$$\begin{cases} \frac{\partial x_2}{\partial t}(\xi,t) = A_2 x_2(\xi,t) \\ x_2(\xi,0) = x_{2_0}(\xi) \end{cases} \quad (27)$$

$y(.,t) = Cx_1(\xi,t)$ (28)

WhereΓ is the coordinate space associated with the basis forX_1, and $C: X_1 \to \mathcal{O}$ in terms of the bases of X_1 and $\mathcal{O}$.

An estimate will now be made of the solutions of (28). A having a pure point spectrum implies that A_2 has a pure point spectrum, while $S_{A_2}(t)$ being a compact operator for some $t > 0$ implies that $S_{A_2}(t)$ is a compact operator. As $S_{A_2}(t)$ is a compact operator, its spectrum consists of only point spectrum, denoted by $\mathcal{P}\sigma(S_{A_2}(t))$ is given by

$e^{\mathcal{P}\sigma(A_2)t}$, as $Re\,\sigma(A_2) \le -\delta$, $\|e^{\mathcal{P}\sigma(A_2)t}\| \le e^{-\delta t}$

Then the spectral radius of $S_{A_2}(t)$ satisfies

$r_\sigma(S_{A_2}(t)) \le e^{-\delta t}$.

using a lemma of Hale [18]. For any $\gamma > 0$ there exists an $M(\gamma) \ge 1$ such that

$$\|S_{A_2}(t) x_{2_0}\| \le M(x_1) e^{(-\delta + x_1)t} \|x_{2_0}\|$$

for all $t \ge 0$ and $x_{2_0} \in X_2$. Thus, (27) is exponentially stable.

4.2 General Reduced System

In the case where the output function (2) gives information about a part of the state vector $x(\xi,t)$, it is necessary to define an exponential estimator enables to construct the unknown part of the state. Consider now $X = X_1 \oplus X_2$ where X_1 and X_2 are subspaces of X. Under the hypothesis of subsection 4.1, the system (1) can be decomposed [16, 20] by

$$A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}, \quad x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \quad B = \begin{bmatrix} B_1 \\ B_2 \end{bmatrix}$$

where $x_1 \in X_1$,$x_2 \in X_2$,$B_1 \in \mathcal{L}(X_1, U)$and $B_2 \in \mathcal{L}(X_2, U)$. Using the decomposition above, the system (1) can be written by the form

$$\begin{cases} \frac{\partial x_1}{\partial t}(\xi,t) = A_{11}x_1(\xi,t) + A_{12}x_2(\xi,t) + B_1u(t) & Q \\ x_1(\eta,t) = 0 & \Theta \\ x_1(\xi,0) = x_{01}(\xi) & \Omega \end{cases} \quad (29)$$

and

$$\begin{cases} \frac{\partial x_2}{\partial t}(\xi,t) = A_{21}x_1(\xi,t) + A_{22}x_2(\xi,t) + B_2u(t) & Q \\ x_2(\eta,t) = 0 & \Theta \\ x_2(\xi,0) = x_{2_0}(\xi) & \Omega \end{cases} \quad (30)$$

augmented with the output function

$$y(.,t) = Cx_1(\xi,t) \quad (31)$$

where $x(\xi,t) = x_1(\xi,t) \oplus x_2(\xi,t)$. The problem consists in constructing a regional exponential estimator that enables one to estimate the unknown part$x_2(\xi,t)$ equivalently; the problem is reduced to define the dynamical system for (31). Thus, equations (30)-(31) allow the following system:

$$\begin{cases} \frac{\partial a}{\partial t}(\xi,t) = A_{22}a(\xi,t) + [B_2u(t) + A_{21}y(.,t)] & Q \\ a(\eta,t) = 0 & \Theta \\ a(\xi,0) = a_0(\xi) & \Omega \end{cases} \quad (32)$$

with the output function

$$\tilde{y}(.,t) = A_{12}a(.,t) \quad (33)$$

where the state a in system (32) plays the role of the state x_2 in system (30).

4.3 Regional Reduced Observability and$\omega_{E\mathcal{R}}$-Detectability

As in previous section 2 we can extend these results to the case of regional reduced ordered system for regional observability andω_E-detectability. In this case, the equation (8) it can be given by define the following operator

$\mathcal{K}: x_2 \in X_2 \to \mathcal{K}x_2 = A_{12}S_{A_{22}}(t)x_2 \in \mathcal{O}$, then$y(.,t) = \mathcal{K}(t)x_{2_0}(.)$, with the adjoint $\mathcal{K}^*: \mathcal{O} \to X_2$ such that

$\mathcal{K}^*y^*(.,t) = \int_0^t S^*(s)\, A_{12}^* y^*(.,s)ds.$

Let $\omega \subset \Omega$ and $\chi_\omega: L^2(\Omega) \to L^2(\omega) = X_2, x_2 \to \chi_\omega = x_{2|_\omega}$

where$x_{2|_\omega}$ is the restriction of the state x_2 to ω.

Definition 4.1: The system (32)-(33) is called exactly regionally reduced-observable (or exactly $\omega_\mathcal{R}$-observable) if

$$\mathrm{Im}\chi_\omega\mathcal{K}^* = L^2(\omega) = X_2$$

Definition 4.2: The system (32)-(33) is called weakly regionally reduced-observable (or weakly $\omega_\mathcal{R}$-observable) if

$$\overline{\mathrm{Im}\chi_\omega\mathcal{K}^*} = L^2(\omega) = X_2$$

Definition 4.3: The suite of sensors (zone or pointwise) $(D_i, f_i)_{1\le i\le q}$ are called regional reduced strategic sensors (or $\omega_\mathcal{R}$- strategic sensors if the system (32)-(33) is weakly $\omega_\mathcal{R}$-observable.

Definition 4.4: The semi-group $(S_{A_{22}}(t))_{t\ge 0}$ is said to be exponential reduced stable (or $\Omega_{E\mathcal{R}}$-stable) if $\exists\, M, \alpha > 0$ such that

$$\left\|S_{A_{22}}(t)\right\|_{L^2(\Omega)} \leq M_{A_{22}} e^{-\alpha_{A_{22}}(t)}, t \geq 0 \qquad (34)$$

Definition 4.5: Let $(S_{A_{22}}(t))_{t\geq 0}$ is an $\Omega_{E\mathcal{R}}$-stable semi-group, then $\forall x_2 \in X_2$ the solution of the associated autonomous system satisfies:

$$\|x_2(.,t)\|_{L^2(\Omega)} = \left\|S_{A_{22}}(t)x_{2_0}(.)\right\|_{L^2(\Omega)} \leq M_{A_{22}} e^{-\alpha_{A_{22}}(t)}\left\|x_{2_0}(.)\right\|_{L^2(\Omega)}$$

and therefore

$$\lim_{t\to\infty}\|x_2(.,t)\|_{L^2(\Omega)} = \lim_{t\to\infty}\left\|S_{A_{22}}(t)x_{2_0}(.)\right\|_{L^2(\Omega)} = 0$$

Definition 4.6: The system (32) is said to be $\Omega_{E\mathcal{R}}$-stable if the operator A_{22} generates a semi-group which is $\Omega_{E\mathcal{R}}$-stable.

Definition 4.7: The system (32)-(33) is said to be exponential reduced detectable on Ω (or $\Omega_{E\mathcal{R}}$-detectable) if there exists an operator $\mathcal{H}: R^q \to L^2(\Omega)$ such that $(A_{22} - \mathcal{H}C)$ generates a strongly continuous semi-group $(S_{A_{22}}(t))_{t\geq 0}$ which is $\Omega_{E\mathcal{R}}$-stable.

Remark 4.8: The relation (34) is true on a given subdomain $\omega \subset \Omega$, i.e.

$$\left\|\chi_\omega S_{A_{22}}(t)\right\|_{\mathcal{L}(L^2(\omega),\, L^2(\Omega))} \leq M_{A_{22}} e^{-\alpha_{A_{22}}(t)}, t \geq 0 \qquad (35)$$

and then

$$\lim_{t\to\infty}\|x_2(.,t)\|_{L^2(\omega)} = 0$$

We refer to this as regional exponential reduced stability (or $\omega_{E\mathcal{R}}$-stability).

Definition 4.9: The system (32) The system is said to be regional exponential reduced stability (or $\omega_{E\mathcal{R}}$-stable) if the operator A_{22} generates a semi-group which is $\omega_{E\mathcal{R}}$ -stable.

In this section, we shall extend the definition of $\Omega_{E\mathcal{R}}$-detectable (35) to the regional case by considering ω as subregion of Ω.

Definition 4.10: The system (32)-(33) is said to be regional exponential reduced detectable (or $\omega_{E\mathcal{R}}$-detectable) if there exists an operator $\mathcal{H}_\omega: R^q \to L^2(\omega)$ such that $(A_{22} - \mathcal{H}_\omega A_{12})$generates a strongly continuous semi-group $(S_{A_{22}}(t))_{t\geq 0}$, which is $\omega_{E\mathcal{R}}$-stable.

From proposition 3.1, we have the dynamical system for (32)-(33) may be given by

$$\begin{cases} \frac{\partial \hat{z}}{\partial t}(\xi,t) = A_{22}\hat{z}(\xi,t) + [B_2u(t) + A_{21}y(.,t)] + \\ \qquad \mathcal{H}_\omega[\tilde{y}(.,t) - A_{12}\hat{z}(\xi,t)] & \mathcal{Q} \\ \hat{z}(\eta,t) \;= 0 & \Theta \\ \hat{z}(\xi,0) \;= \hat{z}_0(\xi) & \Omega \end{cases} \qquad (36)$$

where$(A_{22} - \mathcal{H}_\omega A_{12})$ generates a strongly continuous semi-group $(S_{A_{22}}(t))_{t\geq 0}$ which is $\omega_{E\mathcal{R}}$-stable on the Hilbert space$X_2 \subset X = L^2(\Omega), (B_2 - \mathcal{H}_\omega B_1) \in L(R^p, X_2)$and $(A_{22}\mathcal{H}_\omega - \mathcal{H}_\omega A_{12}\mathcal{H}_\omega - \mathcal{H}_\omega A_{11} + A_{21}) \in L(R^q, X_2)$ [7].

The importance of reduced $\omega_{E\mathcal{R}}$-detectability is possible to define a reduced $\omega_{E\mathcal{R}}$-estimator for system state may be given by the following important result:

Theorem 4.11: If there are q sensors $(D_i, f_i)_{1\leq i\leq q}$ and the spectrum of A_{22} contains J eigenvalues with non-negative real parts. The system (32)-(33) is $\omega_{E\mathcal{R}}$-detectable iff

1. $q \geq m_2$
2. Rank $G_{2_i} = m_{2_i}, \forall i, i = 1, \ldots, J$ with

$$G_2 = G_{2_{ij}} = \begin{cases} \langle \varphi_j(.), f_i(.)\rangle_{L^2(D_i)}, & \text{for zone sensors} \\ \varphi_j(b_i), & \text{for pointwise sensors} \end{cases}$$

wheresup $m_{2_i} = m_2 < \infty$ and $j = 1, \ldots, \infty$.

Proof: The proof is developed to the case of zone sensors in the following stapes:

1) The system (32) can be decomposed by the projections $\mathcal{P}$ and $I - \mathcal{P}$, on two parts, unstable and stable under the assumptions of section 4.2, where $\mathcal{P}$ and $(I - \mathcal{P})$ are play the role of projection as E_1, E_2in section 4.1. The state vector may be given by

$$x_2(\xi,t) = \left[x_{2_1}(\xi,t)x_{2_2}(\xi,t)\right]^{tr}$$

where$x_{2_1}(\xi,t)$ is the state component of the unstable part of system (32), may be written in the form

$$\begin{cases} \frac{\partial x_{2_1}}{\partial t}(\xi,t) = A_{22_1}x_{2_1}(\xi,t) + \mathcal{P}[A_{21_1}x_{1_1}(\xi,t) + B_2u(t)] & \mathcal{Q} \\ x_{2_1}(\eta,t) = 0 & \Theta \\ x_{2_1}(\xi,0) = x_{2_{1_0}}(\xi) & \Omega \end{cases} \quad (37)$$

and$x_{2_2}(\xi,t)$ is the component state of the stable part of system (32), given by

$$\begin{cases} \frac{\partial x_{2_2}}{\partial t}(\xi,t) = A_{22_2}x_{2_2}(\xi,t) + (I-\mathcal{P})[A_{21_2}x_{1_2}(\xi,t) + B_2u(t)] & \mathcal{Q} \\ x_{2_2}(\eta,t) = 0 & \Theta \\ x_{2_2}(\xi,0) = x_{2_{2_0}}(\xi) & \Omega \end{cases} \quad (38)$$

The operator $A_{22_1}$is represented by a matrix of order $\left(\sum_{i=1}^{J} m_{2_i}, \sum_{i=1}^{J} m_{2_i}\right)$ given by

$A_{22_1} = \text{diag}[\lambda_{2_1},\ldots,\lambda_{2_1},\ldots,\lambda_{2_J},\ldots,\lambda_{2_J}]$and

$$\mathcal{P}B_2 = \left[G_{2_1}^{tr}, G_{2_2}^{tr}, \ldots, G_{2_J}^{tr}\right]$$

From condition (2) of this theorem, then the suite of sensors $(D_i, f_i)_{1\le i\le q}$ is $\omega_{\mathcal{R}}$-strategic for the unstable part of the system (32), the subsystem (37) is weakly regionally reduced-observable in ω (or weakly $\omega_{\mathcal{R}}$-observable) and since it is finite dimensional, then it is exactly regionally reduced-observable in ω (or exactly $\omega_{\mathcal{R}}$-observable).

Therefore it is $\omega_{E\mathcal{R}}$-detectable, and hence there exists an operator $\mathcal{H}_\omega^1$ such that $(A_{22_1} - \mathcal{H}_\omega^1 A_{12_1})$ which satisfies the following:

$$\exists M_\omega^1, \alpha_\omega^1 > 0 \text{ such that } \left\| e^{(A_{22_1} - \mathcal{H}_\omega^1 A_{12_1})t} \right\|_{L^2(\omega)} \le M_\omega^1 e^{-\alpha_\omega^1(t)}$$

and we have

$$\|x_{2_1}(\xi,t)\|_{L^2(\omega)} \le M_\omega^1 e^{-\alpha_\omega^1(t)} \|\mathcal{P}x_{2_0}(.)\|_{L^2(\omega)}$$

Since the semi-group generated by the operator A_{22_2} is $\omega_{E\mathcal{R}}$-stable,

$$\exists M_\omega^2, \alpha_\omega^2 > 0 \text{ such that}$$

$$\|x_{2_2}(\xi,t)\|_{L^2(\omega)} \le M_\omega^2 e^{-\alpha_\omega^2(t)} \|(I-\mathcal{P})x_{2_0}(.)\|_{L^2(\omega)}$$

$$+ \int_0^t M_\omega^2 e^{-\alpha_\omega^2(t-\tau)} \|(I-\mathcal{P})x_{2_0}(.)\|_{L^2(\omega)} \|u(\tau)\| d\tau$$

and therefore $x_2(\xi,t) \to 0$ when $t \to \infty$. Thus, the system (32)-(33) is $\omega_{E\mathcal{R}}$-detectable.

2) If the system (32)-(33) is $\omega_{E\mathcal{R}}$-detectable, then

$\exists\ \mathcal{H}_\omega \in \mathcal{L}(L^2(0,\infty,R^q), L^2(\omega))$such that $(A_{22} - \mathcal{H}_\omega A_{12})$ generates an $\omega_{E\mathcal{R}}$-stable, strongly continuous semi-group $(S_{A_{22}}(t))_{t\ge 0}$ on the space $L^2(\omega)$ which satisfies the follwing

$\exists\ M_\omega, \alpha_\omega > 0$such that

$$\|\chi_\omega S_{A_{22}}(t)\|_{L^2(\omega)} \le M_\omega e^{-\alpha_\omega(t)}$$

Thus the unstable subsystem (37) is $\omega_{E\mathcal{R}}$-detectable. Since this subsystem is of finite dimensional, then it is exactly $\omega_{\mathcal{R}}$-observable. Therefore (37) is weakly $\omega_{\mathcal{R}}$-observable and hence it is reduced $\omega_{\mathcal{R}}$-strategic, i.e.

$[\mathcal{K}\chi_\omega^* x_2^*(.,t) = 0 \Longrightarrow x_2^*(.,t) = 0]$. For $x_2^*(.,t) \in L^2(\omega)$we have

$$\mathcal{K}\chi_\omega^* x_2^*(.,t) = (\sum_{j=1}^{J} e^{\lambda_j t} \langle \varphi_j(.), x_2^*(.,t)\rangle_{L^2(\omega)} \langle \varphi_j(.), f_i(.)\rangle_{L^2(\Omega)})_{1\le i\le q}$$

If the unstable system (37) is not $\omega_{\mathcal{R}}$-strategic, $\exists x_2^*(.,t) \in L^2(\omega)$ such that $\mathcal{K}\chi_\omega^* x_2^*(.,t) = 0$ this leads to

$$\sum_{j=1}^{J} \langle \varphi_j(.), x_2^*(.,t)\rangle_{L^2(\omega)} \langle \varphi_j(.), f_i(.)\rangle_{L^2(\Omega)} = 0$$

the state vectors x_{2_i} may be given by

$$x_{2_i}(.,t) = \left[\langle \varphi_1(.), x_2^*(.,t)\rangle_{L^2(\omega)} \langle \varphi_j(.), x_2^*(.,t)\rangle_{L^2(\omega)}\right]^{tr} \ne 0$$

we then obtain $G_{2_i}x_{2_i} = 0, \forall i, i = 1, \dots, J$ and therefore $Rank\ G_{2_i} \neq m_{2_i}$.

Here, we construct the $\omega_{E\mathcal{R}}$-estimator for parabolic distributed parameter system (1), we need to present the following remakes

Remark 4.12: Now, choose the following decomposition:

$$\hat{z} = \begin{bmatrix} \hat{z}_1 \\ \hat{z}_2 \end{bmatrix} = \begin{bmatrix} y \\ \varphi + \mathcal{H}_\omega y \end{bmatrix}$$

which estimates exponentially the state vector

$$x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

Then, the dynamical system (36) is given by the following system:

$$\begin{cases} \frac{\partial \varphi}{\partial t}(\xi,t) = (A_{22} - \mathcal{H}_\omega A_{12})\,\varphi(\xi,t) \\ +[\,A_{22}\mathcal{H}_\omega - \mathcal{H}_\omega A_{12}\mathcal{H}_\omega - \mathcal{H}_\omega A_{11} + A_{21}] \\ y(\xi,t) + [B_2 - \mathcal{H}_\omega B_1]u(t) & \mathcal{Q} \\ \varphi(\eta,t) \ = 0 & \Theta \\ \varphi(\xi,0) \ = \ \varphi_0(\xi) & \Omega \end{cases} \quad (39)$$

which defines an $\omega_{E\mathcal{R}}$-estimator for $T_\omega x_2(\xi,t)$ if

1. $\lim_{t\to\infty} \|\varphi(\xi,t) - T_\omega x_2(\xi,t)\|_{L^2(\omega)} = 0$
2. $T_\omega : D(A_{22}) \to D(A_{22} - \mathcal{H}_\omega A_{12})$ where $T_\omega = \chi_\omega T$ and $\varphi(\xi,t)$ is the solution of system (39).

Remark 4.13: the dynamical system (39) observes the regional reduced state of the system (1) if the following conditions satisfy:

1. $\exists L \in L\big(\mathcal{O}, L^2(\omega)\big)$ and $M \in L\big(L^2(\omega)\big)$ such that:

$$L\,A_{12} + MT_\omega = I_\omega$$

2. $T_\omega A_{22} - (A_{22} - \mathcal{H}_\omega A_{12})T_\omega = \mathcal{H}_\omega \mathrm{A}_{12}$ and $(B_2 - \mathcal{H}_\omega B_1) = T_\omega B_2$
3. The system (39) defines an $\omega_{E\mathcal{R}}$-estimator for the system (1).
4. If $X = X_2$ and $T_\omega = I_\omega$ then, in the above case, we have

$$A_{22} - (A_{22} - \mathcal{H}_\omega \mathrm{A}_{12}) = \mathcal{H}_\omega \mathrm{A}_{12}$$

Remark 4.14: the system (1) is $\omega_{E\mathcal{R}}$-observable if there exists an $\omega_{E\mathcal{R}}$-estimators (39) which estimate the regional exponential reduced state of this system.

Now, we present the sufficient condition of the regional exponential reduced observability notion as in the following main result.

Theorem 4.15: If the system (32)-(33), is $\omega_{E\mathcal{R}}$-detectable, then it is $\omega_{E\mathcal{R}}$-observable by the dynamical system (39), that means

$$\lim_{t\to\infty} \big\|\big(\varphi(\xi,\mathrm{t}) + \mathcal{H}_\omega y(\xi,\mathrm{t})\big) - x_2(\xi,\mathrm{t})\big\|_{L^2(\omega)} = 0,$$

Proof: The solution of the dynamical system (36) is given by

$$\hat{z}(\xi,t) = S_{\mathcal{H}_\omega}(t)\hat{z}_0(\xi) + \int_0^t S_{\mathcal{H}_\omega}(t-\tau)[B_2 u(\tau)$$

$$+A_{21}y(\xi,\tau) + \mathcal{H}_\omega \tilde{y}(\xi,\tau)]d\tau \qquad (40)$$

From the equations (32) and (33), we have

$$\tilde{y}(\xi,t) = A_{12}a(.,t) = \frac{\partial x_1}{\partial t}(\xi,t) - A_{11}x_1(\xi,t) - B_1 u(t) \quad (41)$$

By using (41) and (40), we obtain

$$\hat{z}(\xi,t) = S_{H_\omega}(t)\hat{z}_0(.) + \int_0^t S_{\mathcal{H}_\omega}(t-\tau)\mathcal{H}_\omega \frac{\partial x_1}{\partial t}(\xi,\tau)d\tau$$

$$+ \int_0^t S_{\mathcal{H}_\omega}(t-\tau)\,[B_2 u(\tau) + A_{21}y(\xi,\tau)$$

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$-\mathcal{H}_\omega A_{11}x_1(.,\tau) - \mathcal{H}_\omega B_1 u(\tau)]d\tau.$ (42)

and we can get

$$\int_0^t S_{H_\omega}(t-\tau)\mathcal{H}_\omega \frac{\partial x_1}{\partial t}(\xi,\tau)d\tau = \mathcal{H}_\omega x_1(.,t) - S_{H_\omega}(t)\mathcal{H}_\omega x_{0_1}(.)$$

$$+ (A_{22} - \mathcal{H}_\omega A_{12}) \int_0^t S_{\mathcal{H}_\omega}(t-\tau)\, \mathcal{H}_\omega x_1(.,\tau)d\tau \quad (43)$$

Using Bochnerintegrability properties and closeness of $(A_{22} - \mathcal{H}_\omega A_{12})$, the equation (43) becomes

$$\int_0^t S_{\mathcal{H}_\omega}(t-\tau)\mathcal{H}_\omega \frac{\partial x_1}{\partial t}(\xi,\tau)d\tau = \mathcal{H}_\omega x_1(.,t) - S_{\mathcal{H}_\omega}(t)\mathcal{H}_\omega x_{0_1}(.)$$

$+(\int_0^t S_{\mathcal{H}_\omega}(t-\tau)\,(A_{22} - \mathcal{H}_\omega A_{12})\mathcal{H}_\omega x_1(\xi,\tau)d\tau$ (44)

Substituting (44) into (42), we have

$$\hat{z}(.,t) = S_{H_\omega}(t)\hat{z}_0(.) - S_{H_\omega}(t)H_\omega x_{0_1}(.) + H_\omega x_1(.,t)$$

$$+ \int_0^t S_{H_\omega}(t-\tau)\,[A_{22}H_\omega - H_\omega A_{12}H_\omega - H_\omega A_{11} + A_{21}]x_1(.,\tau)d\tau + \int_0^t S_{H_\omega}(t-\tau)[B_2 - H_\omega B_1]u(\tau)d\tau. \quad (45)$$

Setting $\varphi(.,t) = \hat{z}(.,t) - \mathcal{H}_\omega y(.,t)$,with $\varphi_0(.,0) = \hat{z}_0(.) - \mathcal{H}_\omega x_{0_1}(.)$, where$y_0(.) = x_{0_1}(.)$. Now, assume that $(A_{22}\mathcal{H}_\omega - \mathcal{H}_\omega A_{12}\mathcal{H}_\omega - \mathcal{H}_\omega A_{11}A_{21})$and $(B_2 - \mathcal{H}_\omega B_1)$ are bounded operators , the equation (45) can be differentiated to yield the following system

$$\begin{cases} \frac{\partial \varphi}{\partial t}(\xi,t) = (A_{22} - \mathcal{H}_\omega A_{12})\varphi(\xi,t) + (A_{22}\mathcal{H}_\omega - \mathcal{H}_\omega A_{12}\mathcal{H}_\omega \\ \qquad -\mathcal{H}_\omega A_{11} + A_{21})y(.,t) + (B_2 - \mathcal{H}_\omega B_1)u(t)\mathcal{Q} \\ \varphi(\eta,t) = 0 & \Theta \\ \varphi(\xi,0) = \varphi_0(\xi) & \Omega \end{cases}$$

and therefore

$$\frac{\partial z}{\partial t}(\xi,t) - \frac{\partial x_2}{\partial t}(\xi,t) = (\varphi(\xi,t) + \mathcal{H}_\omega y(\xi,t) - x_2(\xi,t)$$

$$= (A_{22}\hat{z}(\xi,t) + B_2 u(t) + A_{21}y(.,t) + \mathcal{H}_\omega(\tilde{y}(\xi,t)$$

$$-A_{12}\hat{z})(\xi,t) - A_{21}x_1(\xi,t) - A_{22}x_2(\xi,t) - B_2 u(t)$$

$= (A_{22} - \mathcal{H}_\omega A_{12})(\,\hat{z}(\xi,t) - x_2(\xi,t))$(46)

From the relation

$$\|\chi_\omega S_{A_{22}}(t)x_{2_0}(.)\|_{L^2(\omega)} \le M_{A_{22}} e^{-\alpha_{A_{22}}(t)} \|x_{2_0}(.)\|_{L^2(\omega)}$$

we obtain

$$\| \hat{z}(.,t) - x_2(.,t) \|_{L^2(\omega)} \le \| \chi_\omega S_{\mathcal{H}_\omega}(t) \|_{L^2(\omega)} \| \hat{z}(.,0) - x_2(.,0) \|_{L^2(\omega)}$$

$\le M\mathcal{H}_\omega e^{-\alpha H_\omega t} \| \hat{z}(.,0) - x_2(.,0) \|_{L^2(\omega)} \to 0$ as $t \to \infty$ (47)

where the component $\hat{z}(\xi,t)$ is an exponentially estimator of x_2. then, we have the system (36) is a $\omega_{E\mathcal{R}}$-observable for the system (32)-(33).

From the previous theorem 4.15, we can deduce the following definition which characterizes another new strategic sensor:

Definition 4.16: A sensors is$\omega_{E\mathcal{R}}$-strategic sensor if the corresponding systemis$\omega_{E\mathcal{R}}$-observable**.**

5. APPLICATIONS TO EXCHANGE SYSTEMS

Consider the case of two-phase exchange system described by the following coupled parabolic equations:

$$\begin{cases} \frac{\partial x_1}{\partial t}(\xi_1,\xi_2,t) = \alpha \frac{\partial^2 x_1}{\partial \xi^2}(\xi_1,\xi_2,t) + \beta\big(x_1(\xi_1,\xi_2,t) - x_2(\xi_1,\xi_2,t)\big)\mathcal{Q} \\ \frac{\partial x_2}{\partial t}(\xi_1,\xi_2,t) = \gamma \frac{\partial^2 x_2}{\partial \xi^2}(\xi_1,\xi_2,t) + \beta\big(x_2(\xi_1,\xi_2,t) - x_1(\xi_1,\xi_2,t)\big)\mathcal{Q} \\ \quad x_1(\xi_1,\xi_2,0) = x_{0_1}(\xi_1,\xi_2),\ x_2(\xi_1,\xi_2,0) = x_{0_2}(\xi_1,\xi_2)\Omega \\ x_1(\eta_1,\eta_2,t) = 0,\ \ x_2(\eta_1,\eta_2,t) = 0 \qquad \Theta \end{cases} \quad (48)$$

and consider $\Omega = (0,1) \times (0,1)$ with subregion $\omega = (\alpha_1,\beta_1) \times (\alpha_2,\beta_2) \subset \Omega$. Suppose that it is possible to measure the states $x_1(.,t)$, by using q zone sensors$(D_i,f_i)_{1\le i\le q}$. The output function (2) is given by

$$y(t) = Cx_1(.,t)$$
$$= \left[\int_{D_1} x_1(\xi,t) f_1(\xi) d\xi \dots \int_{D_q} x_1(\xi,t) f_q(\xi) d\xi\right]^{tr}.$$

Now, the problem is to estimate exponentially $x_2(\xi,t)$.

Let us consider

$$\frac{\partial x}{\partial t} = \begin{bmatrix} \frac{\partial x_1}{\partial t} \\ \frac{\partial x_2}{\partial t} \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \qquad (49)$$

where

$$A_{11} = \alpha \frac{\partial^2 x_1}{\partial \xi^2}(\xi_1,\xi_2,t) + \beta, A_{22} = \gamma \frac{\partial^2 x_2}{\partial \xi^2}(\xi_1,\xi_2,t) + \beta$$

and$A_{12} = A_{21} = -\beta I$.

From theorem 4.11, we can construct regional reduced estimator for system (48) if the sensors $(D_i, f_i)_{1\le i\le q}$ are ω-strategic for the unstable part of the subsystem

$$\begin{cases} \frac{\partial x_1}{\partial t}(\xi_1,\xi_2,t) = \gamma \frac{\partial^2 x_1}{\partial \xi^2}(\xi_1,\xi_2,t) + \beta(x_1(\xi_1,\xi_2,t) \\ \quad -x_2(\xi_1,\xi_2,t)) & \mathcal{Q} \\ x_1(\xi_1,\xi_2,0) = x_{0_1}(\xi_1,\xi_2) & \Omega \\ x_1(\eta_1,\eta_2,t) = 0 & \Theta \end{cases} \qquad (50)$$

where that $\gamma = 0.1$ and $\beta = 1$. If we choose the sensors$(D_i, f_i)_{1\le i\le q}$ such that

$$y(t) = \left[\int_{D_1} x_1(\xi_1,\xi_2,t) f_1(\xi_1,\xi_2) d\xi_1 d\xi_2 \dots \int_{D_q} x_1(\xi_1,\xi_2,t) f_q(\xi_1,\xi_2) d\xi_1 d\xi_2\right]^{tr} \neq 0,$$

then, there exists $\mathcal{H}_\omega \in \mathcal{L}(R^q, L^2(\omega))$ such that the operator $(A_{22} - \mathcal{H}_\omega A_{12})$ generates a strongly continuous stable semi-group on the space $L^2(\omega)$. Thus we have

$$\lim_{n\to\infty} \|(w(.,t) + \mathcal{H}_\omega x_1(.,t)) - x_2(.,t)\|_{L^2(\omega)} = 0,$$

where

$$\begin{cases} \frac{\partial w}{\partial t}(\xi_1,\xi_2,t) = \gamma \frac{\partial^2 w}{\partial \xi^2}(\xi_1,\xi_2,t) + \beta((1+\mathcal{H}_\omega)w(\xi_1,\xi_2,t) \\ \quad +(\gamma - \alpha\mathcal{H}_\omega)\frac{\partial x_1}{\partial \xi^2}(\xi_1,\xi_2,t) + (\mathcal{H}_\omega^2 - 1)(\xi_1,\xi_2,t)) & \mathcal{Q} \\ w(\xi_1,\xi_2,0) = w_0(\xi_1,\xi_2) & \Omega \\ w(\eta_1,\eta_2,t) = 0 & \Theta \end{cases} \qquad (51)$$

In this section, we give the specific results related to some examples of sensors locations and we apply these results to different situations of the domain, which usually follow from symmetry considerations.

We consider the two-dimensional system defined on $\Omega = (0,1) \times (0,1)$ with the case of system described by the following equations:

$$\begin{cases} \frac{\partial x_2}{\partial t}(\xi_1,\xi_2,t) = \gamma \frac{\partial^2 x_2}{\partial \xi^2}(\xi_1,\xi_2,t) + \beta x_2(\xi_1,\xi_2,t) \\ \quad -\beta x_1(\xi_1,\xi_2,t)\mathcal{Q} \\ x_2(\xi_1,\xi_2,0) = x_{0_2}(\xi_1,\xi_2) & \Omega \\ x_2(\eta_1,\eta_2,t) = 0 & \Theta \end{cases} \qquad (52)$$

augmented with the output function

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$y(t) = Cx_1(.,t)$ (53)

Let $\omega = (\alpha_1, \beta_1) \times (\alpha_2, \beta_2)$, In this case the eigenfunctions and eigenvalues for the dynamic system (52) for Neumann conditions are given by

$$\varphi_{ij}(\xi_1, \xi_2) = \frac{2}{\sqrt{(\beta_1-\alpha_1)(\beta_2-\alpha_2)}} Sin\, i\pi\left(\frac{\xi_1-\alpha_1}{\beta_1-\alpha_1}\right) Sin\, j\pi\left(\frac{\xi_2-\alpha_2}{\beta_2-\alpha_2}\right) \quad (54)$$

$$\lambda_{ij} = -\left(\frac{i^2}{(\beta_1-\alpha_1)^2} + \frac{j^2}{(\beta_2-\alpha_2)^2}\right)\pi^2, \quad i, j \geq 1 \qquad (55)$$

We examine the tow cases illustrated in fig. (2)-(3).

5.1 Internal Rectangular Sensor

For discussing this case, suppose the system (52)-(53) where the sensor supports D_i is the located inΩ. The output function can be written by the form

$$y(t) = \int_{D_i} x_2(\xi_1, \xi_2, t) f_i(\xi_1, \xi_2) d\xi_1 d\xi_2, \qquad (56)$$

where$D \subset \Omega$, is the location of zone sensor as in (Figure 3).

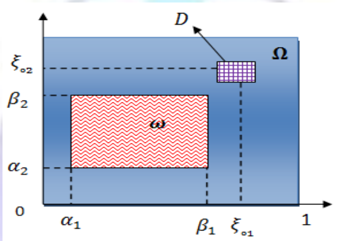

Fig. 3: Rectangular domain, region *ω* and location *D* with rectangular support sensor

Then, the sensor $(D_i, f_i)_{1 \leq i \leq q}$ may be sufficient for $\omega_{E\mathcal{R}}$-observability, and there exists $\mathcal{H}_\omega \in \mathcal{L}(R^q, L^2(\omega))$ such that the operator $(A_{22} - \mathcal{H}_\omega A_{12})$ generates a strongly continuous stable semi-group on the space $L^2(\omega)$. Thus we have

$$\lim_{t\to\infty} \|(w(\xi_1, \xi_2, t) + \mathcal{H}_\omega x_2(\xi_1, \xi_2, t)) - x_1(\xi_1, \xi_2, t)\|_{L^2(\omega)} = 0,$$

where

$$\begin{cases} \frac{\partial w}{\partial t}(\xi_1, \xi_2, t) = \gamma \frac{\partial^2 w}{\partial \xi^2}(\xi_1, \xi_2, t) + \beta((1 + \mathcal{H}_\omega) w(\xi_1, \xi_2, t) \\ \qquad + (\gamma - \alpha\mathcal{H}_\omega) \frac{\partial x_2}{\partial \xi^2}(\xi_1, \xi_2, t) + (\mathcal{H}_\omega^2 - 1)(\xi_1, \xi_2, t)) & \mathcal{Q} \quad (57) \\ w(\xi_1, \xi_2, 0) = w_0(\xi_1, \xi_2) & \Omega \\ w(\xi_1, \xi_2, t) = 0 & \Theta \end{cases}$$

Then, we have the following result

Proposition 5.1: Suppose $\omega = D_i = \pi_{i=1}^2 [\alpha_i - \xi_{i_0}, \beta_i - \xi_{i_0}] \subset \Omega$ as in (Figure 3). Then the sensor $(D_i, f_i)_{1 \leq i \leq q}$ is not $\omega_{E\mathcal{R}}$-observable by the $\omega_{E\mathcal{R}}$-estimator (57), if for any $i_0 \in \{1,2\}, i_0(\xi_{i_0} - \alpha_{i_0})/(\xi_{i_0} - \beta_{i_0}) \in Q$ and f_i is symmetric about the line$x_{i_0} = \xi_{i_0}$.

Proof: suppose $i_0 = 1$, $(\xi_{i_0} - \alpha_{i_0})/(\xi_{i_0} - \beta_{i_0}) \in Q$, then there exists $j_0 \geq 1$ such that$\mathrm{Sin}\,(j_0 \pi c_1/\beta_1\, \alpha_1) = 0$. But

$$y(t) = \langle f_1, \varphi_{i_0 j_0} \rangle = \left(\frac{4}{(\beta_1 - \alpha_1)(\beta_2 - \alpha_2)}\right)^{1/2} \int_{\alpha_2 - \xi_2}^{\alpha_2 + \xi_2} \int_{\alpha_1 - \xi_1}^{\alpha_1 + \xi_1} f_1(\xi_1, \xi_2)$$

$$Sin\left[\frac{j_0 \pi \xi_1}{(\beta_1 - \alpha_1)}\right] Sin\left[\frac{j_0 \pi \xi_2}{(\beta_2 - \alpha_2)}\right] d\xi_1 d\xi_2$$

If f_1 is symmetric about $\xi_1 = x_2$ the integral in the square bracket is zero and hence $y(t) = \langle f_1, \varphi_{i_0 j_0} \rangle = 0$.

5.2 Internal circular sensor

Consider the system (52) augmented with the output function $y(t) = Cx_1(.,t)$ where the sensor supports D_i is located inside the domain Ω. The output $y(t) = Cx_1(.,t)$ can be written by the following form

$$y(t) = \int_{D_i} x_1(r,\theta,t) f_i(r,\theta) d_i d\theta_i, \qquad (58)$$

where $D_i = (r_i, \theta_i) \subset \Omega$, is the location of zone sensor as in (Figure 4).

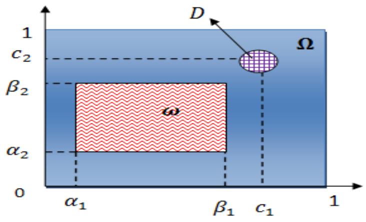

Fig.4: Rectangular domain, region ω and location D with circular support sensor

Then, the sensor $(D_i, f_i)_{1\leq i\leq q}$ may be sufficient for $\omega_{E\mathcal{R}}$-observability, and there exists $\mathcal{H} \in \mathcal{L}(R^q, L^2(\omega))$ such that the operator $(A_{22} - \mathcal{H}_\omega A_{12})$ generates a strongly continuous stable semi-group on the space $L^2(\omega)$. Thus we have

$$\lim_{t\to\infty} \|(w(r,\theta,t) + \mathcal{H}_\omega x_2(r,\theta,t)) - x_1(r,\theta,t)\|_{L^2(\omega)} = 0,$$

where

$$\begin{cases} \frac{\partial w}{\partial t}(r,\theta,t) = \gamma \frac{\partial^2 w}{\partial \xi^2}(r,\theta,t) + \beta((1+\mathcal{H}_\omega) w(r,\theta,t) \\ \qquad + (\gamma - \alpha\mathcal{H}_\omega)\frac{\partial x_2}{\partial \xi^2}(r,\theta,t) + (\mathcal{H}_\omega^2 - 1)(r,\theta,t)) & \mathcal{Q} \quad (59) \\ w(r,\theta,0) = w_0(r,\theta) & \Omega \\ w(r,\theta,t) = 0 & \Theta \end{cases}$$

Then, we have the following result:

Proposition 5.2: Suppose $\omega = D_i = D_i(c,r) \subset \Omega = \pi_{i=1}^2(0,1) c = (c_1, c_2)$. Then the sensor $(D_i, f_i)_{1\leq i\leq q}$ is not $\omega_{E\mathcal{R}}$-observable by the $\omega_{E\mathcal{R}}$-estimator (59), if for any $i_0 \in \{1,2\}, i_0(c_{i_0} - \alpha_{i_0})/(c_{i_0} - \beta_{i_0}) \epsilon Q$ and f_i is symmetric about the line $x_{i_0} = c_{i_0}$.

Proof: suppose $i_0 = 1$, then there exists $j_0 \geq 1$ such that $\cos(j_0\pi c_1/\beta_1\, \alpha_1) = 0$. Consider the output function (53) with the change of variable $x_2 = c_1 + \hat{r}\, cos\,\theta, x_2 = c_2 + \hat{r}\, sin\,\theta$, then

$$y(t) = \langle f_1, \varphi_{i_0 j_0}\rangle = \left(\frac{4}{(\beta_1 - \alpha_1)(\beta_2 - \alpha_2)}\right)^{1/2} \int_0^{2\pi}\int_0^r f_1(c_1 + \hat{r}\mathrm{Cos}\,\theta, c_2 + \hat{r}\mathrm{Sin}\,\theta) \times Sin\left[\frac{j_0\pi(c_1 + \hat{r}\mathrm{Sin}\,\theta)}{(\beta_1 - \alpha_1)}\right] Sin\left[\frac{j_0\pi(c_1 + \hat{r}\mathrm{Sin}\,\theta)}{(\beta_2 - \alpha_2)}\right] \hat{r}\, d\hat{r}\, d\theta$$

Since f_1 is symmetric about $x_2 = c_1$, the function

$$(\hat{r},\theta) \to f_1(c_1 + \hat{r}\,\mathrm{Cos}\,\theta, c_2 + \hat{r}\,\mathrm{Sin}\,\theta) cos\left[\frac{j_0\pi(c_1 + \hat{r}\mathrm{Sin}\,\theta)}{(\beta_2 - \alpha_2)}\right]$$

is symmetric on $[0,\pi]$ about $\theta = \pi/2$ for all $\hat{r}$. But the function

$$(\hat{r},\theta) \to Sin\left[\frac{j_0\pi(c_1 + \hat{r}\mathrm{Sins}\,\theta)}{(\beta_2 - \alpha_2)}\right]$$

is antisymmetric on $[0,\pi]$ about $\pi/2$. By decomposing the integral as a sum on $[0,\pi]$ and $[\pi, 2\pi]$ it is easy to see that

$y(t) = \langle f_1, \varphi_{i_0 j_0}\rangle = 0.$

Remark 5.3: These results can be extended to the following:

(1) Case of Neumann or mixed boundary conditions.

(2) Case of boundary (pointwise, zone) sensors.

6. CONCLUSION

The concept developed in this paper is related to the regional exponential reduced observability in connection with the strategic sensors. Various interesting results concerning the choice of circular sensors are given and illustrated in specific situations. Many questions still opened. This is the case of, for example, the problem of finding the optimal sensor location ensuring such an objective. The result of regional exponential reduced observability concept of hyperbolic linear or semi linear or nonlinear systems is under consideration.

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