COMMON FIXED POINT THEOREMS FOR RATIONAL TYPE CONTRACTION IN PARTIALLY ORDERED METRIC SPACE

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ABSTRACT

In this paper we prove some common fixed point theorems for two and four self-mappings using rational type contraction and some newly notified definitions in partially ordered metric space. In this way we generalized, modify, and extend some recent results due to Chandok and Dinu [14], Shantanwi and Postolache[29] and many others [1, 2, 4, 5, 21, 29, 30], thus generalizing results of Cabrea, Harjani and Sadarangani [12] as well as Dass and Gupta [15] in the context of partial order metric setting.

Keywords

Partially ordered metric space; $\overline{O}$-complete; $\overline{O}$-compatible; common fixed point; $\overline{O}$-weak annihilator; $\overline{O}$-partially weakly increasing.

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1. Introduction

Turinici [31, 32] investigated fixed point theorem for monotone mappings on metrizable uniform spaces and quasi ordered metric spaces. Afterwards, Ran and Reuring [26] extended the Banach contraction principle to partially ordered metric space with some application to matrix equations. Further, Nieto and Rodriguez-Lopez [22, 23] generalized the theorem for increasing mapping not requiring the necessity of continuity and gave application to existence and uniqueness of a lower solution to first order ordinary differential equation. Thereafter, Nieto and Rodriguez-Lopez's theorems were generalized by many researchers [3, 6, 13, 17, 24, 33, 34].

Sessa [27] introduced the concept of weak commutative condition of mappings which are generalization of commutative maps [20] in metric space. Jungck [18] generalized this idea of weak commutative mappings by introducing compatible mappings. Further, Jungck [19] introduced the concept of weakly compatible mappings and proved some common fixed point results for these mappings.

In recent years many researchers have generalized the existence of fixed point and common fixed point theorems for generalized weak contractions in partial ordered metric space [7-11, 21, 25]. Most recently Shatanawi and Postolache [28] proved common fixed point theorems for dominating and weak annihilator mappings in ordered metric space for four self mappings, meanwhile Cabrera, Harjani and Sadarangani generalized the Dass and Gupta's theorem in the partial ordered metric space. Very recently Alam *et al.* [5] identified some more natural definitions in view of Turinici [33, 34] on ordered metric setting.

Our aim in this paper is to modify some recent common fixed point theorems for self-mappings using some newly identified ordered metric definitions, and rational type contraction.

2. Preliminaries

Definition 1. (a) A non empty set X together with partial order relation $\leq$ (reflexive, anti-symmetric and transitive) is said to be an ordered set or partially ordered set. We say x is comparable to y if either $x \leq y$ or $x \geq y$ and denoted as $<>$ symbolically.

(b) A set X is said to be totally ordered or linearly ordered if every pair of elements of X are comparable.

(c) Triplet $(X, d, \leq)$ is said to be an ordered metric space or partially ordered metric space if d is a metric endowed with partial ordering in a nonempty set.

Definition 2. (a) Let X be a nonempty set and A, B be self-maps on X. Then $z \in X$ is said to be coincidence point of A and B if $Az = Bz$. Also if z is a coincidence point of X such that $w = Az = Bz$, then w is a point of coincidence of A and B.

(b) If $z \in X$ be a coincidence point of A and B such that $z = Az = Bz$, then $z \in X$ is said to be common fixed point of A and B.

Definition 3. Let A and B be pair of self mappings defined on an ordered set $(X, \leq)$. We say that A is B-nondecreasing (resp. B-nonincreasing) if for any $x, y \in X, Bx \leq By \Rightarrow Ax \leq Ay$ (resp. $Ax \geq Ay$). In all, A is called B-monotone if A is either B-nondecreasing or B-nonincreasing.

Notice that under the restriction $B = I$, the identity mapping on X, the notions of B-nondecreasing, B-nonincreasing and B-monotone mappings reduce to nondecreasing, nonincreasing and monotone mappings respectively.

Definition 4. Let $(X, \leq)$ be an ordered set and $A, B: X \to X$ then the followings hold :

(a) The pair (A, B) is called commuting if $A(Bz) = B(Az)$,

(b) weakly commuting if $d(ABx, BAx) \leq d(Ax, Bx)$,

(c) compatible if $\lim_{n \to \infty}(ABx_n \ BAx_n) = 0$ whenever $\{x_n\} \subset X$, such that $\lim_{n \to \infty} Ax_n = \lim_{n \to \infty} Bx_n = z$ for some $z \in X$.

(d)   weakly compatible if they commute at their coincidence point i.e. if $Ax = Bx$, then $ABx = BAx$.

Example 1. Let $Ax = 2x^2$ and $Bx = 3x^2$ with usual distance in $X = R$, then the pair (A, B) is compatible but not commuting and even weakly commuting.

(e) Mapping A is said to be weak annihilator of B if $ABx \leq x$ for all $x \in X$ and dominating if $z \leq Az$ for all $z \in X$.

Example 2. Let $X = [0\ 1]$ with partial ordering and usual distance with $Ax = x$ and $Bx = x^2$ then $ABx = x^2 \leq x$ showing that A is weak annihilator of B.

(f) The pair $(A,\ B)$ is said to be weakly increasing if $Ax \leq BAx$ and $Bx \leq ABx$ for all $x \in X$.

(g) The pair $(A,\ B)$ is said to be partially weakly increasing if $Ax \leq BAx$ for all $x \in X$.

Example 3. Let $X = [0\ 1]$ with partial ordering and usual distance with $Ax = x^{\frac{1}{2}}$ and $Bx = x^{\frac{1}{3}}$ then $Bx \leq ABx = x^{\frac{1}{6}}$ and $Ax \leq BAx$ showing that A and B are weakly increasing mappings while if we choose $Ax = x^3$ then mappings become partially weakly increasing accordingly as $Ax \leq BAx$ but $Bx \nleq ABx$.

Thus we conclude that pair $(A,\ B)$ is weakly increasing if and only if pairs $(A,\ B)$ and $(B,\ A)$ are partially weakly increasing.

Some newly notified ordered metric definitions:

Definition 5. (see [5]) Partially ordered metric space $(X, d, \leq)$ is said to be,

(a) $\overline{O}$-complete if every nondecreasing Cauchy sequence $\{x_n\} \subset X$ converges,

(b) $\underline{O}$-complete if every nonincreasing Cauchy sequence $\{x_n\} \subset X$ converges, and

(c) O-complete if every monotone Cauchy sequence $\{x_n\} \subset X$ converges.

Remark 1. In this setting, completeness $\Rightarrow O$-completeness $\Rightarrow \overline{O}$-completeness, as well as $\underline{O}$-completeness.

Definitions 6 [5]. A self mapping S in triplet $(X, d, \leq)$ is called,

(a) $\overline{O}$-continuous for each $z \in X$ and for every nondecreasing sequence $\{x_n\} \subset X$ conversing to z, Sx_n converges to Sz,

(b) $\underline{O}$-continuous for each $z \in X$ and for every nonincreasing sequence $\{x_n\} \subset X$ conversing to z, Sx_n converges to Sz, and

(c) O-continuous for each $z \in X$ and for every monotonic sequence $\{x_n\} \subset X$ conversing to z, Sx_n converges to Sz.

Remark 2. In the above said notions, continuity $\Rightarrow O$-continuity $\Rightarrow \overline{O}$-continuity as well as $\underline{O}$-continuity.

Definitions 7 [5]. In an ordered triplet $(X, d, \leq)$ the pair $(A,\ B)$ is defined as,

(a) $\overline{O}$-compatible if for every nondecreasing sequence $\{x_n\} \subset X$ there exists two nondecreasing sequences Ax_n and Bx_n converging to $z \in X$ implies that $\lim_{n \to \infty} d(ABx_n,\ BAx_n) = 0$,

(b) $\underline{O}$-compatible if for every nonincreasing sequence $\{x_n\} \subset X$ there exists two nonincreasing sequences Ax_n and Bx_n converging to $z \in X$ implies that $\lim_{n \to \infty} d(ABx_n,\ BAx_n) = 0$,

(c) O-compatible if for every monotonic sequence $\{x_n\} \subset X$ there exists two monotonic sequences Ax_n and Bx_n converging to $z \in X$ implies that $\lim_{n \to \infty} d(ABx_n, BAx_n) = 0$.

Remark 3. Thus for the pairs of maps in ordered settings, commutability $\Rightarrow$ weak commutability $\Rightarrow$ compatibility $\Rightarrow O$-compatibility $\Rightarrow \overline{O}$-compatibility as well as $\underline{O}$-compatibility $\Rightarrow$ weak compatibility.

Definitions 8. To emphasize new order theoretic notions we define the following weaker conditions of mappings.

(a) The mapping A is said to be $\overline{O}$-dominating if for every nondecreasing sequence $\{x_n\} \subset X$ converges to z implies that $z \preceq Az$. Analogously, we define $\underline{O}$-dominating and O-dominating for nonincreasing and monotone sequences respectively.

(b) Mapping A is said to be $\overline{O}$-weak annihilator of B if for every nondecreasing sequence $\{x_n\} \subset X$, converges to z, sequences Ax_n and Bx_n are also nondecreasing and converge to z, implies that $ABz \preceq z$. Analogously, we can define $\underline{O}$-weak annihilator and O-weak annihilator for nonincreasing and monotone sequences respectively.

(c) The pair (A, B) is said to be $\overline{O}$-partially weakly increasing if for every nondecreasing sequence $\{x_n\} \subset X$ converges to z, sequences Ax_n and Bx_n are also nondecreasing and converge to z, implies that $Az \preceq BAz$. Analogously, we define $\underline{O}$-partially weakly increasing and O-partially weakly increasing for nonincreasing and monotone sequences respectively.

3. Main results

Theorem 1. Let $(X, d, \preceq)$ be partially ordered metric space. Let A, B, S and T be self maps on X. Suppose that the following conditions hold:

(a) $A(X) \subseteq T(X)$ and $B(X) \subseteq S(X)$,

(b) (T, A) and (S, B) are $\overline{O}$- partially weakly increasing mappings,

(c) $\overline{O}$-dominating maps A and B are $\overline{O}$-weak annihilators of T and S respectively,

(d) for any comparable elements $x, y \in X$, $d(Ax, By) \le \alpha \frac{d(By, Ty)[1+d(Ax, Sx)]}{1+d(Sx, Ty)} + \beta d(Sx, Ty)$ holds, where $\alpha, \beta \ge 0$ and $\alpha + \beta < 1$, (1)

(e) $(X d)$ is $\overline{O}$-complete,

(f) (A, S) are $\overline{O}$-compatible, A or S is $\overline{O}$-continuous and (B, T) are weakly compatible or

(g) (B, T) are $\overline{O}$-compatible, B or T is $\overline{O}$-continuous and (A, S) are weakly compatible and

If for a nondecreasing sequence $\{x_n\} \subset X$ converges to z with $x_n \preceq y_n$ for all n and $y_n \to z$ implies that $x_n \preceq z$. Then A, B, S and T have unique common fixed point in X.

Proof: Let x_0 be an arbitrary point of X. Construct sequences $\{x_n\}$ and $\{y_n\}$ in X such that

$$y_{2n} = Ax_{2n-1} = Tx_{2n}, \quad y_{2n+1} = Bx_{2n} = Sx_{2n+1}. \tag{2}$$

By assumption (b), we have

$$, \qquad x_{2n-1} \preceq Ax_{2n-1} = Tx_{2n} \preceq ATx_{2n} \preceq x_{2n},$$

$$x_{2n} \preceq Bx_{2n} = Sx_{2n+1} \preceq SBx_{2n+1} \preceq x_{2n+1}. \tag{3}$$

Thus for all $n \ge 1$ we have $x_n \preceq x_{n+1}$. Hence by using (d), and putting $x = x_{2n-1}, y = x_{2n}$ we get

$$d(Ax_{2n-1}, Bx_{2n}) \le \alpha \frac{d(Bx_{2n}, Tx_{2n})[1+d(Ax_{2n-1}, Sx_{2n-1})]}{1+d(Sx_{2n-1}, Tx_{2n})} + \beta d(Sx_{2n-1}, Tx_{2n}), \tag{4}$$

$$d(y_{2n}, y_{2n+1}) \le \alpha \frac{d(y_{2n+1}, y_{2n})[1+d(y_{2n}, y_{2n-1})]}{1+d(y_{2n-1}, y_{2n})} + \beta d(y_{2n-1}, y_{2n}),$$

$$= \alpha d(y_{2n+1}, y_{2n}) + \beta d(y_{2n-1}, y_{2n}),$$

i.e. $(1-\alpha)d(y_{2n}, y_{2n+1}) \le \beta d(y_{2n-1}, y_{2n})$,

$$d(y_{2n}, y_{2n+1}) \leq \frac{\beta}{(1-\alpha)} d(y_{2n-1}, y_{2n}). \tag{5}$$

Similarly it can be proved that $d(y_{2n+1}, y_{2n+2}) \leq \frac{\beta}{(1-\alpha)} d(y_{2n}, y_{2n+1}). \tag{6}$

Therefore $(y_n, y_{n+1}) \leq \frac{\beta}{(1-\alpha)} d(y_{n-1}, y_n) \leq \left\{\frac{\beta}{1-\alpha}\right\}^2 d(y_{n-2}, y_{n-1}) \leq \cdots \cdots \cdots \leq \left\{\frac{\beta}{1-\alpha}\right\}^n d(y_0, y_1).$

Put $k = \frac{\beta}{(1-\alpha)}$, we have $\tag{7}$

$$d(y_n, y_{n+1}) \leq k d(y_{n-1}, y_n) \leq k^2 d(y_{n-2}, y_{n-1}) \leq \cdots \cdots \cdots \leq k^n d(y_0, y_1), \text{ for all } n \geq 1. \tag{8}$$

Now for any positive integer m and n with $m \geq n$ we have

$$d(y_m, y_n) \leq d(y_n, y_{n+1}) + d(y_{n+1}, y_{n+2}) + \cdots \cdots \cdots + d(y_{m-1}, y_m) \leq \frac{k^n}{1-k} d(y_0, y_1). \tag{9}$$

Hence we conclude that $\{y_n\}$ is a Cauchy sequence.

Now suppose $\{y_{2n-1}\}$ is a subsequence of $\{y_n\}$ then by assumption (e), there exists $z \in X$ such that $\lim_{n \to \infty} y_{2n-1} = z$.

Therefore, $\lim_{n \to \infty} y_{2n} = \lim_{n \to \infty} T x_{2n} = \lim_{n \to \infty} A x_{2n-1} = z$

$\lim_{n \to \infty} y_{2n+1} = \lim_{n \to \infty} S x_{2n+1} = \lim_{n \to \infty} B x_{2n} = z. \tag{10}$

Assume that S is $\overline{O}$-continuous and in view of assumption (f), we have

$\lim_{n \to \infty} A S x_{2n+1} = \lim_{n \to \infty} S A x_{2n+1} = Sz$ Also, $x_{2n} \leq B x_{2n} = S x_{2n+1}.$

Now $\quad d(A S x_{2n+1}, B x_{2n}) \leq \alpha \frac{d(B x_{2n}, T x_{2n})[1+d(A S x_{2n+1}, T S x_{2n+1})]}{1+d(S S x_{2n-1}, T x_{2n})} + \beta d(S S x_{2n-1}, T x_{2n}). \tag{11}$

Letting $n \to \infty$ in the above inequality, and using (10), we have

$$d(Sz, z) \leq \alpha \frac{d(z, z)[1+d(Az, Tz)]}{1+d(Sz, z)} + \beta d(Sz, z),$$

$$= \beta d(Sz, z), \text{ A contradiction.} \tag{12}$$

Hence $Sz = z$. $\tag{13}$

Now $x_{2n} \leq B x_{2n}$ and $\lim_{n \to \infty} B x_{2n} \to z$, so by assumption we have $x_{2n} \leq z$ and (d) becomes

$$d(Az, B x_{2n}) \leq \alpha \frac{d(B x_{2n}, T x_{2n})[1+d(Az, Sz)]}{1+d(Sz, T x_{2n})} + \beta d(Sz, T x_{2n}). \tag{14}$$

Again letting $n \to \infty$ in the above inequality and using (10), we have

$$d(Az, z) \leq \alpha \frac{d(z, z)[1+d(Az, Sz)]}{1+d(Sz, z)} + \beta d(Sz, z),$$

$$= \beta d(Sz, z)$$

This implies that $Az = z.$ $\tag{15}$

Since $A(X) \subseteq T(X)$ there exists a point $w \in X$ such that $Az = Tw$. Suppose that $Bw \neq Tw$. Since $z \leq Az = Tw \leq ATw \leq w$ implies $z \leq w$. From (d), we obtain

$$d(Tw, Bw) = d(Az, Bw) \leq \alpha \frac{d(Bw, Tw)[1+d(Az, Sz)]}{1+d(Sz, Az)} + \beta d(Sz, Tw), \tag{16}$$

$$= \alpha \frac{d(Bw, Tw)[1+d(Az, Sz)]}{1+d(Sz, Az)} + \beta d(Sz, Tw),$$

$$= \alpha d(Bw, Az) + \beta d(Sz, Az),$$

$$= \alpha d(Bw, \ Az), \text{ A contraction.} \tag{17}$$

Therefore we have $Tw = Bw$. Since B and T are weakly compatible, $Bz = BAz = BTw = TBw = TAz = Tz$ Thus z is a coincidence point of B and T.

Again since $x_{2n-1} \leq Ax_{2n-1}$ and $\lim_{n\to\infty} A x_{2n-1} \to z$, so by assumption we have $x_{2n-1} \leq z$, and (d) becomes

$$d(Ax_{2n-1}, \ Bz) \leq \alpha \frac{d(Bz, \ Tz)[1+d(Ax_{2n-1}, \ Sx_{2n-1})]}{1+d(Sx_{2n-1}, \ Tz)} + \beta d(Sx_{2n-1}, \ Tz). \tag{18}$$

Letting $n \to \infty$, and if $\lim_{n\to\infty} d(Sx_{2n-1}, \ Tz) = 0$ then $d(z, Tz) = 0$ and hence $Tz = z$.

If $d(Sx_{2n-1}, \ Tz) \neq 0$ in the above inequality then using (10), we have

$$d(z, \ Bz) \leq \alpha \frac{d(Bz, \ Tz)[1+d(z, \ z)]}{1+d(z, \ Tz)} + \beta d(z, \ Tz), \tag{19}$$

$$= \beta d(z, \ Tz) \text{ or } d(z, \ Tz) \leq \beta d(z, \ Tz)$$

This is a contradiction. Hence $Tz = z$. $\tag{20}$

Therefore $Az = Bz = Sz = Tz = z$, that is z is a common fixed point of A, B, S and T.

For uniqueness, let us assume $Az = Bz = Sz = Tz = z$ and $Au = Bu = Su = Tu = u$ but $z \neq u$. Consider (1),

$$d(z, \ u) = d(Az, \ Bu) \leq \alpha \frac{d(Bu, \ Tu)[1+d(Az, \ Sz)]}{1+d(Sz, \ Tu)} + \beta d(Sz, \ Tu), \tag{21}$$

$$= \beta d(z, \ u), \text{ A contradiction.} \tag{22}$$

Which means that $z = u$. Thus z is a unique common fixed point of A, B, S and T.

Corollary 1. Let $(X, \ d, \ \leq)$ be partially ordered metric space. Let $A, \ S$ and T be self-maps on X. Suppose that the following conditions hold:

(a) $A(X) \subseteq T(X)$ and $A(X) \subseteq S(X)$,

(b) (T, A) and (S, A) are $\overline{O}$-partially weakly increasing mappings,

(c) $\overline{O}$-dominating map A is $\overline{O}$-weak annihilators of T and S respectively,

(d) for any comparable elements $x, \ y \in X$, $d(Ax, Ay) \leq \alpha \frac{d(Ay, \ Ty)[1+d(Ax, \ Sx)]}{1+d(Sx, \ Ty)} + \beta d(Sx, \ Ty)$ holds, where $\alpha, \beta \geq 0$ and $\alpha + \beta < 1$,

(e) $(X \ d)$ is $\overline{O}$-complete,

(f) $(A, \ S)$ are $\overline{O}$-compatible, A or S is $\overline{O}$-continuous and $(A, \ T)$ are weakly compatible or

(g) $(A, \ T)$ are $\overline{O}$-compatible, A or T is $\overline{O}$-continuous and $(A, \ S)$ are weakly compatible.

If for a nondecreasing sequence $\{x_n\} \subset X$ converges to z with $x_n \leq y_n$ for all n and $y_n \to z$ implies that $x_n \leq z$. Then $A, \ S$ and T have unique common fixed point in X.

Corollary 2. Let $(X, \ d, \ \leq)$ be partially ordered metric space. Let $A, \ B$ and T be self-maps on X. Suppose that the following conditions hold:

(a) $A(X) \subseteq T(X)$ and $B(X) \subseteq T(X)$,

(b) $(T, \ A)$ and $(T, \ B)$ are $\overline{O}$-partially weakly increasing mappings,

(c) $\overline{O}$-dominating maps A and B are $\overline{O}$-weak annihilators of T,

(d) for any comparable elements $x, \ y \in X$, $d(Ax, \ By) \leq \alpha \frac{d(By, \ Ty)[1+d(Ax, \ Tx)]}{1+d(Tx, \ Ty)} + \beta d(Tx, \ Ty)$ holds, where $\alpha, \beta \geq 0$ and $\alpha + \beta < 1$,

(e) $(X\,d)$ is $\overline{O}$-complete,

(f) $(A,\,T)$ are $\overline{O}$-compatible, A or T is $\overline{O}$-continuous and $(A,\,T)$ are weakly compatible or

(g) $(B,\,T)$ are $\overline{O}$-compatible, B or T is $\overline{O}$-continuous and $(B,\,T)$ are weakly compatible.

If for a nondecreasing sequence $\{x_n\} \subset X$ converges to z with $x_n \leq y_n$ for all n and $y_n \to z$ implies that $x_n \leq z$. Then A, B and T have unique common fixed point in X.

Corollary 3. Let $(X,\,d,\,\leq)$ be partially ordered metric space. Let A and T be self-maps on X. Suppose that the following conditions hold:

(a) $A(X) \subseteq T(X)$

(b) (T, A) is $\overline{O}$-partially weakly increasing mappings,

(c) $\overline{O}$-dominating map A is $\overline{O}$-weak annihilators of T,

(d) for any comparable elements $x,\,y \in X$, $d(Ax,\,Ay) \leq \alpha\,\dfrac{d(Ay,\,Ty)[1+d(Ax,\,Tx)]}{1+d(Tx,\,Ty)} + \beta d(Tx, Ty)$ holds, where $\alpha, \beta \geq 0$ and $\alpha + \beta < 1$,

(e) $(X\,d)$ is $\overline{O}$-complete,

(f) $(A,\,T)$ are $\overline{O}$-compatible, and A or T is $\overline{O}$-continuous and $(A,\,T)$ are weakly compatible.

If for a nondecreasing sequence $\{x_n\} \subset X$ converges to z with $x_n \leq y_n$ for all n and $y_n \to z$ implies that $x_n \leq z$. Then A and T have unique common fixed point in X.

Remark 4. Theorem 1 and corollary 1, 2, and 3 remains true if we replace $\overline{O}$-complete, $\overline{O}$-compatible pair, $\overline{O}$-continuous, $\overline{O}$-dominating maps, $\overline{O}$-partially weakly increasing mappings, $\overline{O}$-weak annihilator by O-complete, O-compatible pair, O-continuous, O-dominating maps, O-partially weakly increasing mappings, O-weak annihilating maps respectively.

Remark 5. Theorem 1 and corollary 1, 2, and 3 remains true if we replace $\overline{O}$-complete, $\overline{O}$-compatible pair, $\overline{O}$-continuous, $\overline{O}$-dominating maps, $\overline{O}$-partially weakly increasing mappings, $\overline{O}$-weak annihilator by complete, compatible pair, continuous, dominating maps, partially weakly increasing mappings, weak annihilating maps respectively.

Now we prove common fixed point theorem by relaxing the definitions of $\overline{O}$-dominating, $\overline{O}$-partially weakly increasing, and $\overline{O}$-weak annihilating mappings and enrich some recent common fixed point theorems by ordered metrical definitions.

Theorem 2. Let $(X,\,d,\,\leq)$ be partially ordered metric space. Let $T,\,S$ be self-maps on X. Suppose that the following conditions hold:

(a) $T(X) \subseteq S(X)$,

(b) T is S-nondecreasing,

(c) there exists $x_0 \in X$ such that $Sx_0 \leq Tx_0$,

(d) for any $x,\,y \in X$ and comparable $Sx,\,Sy$ such that $d(Tx,\,Ty) \leq \alpha\,\dfrac{d(Sy,\,Ty)[1+d(Sx,\,Tx)]}{1+d(Sx,\,Sy)} + \beta d(Sx,\,Sy)$ holds, where $\alpha, \beta \geq 0$ and $\alpha + \beta < 1$, (23)

(e) $(X\,d)$ is $\overline{O}$-complete,

(f) $(T,\,S)$ is $\overline{O}$-compatible pair,

(g) S is $\overline{O}$-continuous,

(h) (T, S) is weakly compatible pair.

Then S and T have a unique common fixed point in X.

Proof: Let x_0 be an arbitrary point in X such that $Sx_0 \leq Tx_0$. Since $T(X) \subseteq S(X)$ we can choose $x_1 \in X$ such that $Sx_1 = Tx_0$ again since $Tx_1 \in S(X)$ there exists $x_2 \in X$ such that $Sx_2 = Tx_1$. By induction we can construct a sequence $\{x_n\}$ in X such that $Sx_{n+1} = Tx_n$ for every $n \in N \cup \{0\}$. Since $Sx_0 \leq Tx_0 = Sx_1$, and T is S-nondecreasing mapping we have $Tx_0 \leq Tx_1$. Similarly, since $Sx_1 \leq Sx_2$ we have $Tx_1 \leq Tx_2$. Continuing this process, we obtain $Tx_0 \leq Tx_1 \leq Tx_2 \leq Tx_3 \leq \cdots \ldots \ldots \leq Tx_n \leq Tx_{n+1}$. Suppose that $d(Tx_n Tx_{n+1}) > 0$ for all $n \in N \cup \{0\}$. If not then $Tx_{n+1} = Tx_n$ for some, i.e. $Sx_{n+1} = Tx_{n+1}$ implies that S and T have a coincidence point x_{n+1}. Now using assumption (h), let $z = Sx_{n+1} = Tx_{n+1}$ then $Tz = T(Sx_{n+1}) = S(Tx_{n+1}) = Sz$.

Consider,
$$d(Tx_{n+1}, Tz) \leq \alpha \frac{d(Sz, Tz)[1+d(Sx_{n+1}, Tx_{n+1})]}{1+d(Sx_{n+1}, Sz)} + \beta d(Sx_{n+1}, Sz)$$

$$= \alpha.0 + \beta d(Sx_{n+1}, Sz),$$

$$= \beta d(Tx_{n+1}, Tz), \text{A contradiction.}$$

Hence $d(Tx_{n+1}, Tz) = 0$ implies that $Tz = Sz = z$. (24)

i.e. z is a common fixed point of S and T.

Therefore assuming $d(Tx_n, Tx_{n+1}) > 0$, in view of (d) we have

$$d(Tx_n, Tx_{n+1}) \leq \alpha \frac{d(Sx_{n+1}, Tx_{n+1})[1+d(Sx_n, Tx_n)]}{1+d(Sx_n, Sx_{n+1})} + \beta d(Sx_n, Sx_{n+1}),$$

$$= \alpha d(Tx_n, Tx_{n+1}) + \beta d(Tx_{n-1}, Tx_n),$$

$$d(Tx_n, Tx_{n+1}) \leq \frac{\beta}{1-\alpha} d(Tx_{n-1}, Tx_n). \tag{25}$$

Using mathematical induction we have

$$d(Tx_n, Tx_{n+1}) \leq \left\{\frac{\beta}{1-\alpha}\right\}^n d(Tx_0, Tx_1). \tag{26}$$

Now we shall prove that $\{Tx_n\}$ is a Cauchy sequence. For $m \geq n$, we have

$$d(Tx_m, Tx_n) \leq d(Tx_m, Tx_{m-1}) + d(Tx_{m-1}, Tx_{m-2}) + \cdots \ldots \ldots d(Tx_{n-1}, Tx_n),$$

$$\leq (k^{m-1} + k^{m-2} + \cdots \ldots \ldots k^n)d(Tx_1, Tx_0),$$

$$\leq \frac{k^n}{1-k}d(Tx_1, Tx_0). \text{ where } k = \left(\frac{\beta}{1-\alpha}\right) < 1. \tag{27}$$

Thus $\{Tx_n\}$ is a Cauchy sequence. Also (X, d) is $\overline{O}$-complete, therefore there exists $z \in X$ such that $\lim_{n\to\infty} Tx_n = z$. By $\overline{O}$-continuity of S we have $\lim_{n\to\infty} S(Tx_n) = Sz$. Since $Sx_{n+1} = Tx_n \to z$ and the pair is $\overline{O}$-compatible, we have

$$\lim_{n\to\infty} d(S(Tx_n), T(Sx_n)) = 0. \tag{28}$$

Using triangular inequality and letting $n \to \infty$,

$$d(Tz, Sz) \leq d(Tz, T(Sx_n) + d(T(Sx_n), S(Tx_n)) + d(S(Tx_n), Sz),$$

$$= d(Tz, Tz) + 0 + d(Sz, Sz),$$

$$= 0.$$

Hence $Tz = Sz$ i.e. z is a coincidence point of T and S. Again in view of assumption (h), we arrive at equation (24).

For uniqueness, let us assume that $Tz = Sz = z$, and $Tu = Su = u$ but $z \neq u$. Consider (23),

$$d(z,\ u) = d(Tz,\ Tu) \leq \alpha \frac{d(Su,\ Tu)[1+d(Sz,\ Tz)]}{1+d(Sz,\ Su)} + \beta d(Sz\ Su),$$

$$= \alpha.0 + \beta d(Sz,\ Su),$$

$$= \beta d(z,\ u), \text{A contraction.} \qquad (29)$$

Thus z is a unique common fixed point.

Theorem 3. Theorem 2 remains true if we replace $\overline{O}$-complete, $\overline{O}$-compatible pair, $\overline{O}$-continuous, by $\underline{O}$-complete, $\underline{O}$-compatible pair, $\underline{O}$-continuous, and assumption (c) is replaced by the following (besides retaining rest of the assumptions):

(c') there exists $x_0 \in X$ such that $Sx_0 \geq Tx_0$.

Theorem 4. Theorem 2 remains true if we replace $\overline{O}$-complete, $\overline{O}$-compatible pair, $\overline{O}$-continuous, by O-complete, O-compatible pair, O-continuous, and assumption (c) is replaced by the following (besides retaining rest of the assumptions):

(c'') there exists $x_0 \in X$ such that $Sx_0 <> Tx_0$.

Remark 6. Theorem 2 remains true if we replace $\overline{O}$-complete, $\overline{O}$-compatible pair, $\overline{O}$-continuous, by complete, O-compatible pair, continuous, and assumption (c) replaced by either (c') or (c'') (besides retaining rest of the assumptions).

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