ISSN 2347-1921

EXISTENCE OF NONOSCILLATORY SOLUTIONS OF FIRST ORDER NONLINEAR NEUTRAL DIFFERENCE EQUATIONS

R.Arul[1], K.Alagesan[2] and G.Ayyappan[3]

[1&2]Department of Mathematics, Kandaswami Kandar's College, Velur – 638 182, Namakkal Dt. Tamil Nadu, India.

[3]Department of Mathematics, Periyar University College of Arts and Science College, Pappireddipatti-636 905, Tamil Nadu, India.

Abstract

In this paper, we discuss the existence of nonoscillatory solutions of first order nonlinear neutral difference equations of the from

$$\Delta\left(\left(x(n)-p(n)x(n-\tau)\right)^{\alpha}\right)+Q(n)G\left(x(n-\sigma)\right)=0,.$$

$$\Delta\left(\left(x(n)-p(n)x(n-\tau)\right)^{\alpha}\right)+\sum_{s=c}^{d}Q(n,s)G\left(x(n-s)\right)=0,$$

and

$$\Delta\left(\left(x(n)-\sum_{s=a}^{b}p(n,s)x(n-s)\right)^{\alpha}\right)+\sum_{s=c}^{d}Q(n,s)G\left(x(n-s)\right)=0.$$

We use the Knaster-Tarski fixed point theorem to obtain some sufficient conditions for the existence of nonoscillatory solutions of above equations. Examples are provided to illustrate the main results.

2010 AMS Subject Classification: 39A10

Keywords and Phrases: First order; Nonoscillation; neutral difference equations.

Council for Innovative Research

Peer Review Research Publishing System

Journal: JOURNAL OF ADVANCES IN MATHEMATICS

Vol .11, No. 5

www.cirjam.com , editorjam@gmail.com

ISSN 2347-1921

1. INTRODUCTION

In this paper, we discuss the existence of nonoscillatory solutions of first order nonlinear neutral difference equations of the from

$$\Delta\left(\left(x(n)-p(n)x(n-\tau)\right)^{\alpha}\right)+Q(n)G\left(x(n-\sigma)\right)=0,\ n\in\mathrm{N}_0. \tag{1.1}$$

$$\Delta\left(\left(x(n)-p(n)x(n-\tau)\right)^{\alpha}\right)+\sum_{s=c}^{d}Q(n,s)G\left(x(n-s)\right)=0,\ n\in\mathrm{N}_0 \tag{1.2}$$

and

$$\Delta\left(\left(x(n)-\sum_{s=a}^{b}p(n,s)x(n-s)\right)^{\alpha}\right)+\sum_{s=c}^{d}Q(n,s)G\left(x(n-s)\right)=0,\ n\in\mathrm{N}_0 \tag{1.3}$$

where Δ is the forward difference operator defined by $\Delta x(n)=x(n+1)-x(n)$ and $\mathrm{N}(n_0)=\{n_0,n_0+1,n_0+2,...\}$ and n_0 is a nonnegative integer subject to the following conditions:

(C_1) α is a ratio of odd positive integers;

(C_2) $\sigma,\tau,a,b,c,$ and d are nonnegative integer with $a<b$ and $c<d$;

(C_3) $\{p(n)\},\{Q(n)\}$ and $\{Q(n,s)\}$ are nonnegative real sequences;

(C_4) $G(x)$ is a positive continuous real valued function with $xG(x)>0$ for $x\neq 0$.

Let $\theta=\max\{\tau,\sigma\}$. By a solution of equations (1.1)-(1.3), we mean a real sequence $\{x(n)\}$ defined and satisfying equations (1.1)-(1.3) for all $n\geq n_0-\theta$. Such a solution is said to be oscillatory if it is neither eventually positive nor eventually negative and nonoscillatory otherwise.

In recent years, there has been much research concerning the oscillation of first order neutral delay difference equations, see for example [1-4, 9, 11-13, 16] and the references cited therein. In [2, 5 ,7, 8, 10, 14, 15], the authors investigated the existence of nonoscilatory solutions of first order difference equations. Following this trend, we obtain some new sufficient conditions for the existence of nonoscillatory solutions of equations (1.1)-(1.3).

In Section 2, we establish some sufficient conditions for the existence of nonoscillatory solutions of equations (1.1)-(1.3). In Section 3, we present some examples to illustrate the main results. The results established in this paper are discrete analogue of that in [6].

2. Nonoscillation Theorems

In this section, we present some sufficient conditions for the existence of bounded nonoscillatory solutions of equations (1.1)-(1.3). We begin with the following lemma.

Lemma 2.1. (Knaster-Tarski Fixed Point Theorem)

Let B be a partially ordered Banach space with ordering $\leq$. Let M be a subset of B with the following properties: the infimum of M belongs to M and every nonempty subset of M has a supremum which belongs to M. Let $T:M\to M$ be an increasing mapping, that is, $x\leq y$ implies $Tx\leq Ty$ .Then T has a fixed point in M.

The proof of Lemma 2.1 can be found in [3].

Theorem 2.1. Assume that $0\leq p(n)\leq p<1, G$ is nondecreasing and

$$\sum_{n=n_0}^{\infty}Q(n)<\infty, \tag{2.1}$$

then equation (1.1) has a bounded nonoscillatory solution.

Proof: Let B be the set of all bounded real valued sequence with the supremum norm,

$$\|x\| = \sup_{x_n \in B} |x_n| < \infty.$$

Then clearly B is a Banach space. We can define a partial ordering as follows: for given $x_1, x_2 \in B$, $x_1 \le x_2$ means that $x_1(n) \le x_2(n)$ for $n \ge n_0 \in \mathrm{N}_0$. Define

$$S = \{x \in B : C_1 \le x(n) \le C_2,\ n \ge n_0\},$$

where C_1 and C_2 are positive constants such that

$$C_1 \le \beta < (1-p)C_2.$$

If $\tilde{x}_1(n) = C_1,\ n \ge n_0$, then $\tilde{x}_1 \in S$ and $\tilde{x}_1 = \inf S$. In addition, if $\phi \subset S^* \subset S$, then

$$S^* = \{x \in B :\ \lambda \le x(n) \le \mu,\ C_1 \le \lambda,\ \mu \le C_2,\ n \ge n_0\}.$$

Let $\tilde{x}_2(n) = \mu_0 = \sup\{\mu : C_1 \le \mu \le C_2,\ n \ge n_0\}$. Then $\tilde{x}_2 \in S$ and $\tilde{x}_2 = \sup S^*$. From the condition (2.1) there exists $n_1 \ge n_0$ with

$$n_1 \ge n_0 + \max\{\tau, \sigma\} \tag{2.2}$$

sufficiently large that

$$\sum_{s=n}^{\infty} Q(s) \le \frac{[(1-p)C_2]^{\alpha} - \beta^{\alpha}}{G(C_2)},\ \ n \ge n_1. \tag{2.3}$$

For $x \in S$, we define

$$(Tx)(n) = \begin{cases} p(n)x(n-\tau) + \left[\beta^{\alpha} + \displaystyle\sum_{s=n}^{\infty} Q(s)G(x(s-\sigma))\right]^{1/\alpha}, & n \ge n_1 \\ (Tx_1)(n), & n_0 \le n \le n_1. \end{cases}$$

For $n \ge n_1$ and $x \in S$, by making use of (2.3), we obtain

$$\begin{aligned} (Tx)(n) &\le pC_2 + \left[\beta^{\alpha} + G(C_2)\sum_{s=n}^{\infty} Q(s)\right]^{1/\alpha} \\ &\le pC_2 + \left[\beta^{\alpha} + G(C_2)\frac{[(1-p)C_2]^{\alpha} - \beta^{\alpha}}{G(C_2)}\right]^{1/\alpha} \\ &\le pC_2 + [[(1-p)C_2]]^{1/\alpha} \\ &\le C_2, \end{aligned}$$

and

$$(Tx)(n) \ge \beta \ge C_1.$$

Hence $Tx \in S$ for every $x \in S$. Let $x_1,\ x_2 \in S$ with $x_1 \le x_2$. Since G is nondecreasing, $Tx_1 \le Tx_2$, that is, T is an increasing mapping. Then by the Knaster-Tarski fixed point theorem, there exists a positive $x \in S$ such that $Tx = x$. Thus $\{x(n)\}$ is a bounded nonoscilatory solution of equation (1.1), which completes the proof.

Theorem 2.2. Assume that $1 < p \le p(n) \le p_0 < \infty$, G is nondecreasing and (2.1) holds, then equation (1.1) has a bounded nonoscillatory solution.

Proof: Let B be a Banach space as defined in Theorem 2.1. We can define a partial ordering as follows: for given $x_1, x_2 \in B,\ x_1 \le x_2$ means that $x_1(n) \le x_2(n)$ for $n \ge n_0 \in \mathrm{N}_0$. Define

$$S = \{x \in B : C_3 \le x(n) \le C_4,\ n \ge n_0\},$$

where C_3 and C_4 are positive constants such that

$$(p_0 - 1)C_3 < \beta \le (1 - p)C_4.$$

If $\tilde{x}_1(n) = C_3,\ n \ge n_0,$ then $\tilde{x}_1 \in S$ and $\tilde{x}_1 = \inf S$. In addition, if $\phi \subset S^* \subset S$, then

$$S^* = \{x \in B : \lambda \le x(n) \le \mu,\ C_3 \le \lambda,\ \mu \le C_4,\ n \ge n_0\}.$$

Let $\tilde{x}_2(n) = \mu_0 = \sup\{\mu : C_3 \le \mu \le C_4,\ n \ge n_0\}$. Then $\tilde{x}_2 \in S$ and $\tilde{x}_2 = \sup S^*$. From the condition (2.1) there exists $n_1 \ge n_0$ with

$$n_1 + \tau \ge n_0 + \sigma \tag{2.4}$$

sufficiently large that

$$\sum_{s=n}^{\infty} Q(s) \le \frac{\beta^\alpha - [(p_0 - 1)C_3]^\alpha}{G(C_3)},\ \ n \ge n_1. \tag{2.5}$$

For $x \in S$, we define

$$(Tx)(n) = \begin{cases} \dfrac{1}{p(n+\tau)} + \left[x(n+\tau)\left(\beta^\alpha - \displaystyle\sum_{s=n+\tau}^{\infty} Q(s) G(x(s-\sigma)) \right)^{1/\alpha} \right], & n \ge n_1 \\ (Tx_1)(n), & n_0 \le n \le n_1. \end{cases}$$

For $n \ge n_1$ and $x \in S$, by making use of (2.5), we obtain

$$(Tx)(n) \le \frac{1}{p}\left[C_4 + \left(\beta^\alpha\right)^{1/\alpha} \right] \le \frac{1}{p}[C_4 + \beta] \le \frac{1}{p}[C_4 + (1-p)C_4] \le C_4,$$

and

$$\begin{aligned} (Tx)(n) &\ge \frac{1}{p(n+\tau)}\left[C_3 + \left(\beta^\alpha - G(C_3) \sum_{s=n+\tau}^{\infty} Q(s) \right)^{1/\alpha} \right] \\ &\ge \frac{1}{p(n+\tau)}\left[C_3 + \left(\beta^\alpha - G(C_3) \frac{\beta^\alpha - [(p_0-1)C_3]^\alpha}{G(C_3)} \right)^{1/\alpha} \right] \\ &\ge \frac{1}{p(n+\tau)}\left[C_3 + \left(\beta^\alpha - \beta^\alpha + [(p_0-1)C_3]^\alpha \right)^{1/\alpha} \right] \\ &\ge \frac{1}{p(n+\tau)}\left[C_3 + p_0 C_3 - C_3 \right] \\ &\ge C_3, \end{aligned}$$

Thus $Tx \in S$ for every $x \in S$. Let $x_1,\ x_2 \in S$ with $x_1 \le x_2$. Since G is nondecreasing, $Tx_1 \le Tx_2$, that is, T is an increasing mapping. Then by the Knaster-Tarski fixed point theorem, there exists a positive $x \in S$ such that $Tx = x$. Thus $\{x(n)\}$ is a bounded nonoscilatory solution of equation (1.1), which completes the proof.

Theorem 2.3. Assume that $0 \le p(n) \le p < 1, G$ is nondecreasing and

$$\sum_{n=n_0}^{\infty} \sum_{s=c}^{d} Q(n,s) < \infty, \tag{2.6}$$

then equation (1.2) has a bounded nonoscillatory solution.

Proof: Let B be a Banach space as defined in Theorem 2.1. We can define a partial ordering as follows: for given $x_1, x_2 \in B,\ x_1 \le x_2$ means that $x_1(n) \le x_2(n)$ for $n \ge n_0 \in \mathrm{N}_0$. Define

$$S = \{x \in B : C_5 \le x(n) \le C_6,\ n \ge n_0\},$$

where C_5 and C_6 are positive constants such that

$$C_5 \le \beta < (p-1)C_6.$$

If $\tilde{x}_1(n) = C_5,\ n \ge n_0$, then $\tilde{x}_1 \in S$ and $\tilde{x}_1 = \inf S$. In addition, if $\phi \subset S^* \subset S$, then

$$S^* = \{x \in B : \lambda \le x(n) \le \mu,\ C_5 \le \lambda,\ \mu \le C_6,\ n \ge n_0\}.$$

Let $\tilde{x}_2(n) = \mu_0 = \sup\{\mu : C_5 \le \mu \le C_6,\ n \ge n_0\}$. Then $\tilde{x}_2 \in S$ and $\tilde{x}_2 = \sup S^*$. From the condition (2.6) there exists $n_1 \ge n_0$ with

$$n_1 \le n_0 + \max\{\tau, d\}$$

sufficiently large that

$$\sum_{s=n}^{\infty} \sum_{i=c}^{d} Q(s,i) \le \frac{[(1-p)C_6]^{\alpha} - \beta^{\alpha}}{G(C_6)},\ n \ge n_1.$$

For $x \in S$, we define

$$(Tx)(n) = \begin{cases} p(n)x(n-\tau) + \left[\beta^{\alpha} + \sum_{s=n}^{\infty} \sum_{i=c}^{d} Q(s,i)G(x(s-i))\right]^{1/\alpha}, & n \ge n_1 \\ (Tx_1)(n), & n_0 \le n \le n_1. \end{cases}$$

The remaining part of the proof is similar to that of Theorem 2.1, and hence the details are omitted.

Theorem 2.4. *Assume that* $1 < p \le p(n) \le p_0 < \infty,\ G$ *is nondecreasing and (2.6) holds, then equation (1.2) has a bounded nonoscillatory solution.*

Proof: Let B be a Banach space as defined in Theorem 2.1. We can define a partial ordering as follows: for given $x_1, x_2 \in B,\ x_1 \le x_2$ means that $x_1(n) \le x_2(n)$ for $n \ge n_0 \in \mathrm{N}_0$. Define

$$S = \{x \in B : C_7 \le x(n) \le C_8,\ n \ge n_0\},$$

where C_7 and C_8 are positive constants such that

$$(p_0 - 1)C_7 < \beta \le (p-1)C_8.$$

If $\tilde{x}_1(n) = C_7,\ n \ge n_0$, then $\tilde{x}_1 \in S$ and $\tilde{x}_1 = \inf S$. In addition, if $\phi \subset S^* \subset S$, then

$$S^* = \{x \in B : \lambda \le x(n) \le \mu,\ C_7 \le \lambda,\ \mu \le C_8,\ n \ge n_0\}.$$

Let $\tilde{x}_2(n) = \mu_0 = \sup\{\mu : C_7 \le \mu \le C_8,\ n \ge n_0\}$. Then $\tilde{x}_2 \in S$ and $\tilde{x}_2 = \sup S^*$. From the condition (2.6) there exists $n_1 \ge n_0$ with

$$n_1 + \tau \geq n_0 + d$$

sufficiently large that

$$\sum_{s=n+\tau}^{\infty} \sum_{i=c}^{d} Q(s,i) \leq \frac{\beta^\alpha - [(p_0 - 1)C_7]^\alpha}{G(C_7)}, \ n \geq n_1.$$

For $x \in S$, we define

$$(Tx)(n) = \begin{cases} \dfrac{1}{p(n+\tau)} + \left[x(n+\tau) + \left(\beta^\alpha - \sum_{s=n+\tau}^{\infty} \sum_{i=c}^{d} Q(s,i) G(x(s-i)) \right)^{1/\alpha} \right], n \geq n_1 \\ (Tx_1)(n), \qquad\qquad n_0 \leq n \leq n_1. \end{cases}$$

The remaining part of the proof is similar to that of Theorem 2.2, and hence the details are omitted.

Theorem 2.5. Assume that $0 \leq \sum_{s=a}^{b} p(n,s) \leq p < 1,$ G is nondecreasing and (2.6) holds, then equation (1.3) has a bounded nonoscillatory solution.

Proof: Let B be a Banach space as defined in Theorem 2.1. We can define a partial ordering as follows: for given $x_1, x_2 \in B,$ $x_1 \leq x_2$ means that $x_1(n) \leq x_2(n)$ for $n \geq n_0 \in \mathrm{N}_0$. Define

$$S = \left\{ x \in B : C_9 \leq x(n) \leq C_{10}, \ n \geq n_0 \right\},$$

where C_9 and C_{10} are positive constants such that

$$C_9 \leq \beta < (1-p)C_{10}.$$

If $\tilde{x}_1(n) = C_9, \ n \geq n_0,$ then $\tilde{x}_1 \in S$ and $\tilde{x}_1 = \inf S$. In addition, if $\phi \subset S^* \subset S$, then

$$S^* = \left\{ x \in B : \lambda \leq x(n) \leq \mu, \ C_9 \leq \lambda, \ \mu \leq C_{10}, \ n \geq n_0 \right\}.$$

Let $\tilde{x}_2(n) = \mu_0 = \sup\left\{ \mu : C_9 \leq \mu \leq C_{10}, \ n \geq n_0 \right\}.$ Then $\tilde{x}_2 \in S$ and $\tilde{x}_2 = \sup S^*$. From the condition (2.6) there exists $n_1 \geq n_0$ with

$$n_1 \geq n_0 + \max\{b, d\}$$

sufficiently large that

$$\sum_{s=n}^{\infty} \sum_{i=c}^{d} Q(s,i) \leq \frac{[(1-p)C_{10}]^\alpha - \beta^\alpha}{G(C_{10})}, \ n \geq n_1.$$

For $x \in S$, we define

$$(Tx)(n) = \begin{cases} \sum_{s=a}^{b} p(n,s) x(n-s) + \left[\beta^\alpha + \sum_{s=n}^{\infty} \sum_{i=c}^{d} Q(s,i) G(x(s-i)) \right]^{1/\alpha}, n \geq n_1 \\ (Tx_1)(n), \qquad\qquad n_0 \leq n \leq n_1. \end{cases}$$

The remaining part of the proof is similar to that of Theorem 2.1, and hence the details are omitted.

3. Examples

In this section, we present some examples to illustrate the main results.

Example 3.1. Consider the difference equation

$$\Delta\left(\left(x(n)-\frac{1}{4}x(n-1)\right)^3\right)+\frac{7}{2^{2n+8}}x(n-2)=0,\ n\geq 0. \tag{3.1}$$

Here $p(n)=\frac{1}{4}$, $Q(n)=\frac{7}{2^{2n+8}}$, $\alpha=3$, and $\tau=1$, $\sigma=2$. By taking $G(x)=x$, we see that $\sum_{n=1}^{\infty}Q(n)<\infty$. Further it is easy to verify that all other conditions of Theorem 2.1 are satisfied. Therefore the equation (3.1) has a bounded nonoscillatory solution. In fact, $\{x(n)\}=\left\{\frac{1}{2^n}\right\}$ is one such solution of equation (3.1).

Example 3.2. Consider the difference equation

$$\Delta\left(\left(x(n)-\frac{1}{2}x(n-3)\right)^3\right)+\sum_{s=1}^{2}\frac{1}{n+s}x(n-s)=0,\ n\geq 0. \tag{3.2}$$

Here $p(n)=\frac{1}{2}$, $Q(n,s)=\frac{1}{n+s}$, $\alpha=3$, and $c=1$, $d=2$. By taking $G(x)=x$, we see that all other conditions of Theorem 2.3 are satisfied and hence every solution of equation (3.2) has a bounded nonoscillatory.

Example 3.3. Consider the difference equation

$$\Delta\left(x(n)-\sum_{s=1}^{2}\frac{1}{2(n+s-1)}x(n-s)\right)+\sum_{s=2}^{3}\frac{1}{(n+s)^2}x(n-s)=0,\ n\geq 0. \tag{3.3}$$

Here $p(n)=\frac{1}{2(n+s-1)}$, $Q(n,s)=\frac{1}{(n+s)^2}$, $\alpha=1$, $a=1$, $b=2$,, and $c=2$, $d=3$. By taking $G(x)=x$, we see that all other conditions of Theorem 2.5 are satisfied and hence every solution of equation (3.3) has a bounded nonoscillatory.

References

[1] **R.P. Agarwal**, Difference Equations and Inequalities, Second Edition, Marcel Dekker, New York, 2000.

[2] **R.P. Agarwal, M.Bohner, S.R.Grace** and **D.O'Regan,** Discrete Oscillation Theory, Hindawi Publ. Corp., New York, 2005.

[3] **R.P. Agarwal, S.R.Grace** and **D.O'Regan,** Oscillation Theory for Difference and Functional Differential Equations, Kluwer Academic, 2000.

[4] **R.Arul** and **G.Ayyappan**, Some oscillation results for first order nonlinear neutral difference equations, JMI Int.J. of Math. Sci., 4(1)(2013), 1-9.

[5] **R.Arul** and **T.J.Raghupathi**, Existence of nonoscillatory solutions of nonlinear delay difference equations, Far East J. of Math. Sci., 76(1)(2013), 71-80.

[6] **T.Candan**, Existence of nonoscillatory solutions of first order nonlinear neutral difference equations, Appl. Math. Lett., 26(2013), 1182-1186

[7] **M.P.Chen** and **B.G.Zhang**, The existence of bounded positive solutions of delay difference equations, Pan America Math. I., 3(1)(1993), 79-94.

[8] **S.S.Chang** and **W.T.Patula**, An existence theorem for a nonlinear difference equations, Nonlinear Anal., 20(1993), 193-203.

[9] **J.R.Graef, E.Thandapani** and **S.Elizabeth,** Oscillation of first order nonlinear neutral difference equations, Indian J.Pure Appl. Math., 36(9)(2005), 503-512.

[10] **S.Lourdu Marian, M.Paul Loganathan** and **A.George Maria Selvam,** Existence of nonoscillatory solutions of first order nonlinear neutral delay difference equations, American J. of Math. Sci. and Appl., 1(1)(2013), 21-26.

[11] **O.Ocalan** and **O.Duman**, Oscillation analysis of neutral difference equations with delays, Chaos Solitons and Fractals, 39(2009), 261-270.

[12] **N.Parhi,** Behavior of solutions of delay difference equations, Indian J. Pure Appl. Math., 33(1)(2002), 31-43.

[13] **E.Thandapani, R.Arul** and **P.S.Raja**, Oscillation of first order neutral delay difference equations, Appl. Math. E-Notes, 3(2003), 88-94.

[14] **E.Thandapani, R.Arul** and **P.S.Raja**, The asymptotic behavior of nonoscillatory solutions of nonlinear neutral type difference equations, Math. Comput. Model., 39(13)(2004), 1457-1465.

[15] **E.Thandapani, J.R.Graef and P.W.Spikes**, On existence of positive solutions and oscillations of neutral difference equations of odd order, J.Diff. Equ. Appl., 2(1996), 175-183.

[16] **G.Wei**, Asymptotic behavior results for nonlinear neutral delay difference equations, Appl. Math. Comput., 217(2011), 7184-7190.