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# Finite Groups Having Exactly 34 Elements of Maximal Order

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## ABSTRACT

Let G be a finite group, M(G) denotes the number of elements ofmaximal order of G. In this note a finite group G with M(G) = 34 is determined.

## Indexing terms/Keywords

Finite groups; Classification; Number of elements of maximal order; Thompson's Conjectur.

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## INTRODUCTION

For a finite group G, we denote by M(G) the number of elements of maximal order of G, and the maximal element order in G by k = k(G). There is atopic related to one of Thompson's Conjectures:

**Thompson's Conjecture:** Let G be a finite group. For a positive integerd, define G(d) = |{x $\in$ G|the order of x is d}|. If S is a solvable group,G(d) = S(d) for d = 1; 2; ..., then G is solvable.

Recently, some authors have investigated this topic in several articles(see[3], [7], [8], [9]). In particular, in [2] the authors gave a complete classfication of the finite group with M(G) = 30, and the finite group with M(G) = 24 are classified in [6]. In this paper, we consider a finite group G satisfying M(G) = 34. Our main result of this paper is:

**Main Theorem**: Suppose G is a finite group having exactly 34 elements of maximal order. Then G is solvable and |G| = $2^{\alpha}3^{\beta}$, where $2 \le \alpha \le 7$, and $1 \le \beta \le 4$ .

By the above theorem, we have:

**Corollary:** Thompson's Conjecture holds if G has exactly 34 elements of maximal order.

All groups considered are finite and all unexplained notations are standard and can be found in [4].

## Preliminaries

The following lemma reveals the relationship of M(G) and k.

Lemma 2.1 [9, Lemma 1] Suppose G has exactly n cyclic subgroups of orderl, then the number of elements of order l (de -noted by $n_l$ (G)) is $n_l$ (G) = n$\phi$(l), where $\phi$(l) is the Euler function of l. In particular, if n denotes the number of cyclic subgroups of G of maximal order k, then M(G) = n$\phi$(k) .

By above lemma, we have:

Lemma 2.2 If M(G) = 34 and k is maximal element order of G, thenpossible values of n, k and $\phi$(k) are given in following table:

| n | $\phi$(k) | k |
|---|---|---|
| 34 | 1 | 2 |
| 17 | 2 | 3,4,6 |
| 2 | 17 | null |
| 1 | 34 | null |

In proving our main theorem, the following two results will be frequently used.

Lemma 2.3 [2, Lemma 6] If k is prime, and the number of elements of maximal order k is m, then k divides m + 1.

Lemma 2.4 [2, Lemma 8] If the number of elements of maximal order k is m, then there exists a positive integer $\alpha$ such that |G| divides m$k^{\alpha}$ .

Lemma 2.5 [8, Lemma 2.5] Let P be a p-group with order $p^t$, where p is a prime, and t is a positive integer. Suppose b$\in$ Z(P), where o(b) = $p^u$ = k with u a positive integer. Then P has at least (p -1)$p^{t-1}$ elements of order k.

## Proof of Main Theorem

By the hypothesis M(G) = 34, then k 6= 2 by Lemma 2.3, and and k 6= 4 by [2, Corollary 2]. In the following we prove our theorem case by case for the remaining possible values of k.

Case 1 k = 3. In this case G is a 3-group or a {2,3}-group. If G is a 3-group, then exp(G) = 3. By [5, Theorem 3.8.8], the number of 3-elements M(G) of G satisfies that M(G)$\equiv$4(mod 9), which contradicts with the fact M(G) = 34. Hence G is not a 3-group. If G is a {2,3}-group, then $\pi_e$ (G) = {1,2, 3}. By [1, Theorem] we know that G = N$\succ$Q is a Frobenius group, whe -re N$\cong C_3^t$,Q$\cong C_2$ or N$\cong C_2^{2t}$ ,Q$\cong C_3$. Suppose that Q$\cong C_2$. Then N is an elementary abelian 3-group, By [5, Theorem 3.8.8], we can get a contradiction. If Q$\cong C_3$, then the number of elements of order 3 is two, which contradicts to our assum ption. Thus k $\neq$ 3.

Case 2 k = 6. In this case |G| = $2^{\alpha}3^{\beta}$, where $\alpha$ > 0 and $\beta$ > 0 by Lemma 2.4. Let x be an element of order 6. Then |$C_G$ (x)| = $2^u3^v$. Since there exists no element of order 9 or 4 in $C_G$ (x), we have v≤3 and u≤4 by Lemma 2.5. Since G has exactly 17 cyclic subgroups of order 6, we have |G : $N_G$ (<x>)| = 1; 2; 3; 4; 6; 8; 9; 12 or 16. If there is an element y of order 6 in G such that |G : $N_G$ (<x>)| = 9; 12 or 16, then there exists another element z of order 6 in G such that |G : $N_G$ (<x>)| = 1; 2; 3; 4; 6 or 8. That is to say, G always has an element x of order 6 such that |G : $N_G$ (<x>)| = 1; 2; 3; 4; 6 or 8.Therefore |G| | $2^7 3^4$ since|G| =|G : $N_G$ (<x>)|| $N_G$ (<x>): $C_G$ (x)|| $G_G$ (x)|.The Theorem is proved.

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