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SS-Coprime Modules

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Abstract

Let R be a commutative ring with unity and M is a unitary left R-module . In this paper , we introduce the notion of strongly S-coprime modules, where M is called strongly S-coprime briefly (SS-Coprime) if for each $r \in R$, r^2M is small in M implies $rM=0$. We investigate many properties about this concept. Moreover many relations between it and other related concepts, are given.

Keywords: Coprime module ; S-Coprime module; SS-Coprime module; SSS-Coprime module; τ-noncosingular module.

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1. INTRODUCTION

Let R be a commutative ring with unity and let M be a unital R-module. Recall that a proper submodule N of M is called prime if whenever $r \in R, x \in M, rx \in N$ *implies that* $x \in N$ *or* $r \in [N:M]$,[10] . M is called a prime R-module if the zero submodule (0) is a prime submodule of M. Equivalently, M is a prime R-module if $ann_R M = ann_R N$ for each nonzero submodule N of M,[4].

S. Yassemi in [13] introduced the notions of second submodule and second module, where a submodule N of M is called second if whenever r∈R-{0}, rN=N or rN=0 . M is called a second module if M is a second submodule of M ; that is whenever r∈R-{0}, rM=M or rM=0.

S. Annine in [2] introduced the notion of coprime module as follows: M is called a coprime R-module if annM=ann$\frac{M}{N}$ for each proper submodule N of M. However, it is known that second modules and coprime modules are equivalent.

Moreover I. E. Wijayant in [15], give the following :

An R-module M is coprime if and only if ann_RM=$ann_R \frac{M}{N}$ for each proper fully invariant summodule N of M. and proved that it is equivalent to definition of coprime module (in sense of S. Annine .

Recall that a proper submodule N of M is called small (denoted by N<<M) if whenever N+K=M, K is a submodule of M, then K=M [1]. As a generalization of coprime module. I.M.A.Hadi & R.I.Khalaf in [7] introduce the notion of small coprime (briefly, S-coprime) module, where an R-module M is called S-coprime if $ann_R M = ann_R \frac{M}{N}$ for each N<<M. Equivalently, M is S-coprime if for each r∈R-{0} , rM<<M implies rM = 0 .

Tütuncu, Tribak in [12] introduced an studied the notion of T. non cosingular, where an R-module M is called T-noncosingular if whenever $\varphi \in EndM$, Imφ<<M, implies Im$\varphi = 0$. It is clear that T-noncosingular module is S-coprime and the converse is not true in general, as we shall see later.

In section two of this paper we investigate the notion of strongly S-coprime module (briefly SS-coprime) where an R-module M is called SS-coprime if for any a,b∈R, abM<<M implies aM=0 or bM=0 , it is clear that SS-coprime module is s-coprime but the converse is not true. However we see by examples that the concept SS-coprime and T-noncosingular are independent. However, if M is T-noncosingular and annM is a prime ideal then M is SS-coprime. Moreover many properties of SS-coprime modules and some connections between SS-coprime modules and other related concepts are given.

In section three, the concept of semistrongly S-coprime (briefly, SSS-coprime) is presented, where an R-module is SSS-coprime if whenever r∈R with r^2M<<M, then rM=0. It is clear that every SS-coprime is SSS-coprime, but the converse is true under certain condition. Most of properties of SS-coprime modules generalized to SSS-coprime.

2. SS-Coprime Modules

Definition 2.1 :

An R-module is called strongly S-coprime (briefly SS-coprime) if for each a,b∈R, abM<<M implies aM=0 or bM=0

Remarks and Examples 2.2:

1) It is clear that every SS-coprime module is S-coprime, but the converse is not true in general, for example:

   The Z-module Z_6 is S-coprime since $(\bar{0})$ is the only small submodule in Z_6 and so $rZ_6<<Z_6$ implies $rZ_6=(\bar{0})$. But Z_6 is not SS-coprime, since $2.3Z_6=(\bar{0}) <<Z_6$ and $2.Z_6 \neq (\bar{0})$, $3.Z_6 \neq (\bar{0})$.

2) Let M be an R-module, then M is SS-coprime if and only if M is S-coprime and annM is a prime ideal

   Proof :

   It is easy

3) Every T-noncosingular module is S-coprime , but neither S-coprime nor SS-coprime implies T-noncosingular in general.

   Proof :

   Let M be a T-noncosingular R-module . Let r∈R-{0} , rM<<M. Define φ:M⟶M by φ(m)=rm , $\forall m \in M$. Then Imφ=rM<<M and since M is T-noncosingular , rM=0. Thus M is S-coprime

   Now consider the following example:

   Let M be the Z module $Z_{2^\infty} \oplus Z_2$. Then M is not T-noncosingular Z-module . However we can show that M is SS-coprime so let abM<<M=$Z_{2^\infty} \oplus Z_2$ then ab $Z_{2^\infty} \ll Z_{2^\infty}$ and $abZ_2<<Z_2$. But Z_{2^∞} is divisible so abZ_{2^∞}<<Z_{2^∞} implies ab=0 and hence a=0 or b=0 . Thus aM=0 or bM=0 , that is M is SS-coprime (and hence M is S-coprime)

4) T-noncosingular module need not be SS-coprime, for example: Th Z-module Z_6 is not SS-coprime by part (1), but it is T-noncosingular .

5) If M is T-noncosingular and $ann_R M$ is a prime ideal, then M is SS-coprime .

Proof :

Let a,b∈R, abM<<M . Define f: M⟶M byf(m)=abm, $\forall m \in M$. Then Imf=abM<<M. But M is T-noncosingular , so abM=(0), that is ab∈ $annM$. As annM is a prime ideal, then $a \in annM$ or $b \in annM$; thus aM=(0) or bM=(0) and M is SS-coprime.

6) An R-module M is SS-coprime if and only if M is an SS-coprime $\overline{R}$-module, where $\overline{R} = R/annM$

7) Let M , M' be two isomorphic R-module. Then M is SS-coprime if and only if M' is SS-coprime .

8) A ring R is an SS-coprime R-module if and only if R is an integral domain

Proof :

Let R be an integral domain , let a,b∈R with (a)(b)R<<R. Then (a)(b)=0 and so (a)=0 or (b)=0 . Conversely, if R is SS-coprime R-module , let a,b∈R a.b=0 . Then (a)(b)<<R and R is SS-coprime , either (a)=0 or (b)=0. Thus a=0 or b=0; that is R is an integral domain.

9) Let R be an SS-coprime , then R is K-nonsingular and the converse is not true in general , where R is K-nonsingular if for each f∈R, f≠0 , kerf $\nleq_e$R(ker f is not essential in R) .

Proof :

R is SS-coprime , so part(8), J(R)=(0) . Thus L(R)=Z(R)=(0). Hence kerf$\nleq_e$R .

Also Z_6 as Z_6-module is K-nonsingular , but it is not SS-coprime .

Recall that an R-module M is called a scalar module if for each f∈EndM , there exists r ∈R such that f(m)=rm , $\forall m \in M$ [11].

Proposition 2.3 :

Let M be an R-module. If M is S-coprime and scalar module , then M is T-nonsingular module.

Proof :

Let f∈EndM and Imf<<M . Since M is a scalar R-module , there exists r ∈R such that f(m)=rm , $\forall m \in M$. Thus f(M)=rM<<M and since M is S-coprime f(M)=rM=0.

Therefore M is T-noncosingular.

Recall that an R-module M is called a multiplication R-module , if for each N≤M there exists an ideal I of R such that N=IM. Equivalently, M is a multiplication R-module if for each N≤M, N=[N:M] [3],where[N:M] ={r∈R:rM⊆N} .

Remark 2.4 :

Every multiplication SS-coprime R-module M has (0) as the only small submodule of M

Proof :

Let N<<M . Since M is SS-coprime so M is S-coprime .Hence annM=ann$\frac{M}{N}$ = [N:M].

This implies (0)= (annM)M=[N:M]M=N.

Proposition 2.5 :

Let M be an SS-coprime R-module . Then M is a prime module if and only if M is a primary module.

Proof :

(⟹) It is clear .

(⟸) Let r∈R , x∈M and rx=0 . Since M is primary, either x=0 or r^n ∈annM for some n∈Z_+. But M is SS-coprime implies annM is a prime ideal , hence either x=0 0r r∈annM. Thus M is a prime module.

The following two results are characterizations of SS-coprime modules.

Proposition 2.6 :

Let M be an R-module . Then M is SS-coprime module if and only if for each ideals I,J of R IJM<<M implies IM=0 or JM=0

Proof :

($\Rightarrow$)

Let I,J be ideals of R and IJM<<M. Suppose JM≠(0) . Hence there exists b∈J, b≠0 such that bM≠0. It follows that for each a∈I , abM≤IJM<<M . So that abM <<M But M is SS-coprime and bM≠(0), so that aM=0 for each a∈I. Thus IM=(0).

($\Leftarrow$) It is clear.

Proposition 2.7 :

An R-module M is SS-coprime if and only if for each a,b∈R , abM<<M implies [aM:M]=annM or [bM:M]=annM.

Proof :

It is easy so is omitted

Remark 2.8 :

The homomorphic image of SS-coprime is not necessarily SS-coprime , for example:

The Z-module Z is SS-coprime. Let $\pi: \mathbb{Z} \longrightarrow \mathbb{Z}/<6> \simeq Z_6$ be the natural epimorphism $\pi(Z)=Z_6$ which is not SS-coprime.

Proposition 2.9 :

Let M be an SS-coprime R-module. Let N<<M . Then $\frac{M}{N}$ is an SS-coprime R-module.

Proof :

Let a,b∈R and a.b($\frac{M}{N}$) << $\frac{M}{N}$. Then $\frac{abM+N}{N} \ll \frac{M}{N}$ and since N<<M , we get abM+N <<M , and since N<<M , then abM<<M . But M is SS-coprime, so either aM=0 or bM=0 . It follows that a $\frac{M}{N}$ =(0) or b $\frac{M}{N}$ =(0) . Thus $\frac{M}{N}$ is SS-coprime .

Corollary 2.10 :

Let f:M⟶M' be an epimorphism with kerf<<M. If M is an SS-coprime R-module, then M' is SS-coprime.

Corollary 2.11 :

Let M be an R-module with projective cover f: P ⟶ M. If P is an SS-coprime R-module , then M is SS-coprime.

Corollary 2.12 :

Let be a ring . Then the following statements are equivalent

1) Every projective R-module is SS-coprime
2) Every R-module M having a projective cover is SS-coprime .

Proof :

$(1) \Longrightarrow (2)$

It is following directly by Corollary 2.11 .

$(2) \Longrightarrow (1)$ Let M be a projective R-module . Since there exists the identity mapping $i: \mathrm{M} \longrightarrow \mathrm{M}$ and kerf=0<<M , then M has a projective cover . Hence by (2) , M is SS-coprime.

Proposition 2.13 :

Let M be an R-module. Let N<M such that [N:M] =annM. If $\frac{M}{N}$ is an SS-coprime R-module, then M is SS-coprime.

Proof :

Let a,b∈R and a.bM<<M. It follows that $\frac{abM+N}{N} \ll \frac{M}{N}$, that is ab($\frac{M}{N}$) << $\frac{M}{N}$. But $\frac{M}{N}$ is SS-coprime, so either a $\frac{M}{N}$ =(0) or b $\frac{M}{N}$ =(0) . This implies either aM⊆N or bM⊆N , so either a∈[N:M]=annM or b∈[N:M]=annM . Thus aM=(0) or bM=(0).

Remark 2.14 :

1) A direct summand of SS-coprime module may not be SS-coprime , for example: consider the Z-module $M=Z\oplus Z_6$. It is easy to see that M is SS-coprime , but by Remark and Example 2.2(1) , Z_6 is not SS-coprime .
2) The direct sum of SS-coprime modules need not be SS-coprime module , for example: each of the Z-module $Z_2$and Z_3 is SS-coprime , but $M=Z_2\oplus Z_3 \cong Z_6$ is not SS-coprime.

Proposition 2.15 :

Let M_1 and M_2 be R-modules and $annM_1=annM_2$. Then $M= M_1 \oplus M_2$ is SS-coprime. Particularly, $M \oplus M$ is SS-coprime if M is SS-coprime .

Proof :

Let a,b∈R and $ab(M_1 \oplus M_2) << M_1 \oplus M_2$. Then $abM_1 << M_1$ and $abM_2 << M_2$. As M_1& M_2 are SS-coprime , then (either $aM_1=0$ or $bM_1=0$) and (either $aM_2=0$ or $bM_2=0$) . But $annM_1=annM_2$, hence aM=0 or bM=0. Thus M is SS-coprime .

Proposition 2.16 :

Let $M= M_1 \oplus M_2$. if M is an SS-coprime R-module such that $annM_1$ and $annM_2$ are noncomparable prime ideals , then M_1 and M_2 are SS-coprime modules.

Proof :

Since M is SS-coprime , then M is S-coprime by Remarks and Examples 2.2(2). Hence by [7, Theorem 19] , M_1 and M_2 are S-coprime modules . But ann M_1 and ann M_2 are prime ideals of R , so by Remarks and Examples 2.2(2), M_1 and M_2 are SS-coprime modules.

Recall that an R-module M is called small prime if annM=annN for each (0)≠N<<M. Equivalently M is a small prime R-module if (0) is a small prime submodule , where a proper submodule N of M is called a small prime submodule if whenever r∈R, x∈M and (x)<<M, rx∈N implies x∈N or r∈[N:M] [8].

It is clear that every prime module is a small prime module , and if M is a small prime module, then annM is a prime ideal [8]

Theorem 2.17 :

Let M be an R-module such that every submodule N of M is relatively divisible (i.e. rM∩N=rN , $\forall r \in R$). If M is small prime , then M is SS-coprime.

Proof :

We claim that M is S-coprime . So I shall prove that annM=[N:M] for each N<<M.

Suppose that there is a small submodule N of M and r∈R,r≠0 such that r∈[N:M] and rM≠(0). As M is small prime, we get rN≠(0) . By hypothesis , N is relatively divisible, hence $rM \cap rN=r^2N$ and so $rN=r^2N$. This implies that , for any n∈N , $rn=r^2n_1$ for some $n_1 \in N$, and hence $r(n-rn_1)=0$. But $n-rn_1 \in N<<M$, so that $(n-rn_1)<<M$. On the other hand , M is small prime , so $annM=ann(n-rn_1)$. Hence r∈annM, which is a contradiction . Thus annM = ann(N:M), ∀N<<M, i.e. M is S-coprime . Beside this , M is small prime implies annM is a prime , so by Remark and Example 2.2(2) , M is SS-coprime.

Recall that an R-module M is called F-regular if IM∩N=IN for each N≤M and each ideal I of R [5].

Corollary 2.18 :

Let M be an F-regular R-module .If M is small prime , then M is SS-coprime.

Corollary 2.19 :

Let M be amodule over a regular ring R (i.e. R is regular in sense of VonNeuamann)

Then the following statements are equivalent :

1) M is a small prime R-module
2) M is an SS-coprime R-module
3) M is a prime R-module

Proof :

$(1) \Rightarrow (2)$

Since R is regular ring , R/ann(x) is a regular ring for each x∈M. Hence M is F-regular[14]. Thus the result follows by Corollary 2.18 .

$(2) \Rightarrow (3)$

Since M is SS-coprime , then annM is a prime ideal by Remarks and Examples 2.2(2) , so that $\bar{R}$=R/annM is an integral domain . But R is regular ring implies $\bar{R}$ is regular ring , it is follows that $\bar{R}$ is a field , hence M is a prime $\bar{R}$-module which implies that M is a prime R-module .

$(3) \Rightarrow (1)$

It is clear .

Remark 2.20 :

Let M be a divisible module over an integral domain R . Then M is a faithful SS-coprime.

Proof :

Let a,b∈R and abM<<M . If ab≠0 , then abM=M , so M<<M which is a contradiction . Thus ab=0 and hence a=0 or b=0 . So that aM=0 or bM=0 ; that is M is SS-coprime.

Also, if r∈annM, then rM=0. Since M is divisible, then r=0. Thus M is faithful.

Proposition 2.21 :

Let M be a faithful R-module . Consider the following statements :

1) M is SS-coprime
2) R is SS-coprime
3) R is an integral domain

Then $(1) \Longrightarrow (3)$ ⟺(2) and $(3) \Longrightarrow (1)$ if M is finitely generated multiplication R-module.

Proof :

$(1) \Longrightarrow (3)$

Let a,b∈R such that ab=0 . Then (ab)<<R . So , abM=(0)<<M. But M is SS-coprime , so either aM=0 or bM=0. Since M is faithful , then a=0 0r b=0.

$(3) \Longrightarrow (1)$

Let a,b∈R and abM<<M . Since M is finitely generated faithful multiplication module , then (ab)<<R. But R is an integral domain , so (ab)=(0) and hence either a=0 or b=0. Thus either aM=(0) or bM=(0)

(3) ⟺(2)

See Remarks and Examples 2.2(8)

Let M be an R-module , we say that M is small retractable if Hom(M,N)≠0 for each N<<M .

Proposition 2.22 :

Let M be a small retractable and scalar module . If M is S-coprime , then RadM=(0).

Proof :

Suppose there exists m∈RadM , m≠0 . Hence (m)<<M and since M is small retractable, there exists f:M⟶(m), f≠0 , hence f ∊ EndM. But M is a scalar R-module , so that there exists r∈R such that f(x)=rx , ∀x∈M . Thus f(M)=rM ⊆(m)<<M and as M is S-coprime , we get rM=0. Hence f=0 which is a contradiction , therefore RadM=(0).

Hence it is clear that if M is small retractable scalar module and M is SS-coprime then RadM=0 .

Proposition 2.23 :

Let M be an R-module . If Hom(M,N)=0, for each N<<M , then M is S-coprime

Proof :

Let a∈R and aM<<M. Define f:M⟶M by f(m)=am , ∀m∈M. Hence f(M)=aM<<M, thus f∈Nom(M,aM) and aM<<M, so by hypothesis f=0 . Thus f(M)=aM=0 and M is S-coprime.

Corollary 2.24 :

Let M be an R-module .If Hom(M,N)=0 for each N<<M and annM is a prime ideal . Then M is SS-coprime.

Proof :

It follows by Proposition 2.23 and Remarks and Examples 2.2(2) .

Proposition 2.25 :

Let M be an R-module .Then M is an SS-coprime E-module if and only if Hom(M,N)=0, ∀ N<<M and $\text{ann}_E M$ is a prime ideal in E , where E=End(M).

Proof :

(⟹) Let f∈Hom(M,N), N<<M. Then f(M)⊆ N<<M, so f(M)<<M. But M is SS-coprime E-module , hence M is S-coprime E-module and so f(M)=0 . Thus Hom(M,N)=0. Moreover , since M is SS-coprime E-module , $\text{ann}_E M$ is a prime ideal in E by Remarks and Examples 2.2(2) .

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(⟸) First we shall prove M is an S-coprime E-module . Let f∈Hom(M,N), f(M)<<M . Put f(M)= N , hence f∈Hom(M,N)=0. Thus f=0 and so M is an S-coprime E-module. But $\mathrm{ann}_E M$ is a prime ideal so M is SS-coprime E-module by Remarks and Examples 2.2(2) .

Under the class of multiplication module , we have the following

Theorem 2.26 :

Let M be a multiplication R-module . Then M is an SS-coprime if and only if M is an SS-coprime E-module .

Proof :

(⟹) Let f,g∈EndM, and (f∘g)(M)<<M . Since g(M)≤M and M is multiplication R-module,

g(M)=IM for some ideal I of R. It follows that (f∘g)(M)= f(g (M))= f(IM) = If(M) . But f(M)≤M, so f(M)=JM, for some ideal J of R. Thus (f∘g)(M)=IJM and so IJM<<M. But M is an SS-coprime R-module , hence either IM=0 or JM=0 by Proposition 2.6 . Thus either f(M)=0 or g(M)=0 ; that is M is an SS-coprime E-module.

(⟸)Let abM<<M where a,b∈R . Define f,g:M⟶M by f(m)=am ,g(m)=bm , ∀m∈M. Then (f∘g)(M)=abM<<M . Since M is an SS-coprime E-module, then either f(M)=0 or g(M)=0 and hence either aM=0 or bM=0. Thus M is an SS-coprime R-module.

Recall that an R-module is called hollow module if every proper submodule of M is small[6].

Proposition 2.27 :

Let M be a hollow R-module . Then the following statements are equivalent :

1) M is S-coprime
2) M is coprime
3) M is SS-coprime

Proof :

(1)⟺(2) It is clear .

(1) ⟹ (3) Let abM<<M where a,b∈R. Then either aM or bM is a proper submodule of M. Hence if aM<M, then aM<<M and so aM=0 . Similarly , bM=0 . Thus M is SS-coprime.

(3) ⟹ (1) It follows by Remarks and Examples 2.2(2).

Proposition 2.28 :

Let I be a nil ideal of a ring R . If M is an S-coprime R-module, then IM=0 .

Proof :

Let a ∈I , we claim that aM<<M . Assume aM+K=M for some submodule K of M . Then for each n $\in Z_+$, a^nM+K =M . But a is a nilpotent element , so K=M and aM<<M . Since M is S-coprime , then aM=0 for any a ∈I . Thus IM=(0).

Proposition 2.29:

Let I .J be two ideals of a ring R such that IJ is a nil ideal . If M is an SS-coprime R-module, then IM=0 or JM=0.

Proof :

Since M is an SS-coprime R-module, then M is an S-coprime R-module and hence by Proposition 2.28, IJM=0, so that IJM<<M . But M is SS-coprime , therefore either IM=0 or JM=0.

Recall that a ring R is semilocal if R/J(R) is semisimple.

Proposition 2.30 :

Let R be a semilocal ring and J(R) is nilpotent . Then M is S-coprime if and only if M is semisimple .

Proof :

(⟹) If M is S-coprime . Since R is semilocal , $\frac{R}{J(R)}$ is semilocal, hence RadM=J(R)M and $\frac{M}{\mathrm{RadM}}$ is semisimple by [1. Corollary 15.18]. But J(R) is a nil ideal , so by Proposition 2.28, J(R)M=0 . then RadM =0 .

(⟸) It is clear .

Note 2.31 :

If R is a semilocal ring with J(R) is nilpotent and M is an SS-coprime R-module , then M is semisimple , but the converse is not true for example: consider Z_6 as Z_6-module . The ring Z_6 is semilocal , $J(Z_6)=0$ is a nil ideal . Also Z_6 as Z_6-module is semisimple , but it is not SS-coprime .

3. Semi Strongly S-Coprime Modules

In this section we investigate the notion of semi strongly S-coprime modules and present some of its properties and some of relations between this concept and other related concepts .

Definition 3.1 :

An R-module is called semi strongly S-coprime (briefly, SSS-coprime) if for each $a \in R$, $a^2M << M$ implies $aM=0$.

Remarks and Examples 3.2 :

1) It is clear that every SS-coprime is SSS-coprime , but not conversely , for example : if M is the Z-module Z_6 , then $a^2Z_6 << Z_6$ implies $a^2Z_6=(0)$; that is $a^2 \in ann_z Z_6 = Z_6$ and so $a \in 6Z$. Thus $aZ_6=(0)$ and M is SSS-coprime . But M is not SS-coprime .

2) Every SSS-coprime module is S-coprime

   Proof :

   Let M be an SSS-coprime module , let $a \in R$ with $aM \ll M$. Since $a^2M \subseteq aM$, then $a^2M \ll M$. Hence $aM=(0)$ because M is SSS-coprime . Thus M is S-coprime .

3) It is easy to see that : an R-module M is S-coprime and $ann_R M$ is a semiprime ideal of R if and only if M is SSS-coprime .

4) Let M be a module over a chained ring R . Then M is SS-coprime if and only if M is SSS-coprime .

5) If M and M' are isomorphic R-module. Then M is SSS-coprime if and only if M' is SSS-coprime.

6) The image of SSS-coprime need not be SSS-coprime . As example to show this : The Z-module Z is SSS-coprime , let π: $Z \longrightarrow Z/<4> \simeq Z_4$ be the natural epimorphism , then $\pi(Z)=Z_4$ is not SSS-coprime .

7) For any ring $R \neq 0$. If R is SSS-coprime , then L(R)=J(R)=(0) .

Proof :

Suppose there exists $a \in J(R)$, $a \neq 0$. Then $a^2R \ll R$. Since R is SSS-coprime , then aR=(0) (i.e. a=0) which is a contradiction . Thus J(R)=(0), hence L(R)=(0).

Proposition 3.3 :

The direct sum of two SSS-coprime modules is SSS-coprime .

Proof :

Let $M= M_1 \oplus M_2$, where M_1 and M_2 are SSS-coprime R-module . If $r \in R$ such that $r^2M \ll M$, then $r^2M_1 \ll M_1$ and $r^2M_2 \ll M_2$. By SSS-coprimeness of M_1 and M_2 , $r M_1=(0)$ and $r M_2=(0)$. Thus rM=(0) and M is SSS-coprime.

Remark 3.4 :

A direct summand of SSS-coprime module may be not SSS-coprime , for example : If M is the Z-module $Z \oplus Z_4$, then M is SSS-coprime , but Z_4 is not a SSS-coprime Z-module.

The following result is a characterization of SSS-coprime module .

Proposition 3.3 :

Let M be an R-module. Then the following statements are equivalent >

1) M is SSS-coprime module
2) For any ideal I of R , $I^2M \ll M$ implies IM=(0)
3) For any ideal I of R and $n \in Z_+$, $I^nM \ll M$ implies IM=(0).

Proof :

It is easy , so is omitted .

The following results are analogous to results about SS-coprime modules .

Proposition 3.4 :

Let N<<M. If M is SSS-coprime R-module. Then $\frac{M}{N}$ is an SS-coprime R-module.

Proof :

It is similar to proof of Proposition 2.9 .

Corollary 3.7 :

Let f:M⟶M' be an epimorphism with kerf<<M. if M' is an SSS-coprime R-module, then M is SSS-coprime.

Corollary 3.8 :

Let M be an R-module with projective cover f: P ⟶ M. If P is an SSS-coprime R-module , then M is SSS-coprime.

Proposition 3.9 :

Let M be an R-module. Let N<M such that [N:M] =annM. If $\frac{M}{N}$ is an SSS-coprime R-module, then M is SSS-coprime.

Proof :

It is similar to the proof of Proposition 2.13

Theorem 3.10 :

Let M be a multiplication R-module . Then M is an SSS-coprime if and only if M is an SSS-coprime E-module , where E=End(M)

Proof :

It is similar to the proof of Proposition 2.26

Remark 3.11 :

Since every SSS-coprime is S-coprime by Remark and Example 3.2(2) . If R is a semilocal ring with J(R) is a nilpotent , then every SSS-coprime is semisimple

Next we have

Proposition 3.12 :

Let M be a finitely generated faithful multiplication R-module . Then M is SSS-coprime if and only if R is SSS-coprime .

Proof :

(⟹)Let (a^2)<<R . Since M is faithful finitely generated multiplication , then a^2M<<M, hence aM=0 . But M is faithful so a=0 (i.e. (a)=(0)).

(⟸) Let a∈R and a^2M<<M . Since M is faithful finitely generated multiplication , $[a^2M:_R M]$<<R , hence (a^2)<<R . So (a)=(0). Thus aM=(0).

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