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Numerical Approximation of Internal Temperature in the Cylinder of an IC Engine

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ABSTRACT

In this study, mollification and marching methods are used to solve aninverse problem in combustion engines. With the benefit of 2D mollification, we first propose an algorithm, and then prove some theorems, which ensure us the proposed method in stable and reliable, and then some numerical experiment has been done to show the efficiency of the method.

Indexing terms/Keywords

Inverse problems; Mollification method; Marching method; Combustion Engines.

Academic Discipline And Sub-Disciplines

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SUBJECT CLASSIFICATION

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Introduction

Inverse problems play a vital role in science and engineering and have rightfully received a great deal of attention by applied mathematicians, statisticians, and engineers. The most interesting characteristic of inverse problems is that they cannot be solved analytically; hence how accurate we solve them by computational methods in another important task to do [4].

The majority of the methods that have been proposed for solvinginverse problems were formalized in the last four decades in terms of their capabilities to treat ill-posed unstable problems. The basis of such formal methods resides on the idea of reformulating an inverse problem in terms of an approximate well-posed problem, by utilizing some kind of regularization techniques [1].

Following the model introducedin [1, 3], which the in-cylinder pressure in an internal combustion engine was modeled as a function of crank angle. In this article, wewould like toconsider the temperature as a function ofcrank angle and variation of piston in cylinder during the compression stroke.Basically, we are aim to add more minutedetails to the problem of interest and generate an efficient numerical method to solve the problem.

For convenience we outline our procedure as follows:

In the next section, *the mathematical formulation of* ourinterest problem is described. The next section, the *regularized problem and the marching scheme* isdescribed [2]. In the next section, a marching algorithm is generated to have the optimize solution. After that to proof the reliability of the proposed method, stability and convergence analysis are discussed.After that, for testing the method, two examples are discussed.

Description of the Problem

In internal combustions (IC) engines, the mathematical model of heat transfer process in cylinder wall in general form may be written as follows [1,3],

$$\frac{\partial^2 T}{\partial r^2}+\frac{1}{r}\frac{\partial T}{\partial r}+\frac{1}{r^2}\frac{\partial^2 T}{\partial \theta^2}+\frac{\partial^2 T}{\partial z^2}=k\frac{\partial T}{\partial t} \qquad (1)$$

Where$T=T(r,z,t)$, $R_1<r<R_2$, $0<\theta<2\pi$, $0<z<a$ and $0<t<T_t$. Also the boundary conditions regarding the problem may be written as follows

$$T(R_1,z,t)=P(t,z), \qquad (2)$$

$$k\frac{\partial}{\partial \theta}T(R_2,z,t)=q(t,z), \qquad (3)$$

$$\frac{\partial}{\partial z}T(r,0,t)=\varphi(r,t), \qquad (4)$$

$$\frac{\partial}{\partial z}T(r,b,t)=\psi(r,t), \qquad (5)$$

$$T(r,b,0)=f(r,l). \qquad (6)$$

In cylinder one may consider that the variation of T is not dependent on θ. Therefore, in the equation (1), the term $\frac{\partial^2 T}{\partial \theta^2}$ can be dropped. On the other hand, the crank angle baseofa lot of information in IC engines and it is clear that crank angle varies with time, i.e. $\alpha=\alpha(t)$, which implies,

$$\frac{\partial T}{\partial t}=\frac{\partial T}{\partial \alpha}.\frac{\partial \alpha}{\partial t}. \qquad (7)$$

In IC engines $\frac{\partial \alpha}{\partial t}$ defines the engine angular speed, ω_e, and it is considered as follows,

$$\frac{\partial \alpha}{\partial t}=\omega_e=\frac{2\pi N}{60}, \qquad (8)$$

Where N demonstrates the engine speed. We can rewrite equation (1) using equation (8), we have

$$K\omega_e\frac{\partial T}{\partial \alpha}=\frac{\partial^2 T}{\partial r^2}+\frac{1}{r}\frac{\partial T}{\partial r}+\frac{\partial^2 T}{\partial Z^2}, \qquad (9)$$

Where the initial and boundary conditions may be written as follows,

$$T(R_1,z,\alpha)=P(z,\alpha), \qquad (10)$$

$$K\frac{\partial T}{\partial r}(R_2,z,\alpha)=0, \qquad (11)$$

$$K\frac{\partial T}{\partial z}(r,0,\alpha)=0, \qquad (12)$$

$$K\frac{\partial T}{\partial z}(r,b,0)=f_1(r,b) \qquad (13)$$

In the practical experiences, occasionally, the boundary conditions or some parts ofthem may be unavailable duo to physical situations. Hence having an alternativecondition to solve the problem is essential.

$$T(R_2, z, \alpha) = \eta(t), \quad (14)$$

The problem of interest, then, can be summarized as follows:

$$K\omega_e \frac{\partial u}{\partial \alpha} = \frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{\partial^2 u}{\partial Z^2}, \quad (15)$$

$$T(R_1, z, \alpha) = P(z, \alpha) \quad (16)$$

$$T(R_2, z, \alpha) = \eta(z, \alpha), \quad (17)$$

$$K\frac{\partial v}{\partial r}(R_2, z, \alpha) = 0, \quad (18)$$

Where$K\omega_e$ is given and $\eta(t)$is measured and also.Determining$v(r, z, \alpha)$ and $P(z, \alpha)$ from theproblem is our primal goal.Furthermore, the known data function $\eta(z, \alpha)$is only known approximately as $\eta^{\varepsilon}(z, \alpha)$ such that

$$\|\eta(z, \alpha) - \eta^{\varepsilon}(z, \alpha)\| \le \varepsilon \quad (19)$$

Whereε is a positive tolerance.

Regularized problem and the marching scheme

The mollified problem

The regularized problem, based on mollification, is formulated as follows.

$$K\omega_e \frac{\partial v}{\partial \alpha} = \frac{\partial^2 v}{\partial r^2} + \frac{1}{r}\frac{\partial v}{\partial r} + \frac{\partial^2 v}{\partial Z^2}, \quad (20)$$

$$v(R_1, z, \alpha) = P(z, \alpha) \quad (21)$$

$$v(R_2, z, \alpha) = J_{\delta_1}\eta^{\varepsilon}(z, \alpha), \quad (22)$$

$$K\frac{\partial v}{\partial r}(R_2, z, t) = 0, \quad (23)$$

where$v(r, z, \alpha)$ and $P(z, \alpha)$are needed to be determined.

Marching scheme

Let M, N, and O, positive integers,$h = R_2 - R_1/M$, $l = a/O$, $k = 1/T_t$be theparameters of the finite differences discretization.We introduce the discrete functions$U_{i,j}^n$,$Q_{i,j}^n$,$W_{i,j}^n$,$P_{i,j}^n$ and$S_{i,j}^n$asdiscrete computed approximations of $v(ih,, jl, nk)$, $v_r(ih,, jl, nk)$, $v_\alpha(ih,, jl, nk)$, $v_z(ih,, jl, nk)$and $v_{zz}(ih,, jl, nk)$respectively. Then, the space marching algorithm is defined as follows.

1. Select δ_0.
2. Perform mollification of η^{ε} and set.

$$U_{M,j}^n = J_{\delta_0}\eta^{\varepsilon}(jl, nk), Q_{M,j}^n = 0.$$

3. Perform mollified differentiations. Set

$$U_{M,j}^n = \mathbf{D}_\alpha\left(J_{\delta_0}\eta^{\varepsilon}(jl, nk)\right), S_{M,j}^n = \mathbf{D}_z\left(J_{\delta_0}U_{M,j}^n\right).$$

4. Initialize $i = M$. Do while $i \ge 1$,

i. $$U_{i-1,j}^n = U_{i,j}^n - hQ_{i,j}^n, \quad (24)$$

ii. $$Q_{i-1,j}^n = Q_{i,j}^n - h\left[K\omega_e W_{i,j}^n + \frac{1}{ih}Q_{i,j}^n + S_{i,j}^n\right], \quad (25)$$

iii. $$W_{i-1,j}^n = W_{i,j}^n - h\mathbf{D}_\alpha\left(J_{\delta_i}Q_{i,j}^n\right), \quad (26)$$

iv. $$P_{i-1,j}^n = P_{i,j}^n - k\mathbf{D}_z\left(J_{\delta_i'}W_{i,j}^n\right), \quad (27)$$

v. $$S_{i-1,j}^n = \mathbf{D}_z\left(J_{\delta_i^*}P_{i-1,j}^n\right). \quad (28)$$

where$\mathbf{D}_\alpha$and $\mathbf{D}_z$respectively denote the centered difference operator with respect to α and z [5]. From now on, we denote $|Y_i| = \max_{\mathrm{j,n}} |Y_{i,j}^n|$and $\|Y\|_\infty = \max_{\mathrm{i}} |Y_i|$.

Stability and Convergence of the Scheme

In this section, we analyze the stability of the marching scheme (24)–(28).Without loss of generality, from now on, we assume $|\delta|_{-\infty} = \min\{\delta_i\}$, also $|Y_i| = \max_{\mathrm{j,n}} |Y_{i,j}^n|$ and $\|Y\|_\infty = \max_{\mathrm{i}} |Y_i|$.

Assumption1: For all$(r, z, \alpha) \in I = [R_1, R_2] \times [0, l] \times [0, 2\pi]$, we further assume that $T(r, z, \alpha) \in C^2(I)$.

Theorem 1 [Stability]:If Assumption 1 holds, for the marching scheme (24)–(28), there exists a constantΛ, dependent on $|\delta|_{-\infty}$, such that

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$$\max\{|U_0|,|Q_0|,|W_0|,|P_0|,|S_0|\} \leq \Lambda \max\{|U_M|,|Q_M|,|W_M|,|P_M|,|S_M|\}.$$

Proof:

Usingtheorem 4 and 6 from [5], there exist constants C_1, C_2 and C_3 such that,

$$\mathbf{D}_\alpha\left(Q_{i,j}^n\right) \leq \frac{C_1}{|\delta|_{-\infty}}|Q_i|, \mathbf{D}_z\left(W_{i-1,j}^n\right) \leq \frac{C_2}{|\delta|_{-\infty}}|W_i| \text{ and } \mathbf{D}_z\left(P_{i-1,j}^n\right) \leq \frac{C_3}{|\delta|_{-\infty}}|P_i| \quad (29)$$

We have from (24)-(26) and (29)

$$|U_{i-1,j}^n| \leq |U_{i,j}^n| + h|Q_{i,j}^n| \leq (1+h)\max\{|U_i|,|Q_i|\} \quad (30)$$

$$|Q_{i-1,j}^n| \leq |Q_{i,j}^n| + h\left[K\omega_e|W_{i,j}^n| + \frac{1}{ih}|Q_{i,j}^n| + |S_{i,j}^n|\right] \leq |Q_{i,j}^n| + h\left[K\omega_e|W_{i,j}^n| + \xi|Q_{i,j}^n| + |S_{i,j}^n|\right]$$

$$\leq \left(1 + C_q h\right)\max\{|W_i| + |Q_i| + |S_i|\}. \quad (31)$$

$$|W_{i-1,j}^n| \leq |W_i| + h\frac{C_1}{|\delta|_{-\infty}}|Q_i| \leq \left(1 + h\frac{C_1}{|\delta|_{-\infty}}\right)\max\{|Q_i|,|W_i|\}. \quad (32)$$

Also,

$$|P_{i-1,j}^n| \leq |P_{i,j}^n| + k\frac{C_2}{|\delta|_{-\infty}}|W_{i,j}^n| \leq \left(1 + h\frac{C_1}{|\delta|_{-\infty}}\right)\max\{|P_i|,|W_i|\} \quad (33)$$

$$|S_{i-1,j}^n| \leq \frac{C_3}{|\delta|_{-\infty}}|P_{i-1,j}^n| \leq \left(h\frac{C_3}{|\delta|_{-\infty}}\right)\max\{|P_i|\} \quad (34)$$

Now let $C_\delta = \left\{1, C_q, \frac{C_1}{|\delta|_{-\infty}}, \frac{C_2}{|\delta|_{-\infty}}\right\}$, then from (30)-(34) we can conclude,

$$\max\{|U_{i-1}|,|Q_{i-1}|,|W_{i-1}|,|P_{i-1}|,|S_{i-1}|\} \leq (1 + C_\delta h)\max\{|U_i|,|Q_i|,|W_i|,|P_i|,|S_i|\}, \quad (35)$$

and iterating this last inequality M times, we have

$$\max\{|U_0|,|Q_0|,|W_0|,|P_0|,|S_0|\} \leq (1 + C_\delta h)^M \max\{|U_M|,|Q_M|,|W_M|,|P_M|,|S_M|\} \quad (36)$$

which implies

$$\max\{|U_0|,|Q_0|,|W_0|,|P_0|,|S_0|\} \leq \exp(C_\delta)\max\{|U_M|,|Q_M|,|W_M|,|P_M|,|S_M|\} \quad (37)$$

Setting$\Lambda = \exp(C_\delta)$completes the proof of this statement.

We will now show that the above scheme convergence to the desired $J_\delta v$ for fixed δ. To do so, we define

$$\Delta U_{i,j}^n = U_{i,j}^n - v(ih, jl, nk), \qquad \Delta Q_{i,j}^n = Q_{i,j}^n - v_r(ih, jl, nk), \qquad \Delta W_{i,j}^n = W_{i,j}^n - v_\alpha(ih, jl, nk),$$

$$\Delta P_{i,j}^n = P_{i,j}^n - v_z(ih, jl, nk), \qquad \Delta S_{i,j}^n = S_{i,j}^n - v_{zz}(ih, jl, nk),$$

and denote$\Delta_i = \max\{\Delta U_i, \Delta Q_i, \Delta W_i, \Delta P_{i,}, \Delta S_i\}$.

From the marching scheme introduced in (24)–(28), we can notice that the mollified solution $v(r, z, \alpha)$ satisfies

$$v\big((i-1)h, jl, nk\big) = v(ih, jl, nk) - hv_r(ih, jl, nk), \quad (38)$$

$$v_r\big((i-1)h, jl, nk\big) = v_r(ih, jl, nk) - h\left[K\omega_e v_\alpha(ih, jl, nk) + \frac{1}{ih}v_r(ih, jl, nk) + \frac{\partial}{\partial z}v_z(ih, jl, nk)\right], \quad (39)$$

$$v_\alpha\big((i-1)h, jl, nk\big) = v_\alpha(ih, jl, nk) - h\frac{\partial}{\partial \alpha}v_r(ih, jl, nk), \quad (40)$$

$$v_z\big((i-1)h, jl, nk\big) = v_z(ih, jl, nk) - h\frac{\partial}{\partial z}v_\alpha(ih, jl, nk), \quad (41)$$

$$v_{zz}\big((i-1)h, jl, nk\big) = \frac{\partial}{\partial z}v_z\big((i-1)h, jl, nk\big), \quad (42)$$

Comparing (38)-(42) with the scheme, we can write

$$\Delta U_{i-1,j}^n = \Delta U_{i,j}^n + \left(U_{i-1,j}^n - U_{i,j}^n\right) - \Big(v\big((i-1)h, jl, nk\big) - v(ih, jl, nk)\Big) + O(h^2),$$

$$= \Delta U_{i,j}^n + h\Delta Q_{i,j}^n + O(h^2), \quad (43)$$

$$\Delta Q_{i-1,j}^n = \Delta Q_{i,j}^n + \left(Q_{i-1,j}^n - Q_{i,j}^n\right) - \Big(v_r\big((i-1)h, jl, nk\big) - v_r(ih, jl, nk)\Big) + O(h^2),$$

$$= \Delta Q_{i,j}^n - h\left[K\omega_e \Delta W_{i,j}^n + \frac{1}{ih}\Delta Q_{i,j}^n + \Delta S_{i,j}^n\right] + O(h^2), \quad (44)$$

$$\Delta W_{i-1,j}^n = \Delta W_{i,j}^n - h\Big(v_\alpha\big((i-1)h, jl, nk\big) - v_\alpha(ih, jl, nk)\Big) + O(h^2),$$

$$= \Delta W_{i,j}^n - h\left[\mathbf{D}_\alpha\left(J_{\delta_i}Q_{i,j}^n\right) - v_{r\alpha}(ih, jl, nk)\right] + O(h^2), \quad (45)$$

$$\Delta P_{i-1,j}^{n} = \Delta P_{i,j}^{n} - h\left(v_z\big((i-1)h, jl, nk\big) - v_z(ih, jl, nk)\right) + O(h^2),$$

$$= \Delta W_{i,j}^{n} - h\left[\mathbf{D}_z\left(J_{\delta_i'} W_{i-1,j}^{n}\right) - v_{\alpha z}(ih, jl, nk)\right] + O(h^2), \qquad (45)$$

Also

$$\left|U_{i-1,j}^{n}\right| \le \left|\Delta U_{i,j}^{n}\right| + h\left|\Delta Q_{i,j}^{n}\right| + O(h^2), \qquad (47)$$

$$\left|\Delta Q_{i-1,j}^{n}\right| \le \left|\Delta Q_{i,j}^{n}\right| + h\left[K\omega_e\left|\Delta W_{i,j}^{n}\right| + \frac{1}{ih}\left|\Delta Q_{i,j}^{n}\right| + \left|\Delta S_{i,j}^{n}\right|\right] + O(h^2). \qquad (48)$$

Using theorem 4 and 6 from [5], we have

$$\left|\Delta W_{i-1,j}^{n}\right| \le \left|\Delta W_{i,j}^{n}\right| + h\left(\frac{C}{|\delta|_{-\infty}}\left(\left|\Delta Q_{i,j}^{n}\right| + k\right) + C_\delta k^2\right) + O(h^2), \qquad (49)$$

$$\left|\Delta P_{i-1,j}^{n}\right| \le \left|\Delta P_{i,j}^{n}\right| + h\left(\frac{C}{|\delta|_{-\infty}}\left(\left|\Delta W_{i,j}^{n}\right| + l\right) + C_\delta l^2\right) + O(h^2). \qquad (50)$$

Letting$C_0 = \{1, K\omega_e + 2\}$ and $C_1 = \left\{\frac{Ck}{|\delta|_{-\infty}} + C_\delta k^2, \frac{Cl}{|\delta|_{-\infty}} + C_\delta l^2\right\}$

$$|\Delta_{i-1}| \le (1 + C_0 h)|\Delta_i| + O(h^2). \qquad (51)$$

By calculating L iterations,

$$\Delta_L \le \exp\left(\frac{C r_L}{|\delta|_{-\infty}^2}\right)\left(\Delta_0 + C(l + k + h)\right). \qquad (52)$$

Since

$$\Delta_M \le \frac{C}{|\delta|_{-\infty}^2}(\varepsilon + l + k), \qquad (53)$$

the convergence of the algorithm readily follows. That isΔ_L converges to zero as ε, h, k and l tend to zero.

Numerical Results

In this section to show the ability of the proposed numerical procedure, we present two examples. In all cases, without loss of generality, we set $p = 3$(see [2]). The radii of mollification are always chosen automatically using the mollification and GCV methods. Discretized measured approximations of boundary data are modeled by adding random errors to the exact data functions. For example, for the boundary data function $h(x,t)$, its discrete noisy version is generated by

$$h_{j,n}^{\varepsilon} = h(x_j, t_n) + \varepsilon_{j,n}, j = 0,1,\dots,N, n = 0,1,\dots,T, \qquad (54)$$

where the $\varepsilon_{j,n}$ are random variables uniformly distributed on $[-\varepsilon, \varepsilon]$.

The errors exact and approximate solution are measured by the relative RMS-norm.

Example 1: As the first example, in equations (15)-(18) consider $K = 1$, $\omega_e = 2.576$ and $\eta(z,\alpha) = e^{-0.08}\sin\alpha + z^2$. Also, $R_1 = 0.02, R_2 = 0.08, a = 0.15$and $T_t = 2$.

Since this problem cannot be solved analytically, we follow a scheme which compars the solutions with each other.

Table 1 demonstrates RMS norm of comparing two solutions at three different mesh levels and two different amountsof $\boldsymbol{\varepsilon}$.

Table 1.Comparing solutions with each other

$M = N = O$		ε	$\left\|P(z,\alpha)\|_{M_1} - P(z,\alpha)\|_{M_2}\right\|$
M_1	M_2		
128	256	0.01	0.0831
256	512	0.01	0.0698
512	1024	0.01	0.0357
128	256	0.05	0.0936
256	512	0.05	0.0908
512	1024	0.05	0.0806

Figure 1 illustrates the difference between two levels of noise at $M = 1024$ and for a fixed z.

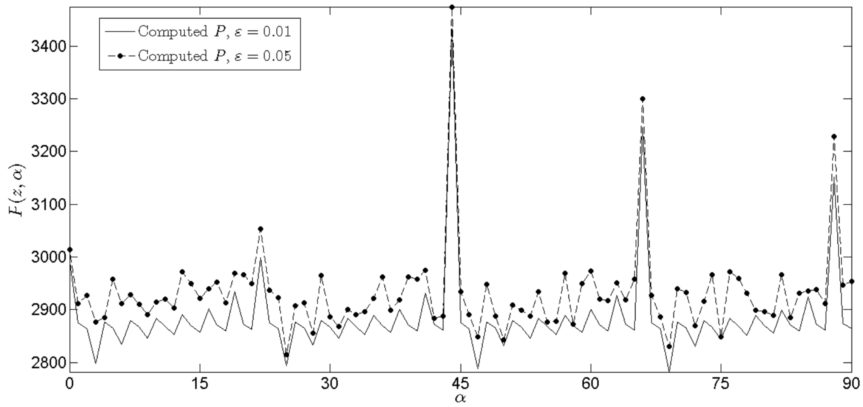

Fig 1: the difference between two levels of noise at $M = 1024$ and for a fixed z.

Example 2: In equations (15)-(18) consider $K = 1$, $\omega_e = 2.576$ and $\eta(z,\alpha) = e^z \cos\alpha$. Also, $R_1 = 0.02, R_2 = 0.1, a = 0.13$and $T_t = 2.4$.

Table 2 demonstrates RMS norm of comparing two solutions at three different mesh levels and two different amountsof $\boldsymbol{\varepsilon}$.

Table 2.Comparing the solution with each other

$M = N = O$		ε	$\lVert P(z,\alpha)\vert_{M_1} - P(z,\alpha)\vert_{M_2} \rVert$
M_1	M_2		
128	256	0.01	0.0809
256	512	0.01	0.0782
512	1024	0.01	0.0719
128	256	0.05	0.1058
256	512	0.05	0.0957
512	1024	0.05	0.0881

Figure 1 illustrates the difference between two levels of noise at $M = 1024$ and for a fixed z.

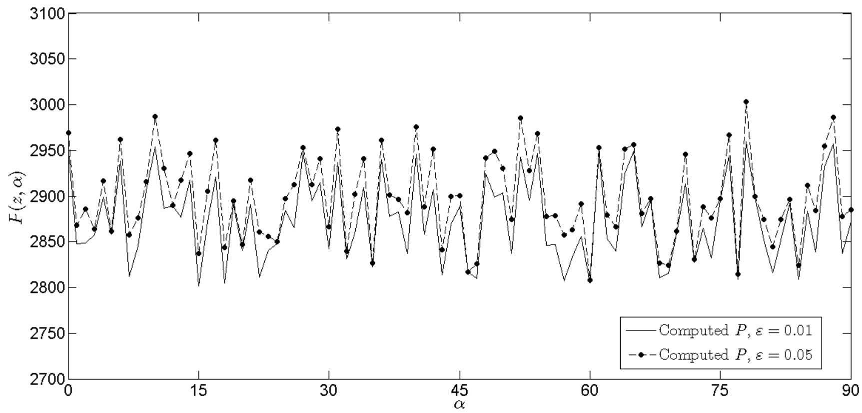

Fig 2: the difference between two levels of noise at $M = 1024$ and for a fixed z.

Conclusion

Overall, we developed a convergence scheme and it demonstrated a reasonable amount of accuracy, although analytical solutions were not available. This conclusion is consistent with the previous studies on this subject of interest [1, 3].

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