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ON THE THREE-PARAMETER BURR TYPE XII DISTRIBUTION AND ITS APPLICATION TO HEAVY TAILED LIFETIME DATA

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ABSTRACT

This paper identifies the characteristics of three-parameter Burr Type XII distribution and discusses its utility in survivorship applications. It addresses the problem of estimating the three-parameter Burr XII distribution and its doubly truncated version. The results are applied on a real dataset by fitting the distribution to the survival time of breast cancer patients in the Gaza Strip. These data are known to have a heavy tailed distribution since patients in this area received different protocols of treatments in different levels of hospitals locally and abroad. The findings indicated that the estimates of the parameters of the truncated distribution are more efficient than those obtained from the original distribution since the distribution is heavy tailed and involves many highly extreme observations.

Key words

Bur type distributions; kernel density estimate; lifetime data; maximum likelihood estimation; truncated distributions.

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1. INTRODUCTION

Burr (1942) introduced twelve different forms of cumulative distribution functions for modeling lifetime data. Burr Type XII

was their most popular one. Several authors considered different aspects of Burr Type XII distributions (including Ahmad et al., 1997; Jaheen, 1995 and 1996; Raqab, 1998; Sartawi & Abu-Salih, 1991; and Surles & Padgett, 1998). Burr Type XII distribution yields a wide range of values of skewness and kurtosis and can be used to fit almost any given set of unimodal data. Burr and Cislak (1968) and Burr (1968) showed that if one chooses the parameters appropriately, the Burr distribution covers a large proportion of curve-shaped characteristics of types I, IV, and VI in the Pearson family of distributions. The goal of this paper is to thoroughly study three-parameter Burr XII distribution as one of the distribution families of survival and life time data. The aim is to identify the characteristics of this distribution and its applications; their utility in survivorship applications will be of special interest. The theoretical results obtained in this paper on Burr XII distribution and their applicability will be examined using simulated and real data sets.

Tadikamalla (1980) established the relationship between Burr Type XII distribution and several other distributions. It can be stated that the Burr distribution has been used extensively to model franchise deductible premium, fixed amount deductible premium, proportional deductible premium, limited proportional deductible premium, and disappearing deductible premium (Burnecki et al., 2004). There are many survival or lifetime distributions that are special cases of the Burr Type XII distribution. Rodriguez (1977) showed that the Burr coverage area on a specific plane is occupied by various well-known and useful distributions, including the normal, log-normal, gamma, logistic, and extreme-value type-I distributions.

Maximum likelihood estimators and approximate estimators of Burr Type XII distribution and the estimators of the variance-covariance matrix have been studied by many authors. Panahi and Asadi (2010) discussed the case when the scale parameter is unknown and observed that the MLEs of the three unknown parameters can be obtained by solving one non-linear equation. When the scale parameter is known, they obtained the maximum likelihood estimator and uniformly minimum variance unbiased estimator. In addition, they obtained Bayes estimator under squared error loss function. The MLE and UMVUE have been observed to be are quite similar in nature, although based on mean squared errors, the performance of the MLEs are marginally better than the rest. Raqab and Kundu (2012) considered two-parameter Burr Type X distribution. They observed several interesting properties of this distribution. It has some interesting relations with the well-studied gamma, Weibull distributions, and with the recently proposed exponentiated exponential and exponentiated Weibull distributions. Feroze and Aslam (2013) dealt with the maximum likelihood estimation of the parameters of Burr Type V distribution based on left censored samples. The maximum likelihood estimators of the parameters have been derived and the Fisher information matrix for the parameters of the said distribution has been obtained explicitly. The confidence intervals for the parameters have been also discussed.

Al-Yousef (2002) discussed the problem of estimating the parameters of a doubly truncated Burr distribution when truncation points are unknown. The estimation was based on type II censored data. Adeyemi and Adebanji (2007) obtained results on the recurrence relations for single and product moments of order statistics from a doubly truncated Burr XII distribution. These results complemented earlier results of Begum and Parvin (2002) and generalized results obtained by Balakrishnan and Gupta (1998), Balakrishnan et al. (1994), and Saran and Pushkarna (1999). Simulation results are consistent with those obtained by Begum and Parvin (2002) and are given for single and product moments in two tables. Olapade (2008) obtained the cumulative distribution function and the n^{th} moment of the generalized distribution and established the distribution of some order statistics of the distribution. A theorem that relates the new distribution to another statistical distribution was established. Paranaíba et al. (2011) derived the moment generating function of Burr XII distribution. Moments, mean deviations, Bonferroni and Lorenz curves, and reliability were provided.

2. Properties of Burr Type XII Distribution with Three Parameters

Let X be a random variable having Burr XII (c, k, α) distribution, and then the cumulative distributions Function (cdf) have the form given by Tejeda and Barry (2008) as follows:

$$F(x) = 1-\left(1+\left(\frac{x}{\alpha}\right)^{c}\right)^{-k}; \qquad x>0,\ c>0,\ k>0,\ \alpha>0 \quad , \tag{1}$$

where c >0 and k >0 are the shape parameters and α is the scale parameter of the distribution. The probability density functions (pdf) of Burr XII (c, k) distribution have the form:

$$f(x) = \frac{ck}{\alpha}\left(\frac{x}{\alpha}\right)^{c-1}\left(1+\left(\frac{x}{\alpha}\right)^{c}\right)^{-(k+1)}; \qquad x>0,\ c>0,\ k>0,\ \alpha>0 \tag{2}$$

Fig. (1.a) illustrates a graph of various shapes of the density function of Burr XII (c, k, α) distribution with different values of α and fixed values of c =5 and k =2. Whenever the values of α increased, the flattening form of the probability density function increased under the fixed values of c and k.

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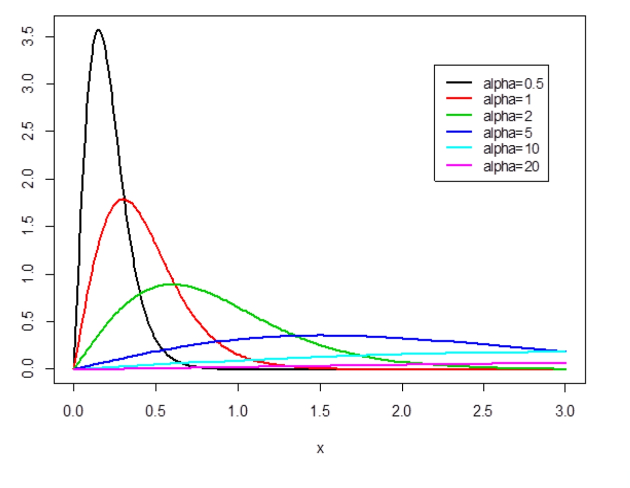

Figure (1): Different density functions of Burr XII (c, k, α) distribution with different values of α and fixed values of c =5 and k =2

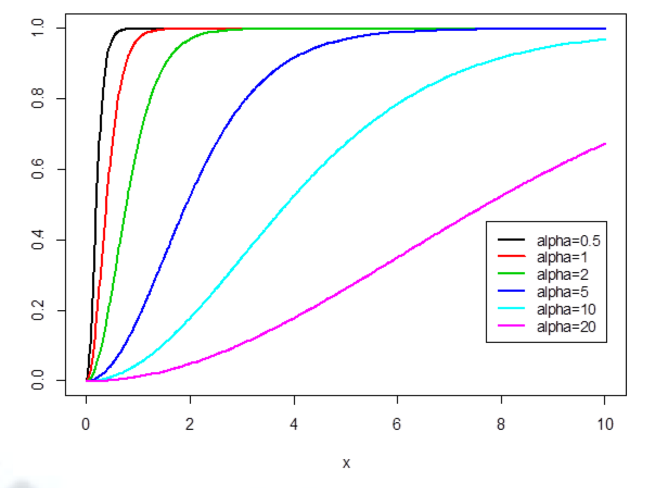

Figure (2): Different cumulative functions of Burr XII (c, k, α) distribution with different values of α and fixed values of c =5 and k =2

2.1. The r^{th} moment of Burr XII (c, k, α)

The r^{th} moment of Burr XII (c, k, α) distribution is

$$E\left(x^r\right) = \mu' = \int_0^\infty x^r f\,(x)dx$$

$$= \int_0^\infty x^r \frac{ck}{\alpha}\left(\frac{x}{\alpha}\right)^{c-1}\left(1+\left(\frac{x}{\alpha}\right)^c\right)^{-(k+1)} dx$$

$$= k\,\alpha^r \int_0^\infty (\mathrm{u})^{(\frac{r}{c})+1-1} \frac{1}{(1+u)^{k+1}}\, du$$

$$= k\,\alpha^r \beta\left(\frac{r}{c}+1, k-\frac{r}{c}\right) = \frac{\alpha^r \lambda_r}{\Gamma(k)},\quad r < c,\ \ r < \alpha \qquad (3)$$

where $\lambda_r = \frac{r}{c}\Gamma\left(\frac{r}{c}\right)\Gamma\left(k-\frac{r}{c}\right) \quad , \ r = 1, 2, \ldots, i$

2.2. Mean value of Burr XII (c, k, α)

Let X be a random variable having Burr XII (c, k, α) distribution, and then

$$\mu = E\left(x\right) = \frac{\alpha\,\lambda_1}{\Gamma(k)} = \frac{\alpha\,\Gamma\left(\frac{1}{c}\right)\Gamma\left(k-\frac{1}{c}\right)}{c\ \ \Gamma(k)},\quad c > 1,\ \ \alpha > 1$$

then $$\mu = E\left(x\right) = \frac{\alpha}{c}\ \beta\left(\frac{1}{c}, k-\frac{1}{c}\right),\quad c > 1,\ \ \alpha > 1 \qquad (4)$$

2.3. Variance of Burr XII (c, k, α)

$$var(x) = E\left(x^2\right) - \left(E\left(x\right)\right)^2$$

From equation (3) and for the $r = 2$ we find

$$E\left(x^2\right) = \frac{\alpha^2\,\lambda_2}{\Gamma(k)} = \frac{2\,\alpha^2}{c}\ \beta\left(\frac{2}{c}, k-\frac{2}{c}\right) \qquad ,c > 2, \alpha > 2$$

$$\sigma^2 = var(x) = \frac{\alpha^2\lambda_2}{\Gamma(k)} - \left(\frac{\alpha\lambda_1}{\Gamma(k)}\right)^2 = \frac{\Gamma(k)\,\alpha^2\lambda_2 - \alpha^2\lambda_1^{\,2}}{\Gamma(k)^2}$$

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$$= \frac{2\alpha^2}{c}\ \beta\left(\frac{2}{c}, k-\frac{2}{c}\right) - \frac{\alpha^2}{c^2}\left(\beta\left(\frac{1}{c}, k-\frac{1}{c}\right)\right)^2, \quad c>2,\ \alpha>2 \tag{5}$$

2.4. Mode of Burr XII (c, k, α)

The mode of Burr XII (c, k, α) is the value x at which its pdf has the maximum value. Thus,

$$\frac{df\ (x;c,k)}{dx} = \frac{d}{dx}\left[\frac{ck}{\alpha}\ \left(\frac{x}{\alpha}\right)^{c-1}\ \left(1+\left(\frac{x}{\alpha}\right)^c\right)^{-(k+1)}\right] = 0$$

$$\frac{c\,k}{\alpha^{2c}}\ x^{c-2}\left(1+\left(\frac{x}{\alpha}\right)^c\right)^{-(k+2)}\ \left[\ \alpha^c(c-1) - x^c(1+\ ck\ )\right] = 0$$

since $x>0,\ \ \alpha>0 \quad x^c = \dfrac{\alpha^c\ (c-1)}{(ck+1)}$ then,

$$x = \alpha\ \left(\frac{c-1}{ck+1}\right)^{\frac{1}{c}};\ \ c\,k \ge 1,\ c>0,\ k>0,\ \alpha>0 \tag{6}$$

2.5. Median of Burr XII (c, k, α)

The median of Burr XII (c, k, α) is the value x at which $P(X \le x) = 0.5$.

$$P(X \le x) = F(x) = \ 0.5$$

$$1-\left(1+\left(\frac{x}{\alpha}\right)^c\right)^{-k} \quad = \quad 0.5$$

$$\left(1+\left(\frac{x}{\alpha}\right)^c\right) \quad = \quad (2)^{\frac{1}{k}}$$

$$x = \alpha\left((2)^{\frac{1}{k}} - 1\right)^{\frac{1}{c}} \tag{7}$$

2.6. Moment-generating function of Burr XII (c, k, α)

For a random variable X that follows Burr XII (c, k, α) distribution, the moment-generating function (mgf) is given by Rodrigo, et al. (2013) and can be obtained as follows:

$$\mathrm{M}_x\ (t)\ =\ E\left(e^{tx}\right) = \int_0^\infty e^{tx} f\ (x)dx$$

$$\mathrm{M}_x\ (t)\ =\ E\left(e^{tx}\right) = \frac{ck}{\alpha}\ \int_0^\infty e^{t\alpha\left(\frac{x}{\alpha}\right)}\ \left(\frac{x}{\alpha}\right)^{c-1}\ \left(1+\left(\frac{x}{\alpha}\right)^c\right)^{-(k+1)}\ dx$$

Using the following result

$$\mathrm{I}\left(p,\mu,\frac{m}{k},\nu\right)\ =\ \int_0^\infty e^{(-px)}\ x^{\mu}\left(1+x^{\frac{m}{k}}\right)^{\nu}\ dx \quad ,$$

where *m* and *k* positive integers, μ >-1 and p >0. Let $c = \dfrac{m}{k}$, and then

$$\mathrm{M}_x\ (t)\ =\ E\left(e^{tx}\right)\ =\ \frac{mk}{k\,\alpha}\ \int_0^\infty e^{t\alpha\left(\frac{x}{\alpha}\right)}\ \left(\frac{x}{\alpha}\right)^{\frac{m}{k}-1}\ \left(1+\left(\frac{x}{\alpha}\right)^{\frac{m}{k}}\right)^{-(k+1)}\ dx$$

$$\therefore \mathrm{M}_x\ (t)\ =\ \frac{m}{\alpha}\ \mathrm{I}\left(p,\mu,\frac{m}{k},\nu\right)\ =\ \frac{m}{\alpha}\ \mathrm{I}\left(-t\alpha,\frac{m}{k}-1,\frac{m}{k},-k-1\right) \tag{8}$$

2.7. Probability density functions (pdf) of the r^{th} order statistics of Burr XII (c, k, α)

Let $X_1, X_2, \dots, X_n$ be n independent and continuous random variables from Burr XII (c, k, α) and let $X_{1:n} \le X_{2:n} \le \dots \le X_{n:n}$ be the corresponding order statistics. Let $f_{X_{(r:n)}}(x)$ denote the probability density function of the r^{th} order statistics $X_{(r:n)}$. David (1970) gives the probability density function of $X_{(r:n)}$ as:-

$$f_{X_{(r:n)}}(x) = \frac{1}{\beta(r, n-r+1)} f(x)\left(F(x)\right)^{r-1}\left(1-F(x)\right)^{n-r}$$

$$= \frac{1}{\beta(r,n-r+1)} \frac{ck}{\alpha}\left(\frac{x}{\alpha}\right)^{c-1}\left(1+\left(\frac{x}{\alpha}\right)^{c}\right)^{-(k+1)}\left(1-\left(1+\left(\frac{x}{\alpha}\right)^{c}\right)^{-k}\right)^{r-1}\left(1-1+\left(1+\left(\frac{x}{\alpha}\right)^{c}\right)^{-k}\right)^{n-r}$$

$$f_{X_{(r:n)}}(x) = \frac{ck\,(x)^{c-1}}{\alpha^{c}\,\beta(r,n-r+1)}\left(1+\left(\frac{x}{\alpha}\right)^{c}\right)^{-k\,(n-r+1)-1}\left(1-\left(1+\left(\frac{x}{\alpha}\right)^{c}\right)^{-k}\right)^{r-1} \qquad (9)$$

3. Estimation of the Parameters of Burr XII (*c, k, α*)

3.1. Maximum likelihood estimation

Let $X_1, X_2, \dots, X_n$ be n independent identical distributed (i.i.d) continuous random variables from Burr XII (c, k, α), and then

$$L(\underline{X};c,k) = \frac{c^n k^n}{\alpha^{nc}}\left(\prod_{i=1}^{n} x_i\right)^{c-1}\left(\prod_{i=1}^{n}\left(1+\left(\frac{x_i}{\alpha}\right)^{c}\right)\right)^{-(k+1)} \quad ; \quad x>0 \;, i=1,2,\dots,n$$

$$\ell(c,k) = n\ln c + n\ln k - nc\ln\alpha + (c-1)\left(\sum_{i=1}^{n}\ln(x_i)\right) - (k+1)\left(\sum_{i=1}^{n}\ln\left(1+\frac{x_i^{\,c}}{\alpha^{c}}\right)\right) \qquad (10)$$

The derivation of the equation (10) for c, k, and α, and then

$$\frac{\partial\ell(c,k)}{\partial k} = \frac{n}{k} - \left(\sum_{i=1}^{n}\ln\left(1+\left(\frac{x_i}{\alpha}\right)^{c}\right)\right) = 0 \qquad (11)$$

$$\frac{\partial\ell(c,k)}{\partial c} = \frac{n}{c} - n\ln\alpha + \sum_{i=1}^{n}\ln(x_i) - (k+1)\left(\sum_{i=1}^{n}\left(\frac{\left(\frac{x_i}{\alpha}\right)^{c}}{1+\left(\frac{x_i}{\alpha}\right)^{c}}\right)\ln\left(\frac{x_i}{\alpha}\right)\right) = 0 \qquad (12)$$

and

$$\frac{\partial\ell(c,k)}{\partial\alpha} = \frac{-nc}{\alpha c} - \frac{c(k+1)}{\alpha}\left(\sum_{i=1}^{n}\frac{\left(\frac{x_i}{\alpha}\right)^{c}}{1+\left(\frac{x_i}{\alpha}\right)^{c}}\right) . \qquad (13)$$

Solving equations (11), (12), and (13) together, we may find that

$$\hat{k} = \frac{n}{\left(\sum_{i=1}^{n}\ln\left(1+\left(\frac{x_i}{\hat{\alpha}}\right)^{\hat{c}}\right)\right)} \qquad (14)$$

and by numerical methods such as Newton-Raphson iteration techniques, parameters c and α could be estimated using initial values like those referred to in (SAS Institute Inc, 2015).

3.2. Sampling distribution of the estimate of the parameters - a simulation study

A simulation study has been conducted to estimate the means and the standard errors of the estimates of shape1(a), shape2(b), and scale (α) of the three-parameter Burr XII distribution at different sample sizes when the true parameters of shape1(a)=5 and shape2(b)=2. From Figures (3) to (5) and Table (1), we can observe that the estimates of parameters c and k are biased. However, as can be seen in Table (1), when the sample size increases, the amount of bias decreases and the standard error decreases, which indicates that the estimates of the parameters of the Burr XII distribution, $\hat{c}$, $\hat{k}$, and $\hat{\alpha}$ are asymptotically unbiased and consistent.

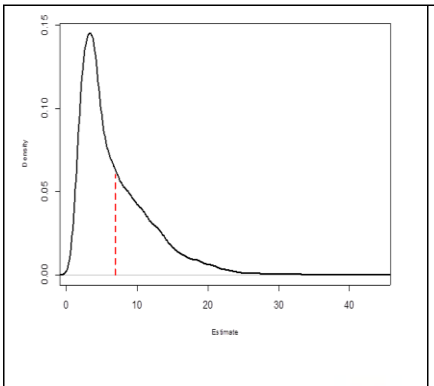

Figure (3): Sampling distribution of the estimate of the c parameter for simulated samples (n=100) from Burr XII (c, k, α) distribution with c =5, k =2, and α= 5.

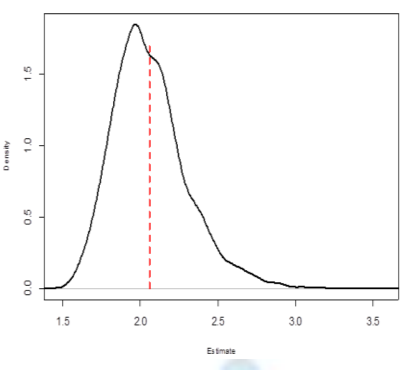

Figure (4): Sampling distribution of the estimate of the k parameter for simulated samples (n=100) from Burr XII (c, k, α) distribution with c =5, k =2, and α= 5.

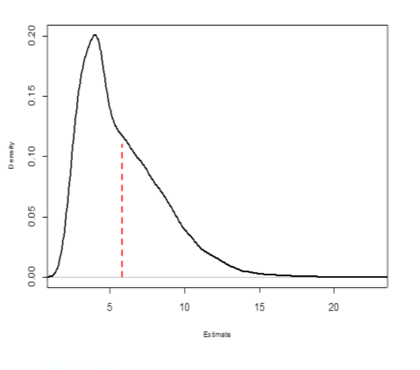

Figure (5): Sampling distribution of the estimate of the α parameter for simulated samples (n=100) from Burr XII (c, k, α) distribution with c =5, k =2, and α= 5.

Table (1): Expected values and standard errors of the estimates of shape1(a), shape2(b), and scale (alpha) of the three-parameter Burr XII distribution at different sample sizes when the true parameters are shape1(a)=5 and shape2(b)=2.

Sample size (n)	shape1(a)		shape2(b)		scale (alpha)	
	Mean	Standard Error	Mean	Standard Error	Mean	Standard Error
20	9.108531	8.080529	2.285706	0.6419146	6.616357	4.862357
30	8.438552	7.051016	2.188314	0.470612	6.34186	3.896112
50	7.84571	6.273592	2.115881	0.3525366	6.15295	3.384502
100	6.922269	5.035089	2.062169	0.242943	5.805313	2.720025
150	6.526655	4.329892	2.0428	0.1999174	5.660411	2.392129
200	6.252571	3.801473	2.031032	0.1709096	5.554979	2.126072
500	5.65462	2.558691	2.011835	0.1103243	5.303005	1.48353
1000	5.336365	1.763947	2.007569	0.0800266	5.15395	1.070736

3.3. Estimation by the method of moments

Let $X_1, X_2, \ldots, X_n$ be n independent identical distributed *iid* continuous random variables from Burr XII (c, k), and then the method of moments estimates of parameters c, k, and α are defined by:

$$E\left(x^j\right) = m(j), \quad \text{where } m(j) = \frac{1}{n}\sum_{i=1}^{n} x_i^j, \quad j = 1,2$$

$$\mu_1 = m_1 = \frac{1}{n}\sum_{i=1}^{n} x_i = \bar{X},$$

$$\mu_2' = m_2' = \frac{1}{n}\sum_{i=1}^{n} x_i^2,$$

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and $\mu_3' = m_3' = \frac{1}{n}\sum_{i=1}^{n} x_i^3.$

then we get the following system of equations:

$$\frac{n\,\alpha\,\Gamma\left(\frac{1}{c}\right)\Gamma\left(k-\frac{1}{c}\right)}{c\sum_{i=1}^{n} x_i} = \Gamma(k) \tag{15}$$

$$\frac{2n\,\alpha^2\,\Gamma\left(\frac{2}{c}\right)\Gamma\left(k-\frac{2}{c}\right)}{c\sum_{i=1}^{n} x_i^2} = \Gamma(k) \tag{16}$$

$$\frac{3n\,\alpha^3\,\Gamma\left(\frac{3}{c}\right)\Gamma\left(k-\frac{3}{c}\right)}{c\sum_{i=1}^{n} x_i^3} = \Gamma(k) \tag{17}$$

Again, we need to solve equations (15), (16), and (17) using numerical methods such as the Newton-Raphson iteration method and find an estimate of parameters c, k, and α .

4. The Doubly Truncated Three-Parameter Burr XII(c,k,α) Distribution

The truncated *pdf* of any variable takes the form given by Hattaway (2010) as follows:

$$f(x;c,k,\alpha \mid a<x<b) = \frac{f(x;c,k,\alpha)}{\int_a^b f(x;c,k,\alpha)dx}; \quad a \le x \le b, c>0, k>0, \alpha>0$$

$$= \frac{f(x;c,k,\alpha)}{F(b;c,k,\alpha)-F(a;c,k,\alpha)} \tag{18}$$

Let X be a random variable having the three-parameter doubly truncated Burr XII(c,k,α) distribution and taking values x in the interval [*a*,*b*]. The distribution function of X takes the following form:

$$f(x;c,k,\alpha,a,b) = \frac{\frac{ck}{\alpha}\left(\frac{x}{\alpha}\right)^{c-1}\left(1+\left(\frac{x}{\alpha}\right)^c\right)^{-(k+1)}}{1-\left(1+\left(\frac{b}{\alpha}\right)^c\right)^{-k}-1+\left(1+\left(\frac{a}{\alpha}\right)^c\right)^{-k}}$$

$$= \frac{ck\;x^{c-1}\left(\alpha^c+x^c\right)^{-(k+1)}}{\left(\alpha^c+a^c\right)^{-k}-\left(\alpha^c+b^c\right)^{-k}} \tag{19}$$

then $f(x;c,k,\alpha,a,b) = \gamma\, x^{c-1}\left(\alpha^c+x^c\right)^{-(k+1)}$ (20)

where the constant γ is

$$\gamma = \frac{ck}{\left(\alpha^c+a^c\right)^{-k}-\left(\alpha^c+b^c\right)^{-k}} \tag{21}$$

4.1. Cumulative distribution function of the doubly truncated Burr XII(c,k,α)

The *cdf* of the three-parameter doubly truncated *Burr XII* distribution takes the form:

$$F(x;c,k,\alpha,a,b) = \int_a^x f(x;c,k,\alpha,a,b)\;dx$$

$$= \frac{\gamma}{k\,c}\left[\left(\alpha^c+a^c\right)^{-k}-\left(\alpha^c+x^c\right)^{-k}\right] \tag{22}$$

4.2. Moment generating function of the truncated Burr XII(c,k,α) distribution

For random X that follows a Burr XII (c,k,α) distribution, the moment generating function (*mgf*) is given by

$$\mathrm{M}_x(t) = \frac{m}{\alpha}\,\mathrm{I}\left(p,\mu,\frac{m}{k},\nu\right) = \frac{m}{\alpha}\,\mathrm{I}\left(-t\alpha,\frac{m}{k}-1,\frac{m}{k},-k-1\right)$$

Now, we consider Y a random variable that follows a doubly truncated version of X with a lower truncation point, a, and an upper truncation point, b. The *mgf* of Y is

$$\mathrm{M}_y(t) = E\left(e^{ty}\right) = \int_a^b e^{ty} f\,(y;c,k,\alpha,a,b)\,dy$$

$$\mathrm{M}_y(t) = E\left(e^{ty}\right) = \int_a^b e^{ty}\,\frac{f\,(y;c,k,\alpha)}{F\,(b;c,k,\alpha)-F\,(a;c,k,\alpha)}\,dy$$

$$\mathrm{M}_y(t) = \frac{k}{F\,(b;c,k,\alpha)-F\,(a;c,k,\alpha)}$$

$$\sum_{n=0}^{\infty}\frac{(\alpha t)^n}{n!}\left[\beta\left(\left(\frac{b}{\alpha}\right)^c,\frac{n}{c}+1,k-\frac{n}{c}\right)-\beta\left(\left(\frac{a}{\alpha}\right)^c,\frac{n}{c}+1,k-\frac{n}{c}\right)\right] \qquad (23)$$

4.3. Moments and moment generating function of the truncated Burr XII (c,k,α) distribution

The r^{th} moment

The r^{th} moment of the truncated Burr XII (c,k,α) distribution is

$$E\left(x^r;c,k,\alpha,a,b\right) = \int_a^b x^r \gamma\, x^{c-1}\left(\alpha^c + x^c\right)^{-(k+1)} dx$$

$$E\left(x^r;c,k,\alpha,a,b\right) = \frac{\gamma\,\alpha^{r-ck}}{c}\left[\beta\left(\left(\frac{b}{\alpha}\right)^c,\frac{r}{c}+1,k-\frac{r}{c}\right)-\beta\left(\left(\frac{a}{\alpha}\right)^c,\frac{r}{c}+1,k-\frac{r}{c}\right)\right] \qquad (24)$$

Mean

The expected value of the truncated Burr XII (c,k,α) distribution is

$$E\left(x;c,k,\alpha,a,b\right) = \frac{\gamma\,\alpha^{1-ck}}{c}\left[\beta\left(\left(\frac{b}{\alpha}\right)^c,\frac{1}{c}+1,k-\frac{1}{c}\right)-\beta\left(\left(\frac{a}{\alpha}\right)^c,\frac{1}{c}+1,k-\frac{1}{c}\right)\right] \qquad (25)$$

Variance

Now, to obtain the variance of the truncated Burr XII (c,k,α) distribution, when r = 2, we have:

$$E\left(x^2;c,k,a,b\right) = \frac{\gamma\,\alpha^{2-ck}}{c}\left[\beta\left(\left(\frac{b}{\alpha}\right)^c,\frac{2}{c}+1,k-\frac{2}{c}\right)-\beta\left(\left(\frac{a}{\alpha}\right)^c,\frac{2}{c}+1,k-\frac{2}{c}\right)\right]$$

Therefore,

$$Var\left(x;c,k,a,b\right) = E\left(x^2;c,k,a,b\right) - \left(E\left(x;c,k,a,b\right)\right)^2$$

$$= \frac{\gamma\,\alpha^{2-ck}}{c}\left[\beta\left(\left(\frac{b}{\alpha}\right)^c,\frac{2}{c}+1,k-\frac{2}{c}\right)-\beta\left(\left(\frac{a}{\alpha}\right)^c,\frac{2}{c}+1,k-\frac{2}{c}\right)\right]$$

$$-\left(\frac{\gamma\,\alpha^{1-ck}}{c}\right)^2\left[\beta\left(\left(\frac{b}{\alpha}\right)^c,\frac{1}{c}+1,k-\frac{1}{c}\right)-\beta\left(\left(\frac{a}{\alpha}\right)^c,\frac{1}{c}+1,k-\frac{1}{c}\right)\right]^2 \qquad (26)$$

4.4. Mode of the truncated Burr XII (c,k,α)

The mode of the truncated Burr XII (c,k,α) distribution is the value x at which its *pdf* has its maximum value.

$$\frac{df\ (x;c,k,\alpha)}{dx}=\frac{d}{dx}\left[\gamma\, x^{c-1}\left(\alpha^{c}+x^{c}\right)^{-(k+1)}\right]=0$$

$$\gamma\ x^{c-2}\left(\alpha^{c}+x^{c}\right)^{-(k+2)}\left[-(ck+1)x^{c}+\alpha^{c}(c-1)\right]=0$$

then $x^{c}=0,\ x=0,\ x^{c}=-1$

$$\left[-(ck+1)\,x^{c}+\alpha^{c}\,(c-1)\right]=0,$$

$$(ck+1)\,x^{c}=\alpha^{c}\,(c-1)$$

Therefore, $x^{c}=\dfrac{\alpha^{c}(c-1)}{(ck+1)}$

Thus, $$x=\alpha\left(\frac{c-1}{c\,k+1}\right)^{\frac{1}{c}};\qquad c\ k\ \geq 1, c>0, k>0, \alpha>0 \tag{27}$$

All other roots were rejected since $x>0$. Therefore, we get the mode of the truncated distribution at $x=\alpha\left(\frac{c-1}{c\,k+1}\right)^{\frac{1}{c}}$, which is exactly the same as the mode of the original Burr XII (c,k,α) distribution.

4.5. Median of the truncated Burr XII (c,k,α)

The median of the truncated Burr XII (c,k,α) distribution is the value x at which the probability $P(X\leq x)=0.5$. Then we have:

$$\frac{\gamma}{k\ c}\ \left(\alpha^{c}+a^{c}\right)^{-k}-\left(\alpha^{c}+x^{c}\right)^{-k}=0.5$$

$$\left[\left(\alpha^{c}+a^{c}\right)^{-k}-\frac{c\ \ k}{2\,\gamma}\right]^{\frac{-1}{k}}=\left(\alpha^{c}+x^{c}\right)$$

Thus, $$x=\left(\left[\left(\alpha^{c}+a^{c}\right)^{-k}-\frac{c\ \ k}{2\,\gamma}\right]^{\frac{-1}{k}}-\alpha^{c}\right)^{\frac{1}{c}} \tag{28}$$

5. Maximum Likelihood Estimation of Parameters of Truncated Burr XII Distribution

Let $X_1, X_2, \ldots, X_n$ be *n iid* continuous random variables from the truncated Burr XII *(c,k,α)* distribution, and then the likelihood function is denoted by

$$L(\underline{X};c,k,\alpha)\ =\ \gamma^{n}\left(\prod_{i=1}^{n}x_i\right)^{c-1}\left(\prod_{i=1}^{n}\left(\alpha^{c}+x_i^{\,c}\right)\right)^{-(k+1)}\quad;\quad a<x<b\,, i=1,2,\ldots,n$$

The log-likelihood function is

$$\ell(c,k,\alpha)\ =\ n\ln\gamma+(c-1)\left(\sum_{i=1}^{n}\ln(x_i)\right)-(k+1)\left(\sum_{i=1}^{n}\ln\left(\alpha^{c}+x_i^{\,c}\right)\right)$$

$$\ell(c,k,\alpha)=n\ \ln ck-n\ \ln\left(\left(\alpha^{c}+a^{c}\right)^{-k}-\left(\alpha^{c}+b^{c}\right)^{-k}\right)+(c-1)\left(\sum_{i=1}^{n}\ln(x_i)\right)-(k+1)\left(\sum_{i=1}^{n}\ln\left(\alpha^{c}+x_i^{\,c}\right)\right) \tag{29}$$

The derivation of the equation (29) with respect to the three parameters is:

$$\frac{\partial\ell(c,k,\alpha)}{\partial k}\ =\ \frac{n}{k}\ -\frac{n\ \left(\left(\alpha^{c}+b^{c}\right)^{-k}\ln\left(\alpha^{c}+b^{c}\right)-\left(\alpha^{c}+a^{c}\right)^{-k}\ln\left(\alpha^{c}+a^{c}\right)\right)}{\left(\alpha^{c}+a^{c}\right)^{-k}-\left(\alpha^{c}+b^{c}\right)^{-k}}\ -\ \left(\sum_{i=1}^{n}\ln\left(\alpha^{c}+x_i^{\,c}\right)\right) \tag{30}$$

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$$\frac{\partial \ell(c,k,\alpha)}{\partial c} = \frac{n}{c} - \frac{n\,k\left(\left(\alpha^{c}\ln\alpha + b^{c}\ln b\right)\left(\alpha^{c}+b^{c}\right)^{-(k+1)} - \left(\alpha^{c}\ln\alpha + a^{c}\ln a\right)\left(\alpha^{c}+a^{c}\right)^{-(k+1)}\right)}{\left(\alpha^{c}+a^{c}\right)^{-k} - \left(\alpha^{c}+b^{c}\right)^{-k}}$$

$$+\sum_{i=1}^{n}\ln(x_i) - (k+1)\left(\sum_{i=1}^{n}\frac{\left(\alpha^{c}\ln(\alpha) + x_i^{\,c}\ln(x_i)\right)}{\alpha^{c}+x_i^{\,c}}\right) \qquad (31)$$

$$\frac{\partial \ell(c,k,\alpha)}{\partial \alpha} = \frac{c\;k\;n\;\alpha^{c-1}\left(\left(\alpha^{c}+a^{c}\right)^{-(k+1)} - \left(\alpha^{c}+b^{c}\right)^{-(k+1)}\right)}{\left(\alpha^{c}+a^{c}\right)^{-k} - \left(\alpha^{c}+b^{c}\right)^{-k}} - (k+1)\left(\sum_{i=1}^{n}\frac{\left(c\alpha^{c-1}\right)}{\alpha^{c}+x_i^{\,c}}\right) \qquad (32)$$

Equations (30), (31), and (32) may be solved using numerical methods to get the estimates of parameters *c, k*, and α.

6. Application of the Burr Burr XII (c,k,α) Distribution to Lifetime of Breast Cancer Patients in Gaza Strip

6.1. Data

The Burr distribution has been shown to be very important in modeling heavy tailed data since it offers greater flexibility than many other lifetime distributions. The Burr distribution has been used extensively to model deductible premiums (Burnecki et al., 2004). In this section, we apply the three-parameter Burr Type XII distribution and its truncated distribution to the survival time of breast cancer patients in Gaza Strip. We believe that these data are heavy tailed and can be a good example in which the three-parameter Burr Type XII distribution can be useful because patients of this fatal disease in this poor area receive different types of treatment locally and abroad. The data have been obtained from the Ministry of Health in Gaza City on the incidence dates and death dates of about 1,000 breast cancer patients within a period of 5 years starting from beginning of 2009 to end of 2013. The survival times for those patients were computed. Among them, 703 people were still alive at the end of 2013 and 55 patients had a zero lifetime and were believed to be wrongly reported or their records were absent upon death and thus excluded from the analysis. The remaining 242 patients have a lifetime as shown in Table (2).

Table (2) Data of lifetime (days) of breast cancer patients in Gaza Strip

11	17	25	27	29	29	29	29	29	31	34	38	38	41	47
49	57	59	59	59	59	60	65	66	73	82	89	90	104	106
118	118	118	118	139	145	145	146	147	148	148	148	158	160	161
161	164	168	173	177	178	181	189	190	196	198	205	207	207	213
219	227	231	234	236	236	237	251	257	272	280	283	284	290	292
293	294	298	300	302	314	323	323	325	326	334	337	339	347	347
351	354	355	362	363	367	367	379	402	403	403	406	406	422	423
434	439	444	449	457	457	463	466	468	470	482	484	487	494	498
499	499	502	503	504	516	520	530	532	537	538	543	548	555	558
562	574	578	583	591	593	597	607	609	616	616	629	646	647	653
663	668	674	676	679	680	682	686	698	699	716	716	722	722	726
726	729	732	735	742	745	749	752	752	764	764	770	771	776	779
784	786	795	822	830	831	837	848	889	898	906	912	916	933	962
992	995	1004	1029	1048	1052	1053	1069	1091	1120	1140	1142	1147	1156	1164
1212	1231	1238	1267	1285	1307	1332	1361	1364	1368	1380	1412	1416	1456	1554
1597	1614	1756	1790	1813	1835	1997								

Fig. (6) exhibits a histogram of the above data where it is clear that the distribution of the data is so heavy tailed and can be close to the Burr Type XII distribution with three parameters since values range from 0 to almost about 2000. Thus we considered the original three-parameter Burr Type XII distribution for these data.

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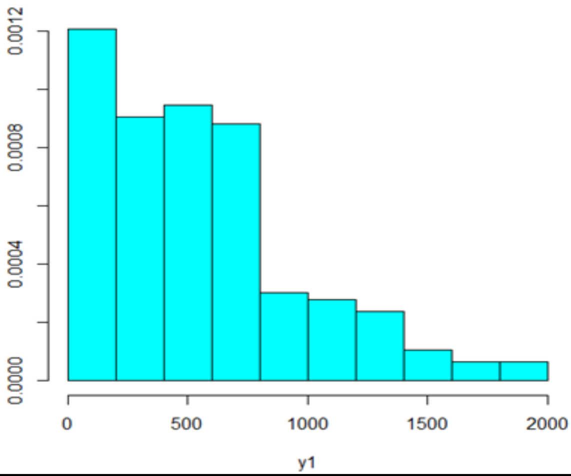


Figure (6): Histogram of the lifetime of breast cancer patients in Gaza Strip data

6.2. Estimation of the parameters of the Burr XII (c, k, α) distribution from the lifetime breast cancer patients' data in Gaza Strip

If the breast cancer patients in the Gaza Strip data follow Burr XII (c, k, α) distribution, then the estimates of three parameters using the R software and initial value= 470 are as follows:

c= 10809.77 k= 1.258115 α = 953601.9

6.3. Expected lifetime of breast cancer patients from original lifetime data

The expected mean survival time of breast cancer patients based on these data and the three-parameter Burr Type XII distribution using the R software is 551.6342 day, which is roughly a year and a half; and the variance and standard deviation are 194820.8 and 441.3851, respectively. The mean survival time of breast cancer patients seems to be high for several reasons, including the type of cancer and the benign or malignant nature and type of treatment.

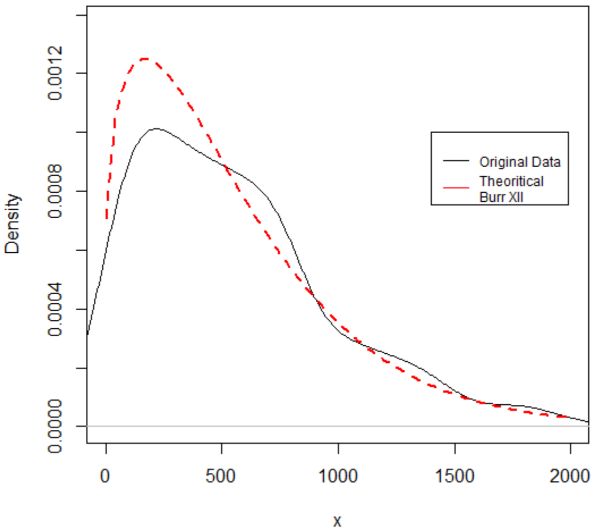
6.4. Estimation of the parameters of truncated Burr XII (c, k, α) distribution from the lifetime breast cancer patients' data in Gaza Strip

If we consider the case that breast cancer patients in Gaza Strip data follow the doubly truncated Burr XII (c, k, α) distribution at (a= 100, b= 1000) by trimming the original data, then the estimations of three parameters based on the truncated three-parameter Burr Type XII distribution and using the R software and initial value= -1142 are:

c= 10827.86 k= 1.188897 α= $5.521534 \times (10)^{-7}$

6.5. Expected lifetime of breast cancer patients from truncated data

The expected mean survival time for breast cancer patients in Gaza based on the truncated data and three-parameter truncated Burr Type XII distribution is 479.155 days, with a variance and standard deviation of 61751.89 and 248.4993, respectively. We note that the mean and variance of the truncated data are much lower than those of the original data. This is because of the heavy tail of the distribution of the data and the existence of some extreme values with a lifetime more than 1500 days inflated both the expected mean survival time and its standard deviation above those of the original distribution. Fig. (7) compares the estimated kernel density estimate of the distribution of the original breast cancer patients' data with a (black) solid line and the theoretical distribution of Burr XII with a (red) dotted line; this shows how close the distribution of the original data is to the theoretical Burr XII distribution.



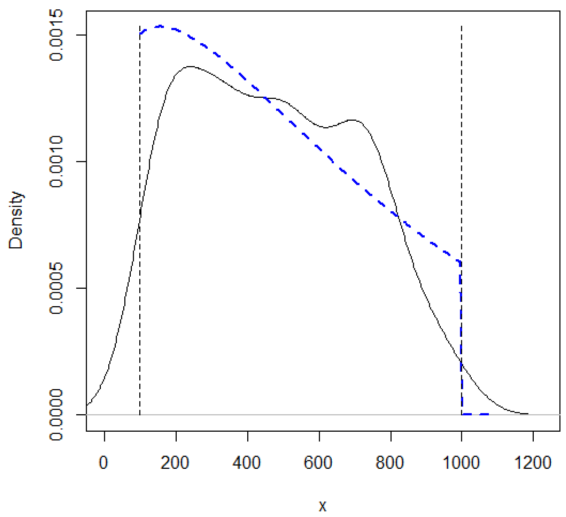


Figure (7): Comparison between the distribution of the original data and the estimated Burr XII distribution.	Figure (8): Comparison between the distribution of truncated data and the estimated truncated Burr XII distribution

Fig. (8) however, compares the kernel density estimate of truncated breast cancer patients' data with a (black) solid line with the theoretical truncated distribution of Burr XII with a (blue) dotted line, where they show that the distribution of the truncated data so close to the theoretical truncated three-parameter Burr XII distribution and much closer than the curves in Fig. (7) of the original distribution.

Now, if we used the estimated values of parameters c, k, and α of the three-parameter truncated Burr XII (c, k, α) distribution and substituted them in the original Burr XII (c, k, α) distribution, we get a closer fit to the data. This means that the estimated values of parameters c, k, and α using the truncated Burr XII (c, k, α) distribution are more accurate than the estimated values of c, k, and α parameters using the original Burr XII (c, k, α) distribution. To illustrate this, Fig. (8) shows the kernel density of the original data with a (black) solid line and the theoretical distribution of the original Burr XII with (red) dotted lines with parameters equal to the estimated values of c, k, and α parameters using the three-parameter truncated Burr XII (c, k, α) distribution. This means that if we trim the data and estimate parameters c, k, and α using the parameter estimators of the three-parameter truncated Burr XII (c, k, α) distribution, we get more accurate estimates of parameters c, k, and α.

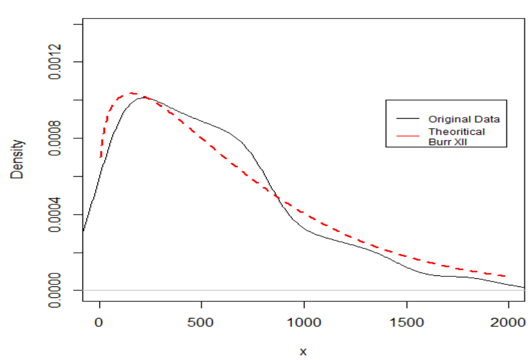


Figure (9): Comparison between the kernel density of the original data and the estimates of the theoretical three-parameter Burr XII using the parameters' estimates from the truncated data.

7. Concluding remarks:

Based on all the above illustrations of the importance of the three-parameter Burr XII distribution in applied statistics, especially in lifetime data analysis, we may conclude the following points:

1. The three-parameter Burr XII distribution provides a very good fit to breast cancer patients' data in Gaza Strip.
2. The analysis of breast cancer patients' data supported all the theatrical results.

3. The expected mean survival time of breast cancer patients based on the data and the three-parameter Burr Type XII distribution is 551.63 days.
4. The expected mean survival time for breast cancer patients in Gaza Strip based on the truncated data and three-parameter truncated Burr Type XII distribution is 479.16 days. This result seems to be more accurate than the previous result.
5. The Burr XII distributions and truncated Burr XII distributions are important to use in many applications.
6. The three-parameter Burr XII distribution should be used to fit survival and lifetime data in survival analysis and many applications such as economics, quality assurance, and environmental applications.
7.

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