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Additive Lie derivations on the algebras of locally measurable operators

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Abstract

Let M be a von Neumann algebra without central summands of type I. *We* are studying conditions that an additive map L on the algebra of locally measurable operators has the standard form, that is equal to the sum of an additive derivation and an additive center-valued trace.

Key words: von Neumann algebras, locally measurable operator, derivation, additive derivation, additive Lie derivation, center-valued trace.

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INTRODUCTION

The structure of Lie derivations on C*-algebras and on more general Banach algebras has attracted some attention over the past years. Let A be an algebra over the complex number. An additive (linear) operator $D: A \to A$ is called an *additive derivation (linear derivation)* if $D(xy) = D(x)y + xD(y)$ for all $x, y \in A$ (Leibniz rule). Each element $a \in A$. defines linear associative a derivation D_a on A given as $D_a(x) = ax - xa$, $x \in A$. Such derivations D_a are said to be *inner derivations.* If the element *a* implementing the derivation D_a on A, belongs to a larger algebra B, containing A (as a proper ideal as usual) then D_a is called a *spatial derivation.* An additive (linear) operator $L: A \to A$ is called an *additive Lie derivation (linear Lie derivation)*if $L([x, y]) = [L(x), y] + [x, L(y)]$, for all $x, y \in A$, where $[x, y] = xy - yx$.

Denote by $Z(A)$ the center of A.

An additive (linear) operator $\tau : A \to Z(A)$ is called *an additive centervalued trace (a linear center-valued trace*) if $\tau(xy) = \tau(yx)$), $\forall x, y \in A$. The problem of the standard decomposition for a Lie derivation in rings theory was studied in work by W. S. Martindale [9]. W. S. Martindale solved this problem for primitives rings containing nontrivial idempotents and with the characteristic unequal to 2. Following these results obtained for rings, C. Robert Miers in [11] solved the problem of the standard decomposition for the case of von Neumann algebras. In the present work we are studyingconditions that an additive map L on $LS(M)$ has the standard form, that is equal to the sum of an additive derivation and an additive center-valued trace.

Development of the theory of algebras measurable operators $S(M)$ and the algebra of locally measurable operators $LS(M)$ affiliated with von Neumann algebra or AW^* algebras M [6], [10] provided an opportunity to construct and learn new interesting examples of $*$ -algebras unbounded operators.

We us terminology and notations from the von Neumann algebra theory [7] and the theory of locally measurable operators from [10].

Let H be a complex Hilbert space, $B(H)$ be the algebra of all bounded linear operators acting in H, M be a von Neumann algebra in $B(H)$, $P(M)$ be a complete lattice of all orthoprojections in M .

Let *H* be a Hilbert space, *B*(*H*) be the algebra of all bounded linear operators acting in *H*, *M* be a von Neumann subalgebra in *B*(*H*), *P*(*M*) be a complete lattice of all orthoprojections in *M*.

A linear subspace *D* on *H* is said to be *affiliated* with*M* (denoted as *D*η*M*), if *u*(*D*)⊆*D* for every unitary operator *u* from the commutant $M' = \{y \in B(H) : xy = yx, \forall x \in M\}$ of the algebra *M*.

A linear operator *x* on *H* with the domain *D*(*x*) is said to be *affiliated* with*M* (denoted as *x*η*M*), if and *ux*(ξ)=*xu*(ξ) for every unitary operator *u*∈*M*', and all ξ∈*D*(*x*).

A linear subspace *D* in *H* is said to be *strongly dens* in *H* with respect to the von Neumann algebra *M*, if

1) *D*η*M*,
2) there exists a sequence of projections $\{p_n\}_{n=1}^{\infty} \subset P(M)$, such that $p_n \uparrow \mathbf{1}$, $p_n(H) \subset \mathrm{D}$, and $p_n^{\perp} = \mathbf{1} - p_n$ is finite in M for all $n \in \mathrm{N}$, where $\mathbf{1}$ is the identity M.

A closed linear operator *x*, on a *H*, is said to be *measurable* with respect to the von Neumann algebra *M*, if *x*η*M*, and *D*(*x*) is strongly dens in *H*. Denote by *S*(*M*) the set of all measurable operators affiliated with *M* (see. [5,11]) and the center of an algebra *S*(*M*) by *Z*(*S*(*M*)).

A closed linear operator x in H is said to be locally measurable with respect to the von Neumann algebra M ; if *x*η*M*, and there exists a sequence $\{z_n\}_{n=1}^{\infty}$ of central projections in M such that $z_n \uparrow 1$ and $xz_n \in S(M)$ for all $n \in N$. It is well-known [11] that the set $LS(M)$ of all locally measurable operators with respect to M is a unital$*$-algebra when equipped with the algebraic operations of strong addition and multiplication and taking the adjoint of an operator. Note that if M is a finite von Neumann algebra then $S(M) = LS(M)$.

Denote by $Z(LS(M))$ the center of $LS(M)$.

Let M be a von Neumann algebra without central summands of type I_1.

Let $L: LS(M) \to LS(M)$ is an additive map.If p_i, p_j are projectors in $S(M)$, then $p_i LS(M) p_j = \{p_i A p_j : A \in LS(M)\}, i, j = 1,2.$ Set $p_1 = p$ and $p_2 = 1 - p.$ Then $LS(M) = \sum_{i=1}^{2}\sum_{j=1}^{2} p_i LS(M) p_j.$ Let further $S_{ij} = p_i LS(M) p_j, i, j = 1,2.$ Recall that $S_{ij} = S_{ik}S_{kj}$, for $i, j = 1,2$.

In this paper is established of the standard form of additive Lie derivation, acting on algebra of LS(M) when M be a von Neumann algebra without central summands of type I.

In particular, it follows that the properly infinite von Neumann algebras M., all additive Lie derivations operations on the arbitrarily algebras LS(M), is the linear derivations

RESULTS

Lemma 1. *If* $x \in S_{ij}$ *and* $xy = 0$ *for all* $y \in S_{jk}$, then $x = 0$.

Lemma 2. $\mathrm{pL}(\mathrm{p})\mathrm{p} + (1\text{-p})\mathrm{L}(\mathrm{p})(1\text{-p}) \in \mathrm{Z}(\mathrm{LS}(\mathrm{M})).$

Let $\delta: LS(M) \to S(M)$ defined as follows: $\delta(x) = \mathrm{L}(x) + sx - xs$ for each $x \in LS(M)$.

We have the following

Lemma 3. $\mathrm{p}\delta(1)(1\text{-p}) = (1\text{-p})\delta(1)\mathrm{p}=0$.

Lemma 4. $L(S_{ij}) \subset S_{ij}$, *where* *I, j=1,2,* $i \neq j$.

Lemma 5. *There exists a map* $f_i : S_{ii} \to Z(LS(M))$ such *that* $\delta(x_{ii}) \in S_{ii} + f_1(x_{ii})$ all $x_{ii} \in S_{ii}$, $i, j = 1,2$.

Now defined the mappings $f: LS(M) \to Z(LS(M))$ and $d: LS(M) \to S(M)$ as follows: $f(x) = f_1(x_{11}) + f_2(x_{22})$ and $\mathrm{d}(x) = \delta(x) - f(x)$ all $x_{11} + x_{12} + x_{21} + x_{22} \in LS(M)$. Then by Lemma 4 and 5, we obtain $d(S_{ij}) \subseteq S_{ij}, d(S_{ii}) \subseteq S_{ii}$, $d(S_{ij}) = \delta(S_{ij})$, $1 \le i \ne j \le 2$

Lemma 6. d *and* f are *additive.*

Lemma 7. The *mapping* d is *derivation.*

Lemma 8. $f([x, y]) = 0$ *for all* $x, y \in LS(M)$ *,where* $xy = 0$

Now we can formulate the main theorem.

Theorem 1. Let $LS(M)$ be *of all locally measurable operators affiliated with a von Neumann algebra* M *without central summands of type* I_1. *Let* $L: LS(M) \to LS(M)$ *additive mapping. Then* $L([x, y]) = [L(x), y] + [x, L(y)]$, *for all* $x, y \in LS(M)$ *,where* $xy = 0$, *if and only if there exists an additive derivation* $\varphi: LS(M) \to LS(M)$ *and an additive map* $f: LS(M) \to Z(LS(M))$ *where* $f([x, y]) = 0$, such *that* $L(x) = \varphi(x) + f(x)$, $x \in LS(M)$, where $Z(LS(M))$ *center of* $LS(M)$.

Now Theorem 1 implies the following

*Corollary.*Let $LS(M)$ be of all locally measurable operators affiliated with a von Neumann algebra M without central summands of type I_1. Suppose that $L: LS(M) \to LS(M)$ is an additive map. Then is a Lie derivation if and only if there exists an additive derivation

$\varphi: LS(M) \to LS(M)$ and an additive map $f: LS(M) \to Z(LS(M))$, where $f([x,y]) = 0$, such that $L(x) = \varphi(x) + f(x)$ for all $x \in LS(M)$, *where*

$Z(LS(M))$ center of $LS(M)$.

Theorem 2. Let $LS(M)$ be *of all locally measurable operators affiliated with a von Neumann algebra* M *without central summands of type* I_1. *Then any additive Lie derivation* $L: LS(M) \to LS(M)$ *can be represented in the form* $L = \varphi + f$, *where* φ *- additive derivation on the algebra* $LS(M)$ *and* f *- additive* $Z(LS(M))$*-valued trace on the* $LS(M)$.

Theorem 3. *If* M *is a type* I or III *von Neumann algebra, then any additive Lie derivation* $L: LS(M) \to LS(M)$ is *linear Lie derivation and has the form* $L = D_a + f$, *where* D_a *- is inner derivation on the algebra* $LS(M)$ *and* f *-is linear* $Z(LS(M))$*-valued trace on the* $LS(M)$.

Corollary.*Let* M *be a von Neumann algebra of type* I_∞. *Then any additive Lie derivation* $L: LS(M) \to LS(M)$ is *linear derivation.*

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