ISSN 2347-1921

Neumann-Boundary Stabilization of the Wave Equation with Internal Damping Control and applications

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Abstract

This paper is devoted to the Neumann boundary stabilization of a non-homogeneous n-dimensional

wave equation subject to static or dynamic boundary conditions. Using a linear feedback law involving only an internal term, we prove the well-posedness of the considered systems and provide a simple method to obtain an asymptotic convergence result for the solutions. The method consists of proposing a new energy norm, and applying the semigroup theory and LaSalle's principle. Finally, the method presented in this work is also applied to several distributed parameter systems such as the Petrovsky system, coupled wave-wave equations and elastic system.

Keywords

Stability; semigroup; LaSalle's principle; Petrovsky system; coupled wave wave equations and elastic system

Council for Innovative Research

Peer Review Research Publishing System

Journal: JOURNAL OF ADVANCES IN MATHEMATICS

Vol .10, No.4

www.cirjam.com , editorjam@gmail.com

1 Introduction

Let Ω be a bounded open connected set in $\Re^n$ having a smooth boundary $\Gamma = \partial\Omega$ of class C^2. Consider the following wave equation:

$$y_{tt}(x,y) - Ay(x,t) + a(x)y_t(x,t) = 0. \qquad \text{in } \Omega\times(0,\infty) \qquad (1.1)$$

with either static Neumann boundary conditions and initial conditions

$$\begin{cases} \partial_A y(x,t) = 0, & \text{on } \Gamma\times(0,\infty) \\ y(x,0) = y_0(x),\ y_t(x,0) = z_0(x) & \text{in } \Omega, \end{cases} \qquad (1.2)$$

or dynamical boundary conditions and initial conditions

$$\begin{cases} m(x)y_{tt}(x,t) + \partial_A y(x,t) = 0 & \text{on } \Gamma\times(0,\infty) \\ y(x,0) = y_0(x),\ \ y_t(x,0) = z_0(x) & \text{in } \Omega \\ y_t\big|_\Gamma (x,0) = w_0(x) & \text{in } \Omega \end{cases} \qquad (1.3)$$

Where $A = \sum_{i,j=1}^{n} \partial_i (a_{ij}\partial_j),\ \partial_A = \sum_{i,j=1}^{n} a_{ij}\nu_i\partial_j, \partial_k = \dfrac{\partial}{\partial x_k},\ \ \nu = (\nu_1,....,\nu_n)$ is the unit normal of Γ pointing towards the exterior of Ω and $a_{ij} \in C^1(\bar{\Omega})$, with $a_{i,j} = a_{j,i},\ \forall i,j = 1,....,n$ and satisfying, for

$$\alpha_0 > 0, \sum_{i,j=1}^{n} a_{i,j}\varepsilon_i\varepsilon_j \geq \alpha_0 \sum_{i=1}^{n} \varepsilon_i^2,\ \ \forall(\varepsilon_1,...,\varepsilon_n) \in \mathbb{R}^n.$$

Moreover, there exist two positive constants a_0 and m_0 for which

$$a \in L^\infty(\Omega); a(x) \geq a_0,\ \ a.e\, x \in \Omega. \qquad (1.4)$$

$$m(x) \in L^\infty(\Gamma) : m(x) \geq m_0, \quad a.e\, x \in \Gamma. \qquad (1.5)$$

The main results for this article are:

1) For any initial data $(y_0, z_0) \in H^1(\Omega)\times L^2(\Omega),$ the solution of the closed loop system (1.1) - (1.2) and (1.5) satisfy $(y(t), y_t(t)) \to (\chi, 0)$ in $H^1(\Omega)\times L^2(\Omega)$ as $t \to \infty$, where

$$\chi = \left(\int_\Omega a\, dx\right)^{-1} \left(\int_\Omega (ay_0 + z_0)\, dx\right). \qquad (1.6)$$

2) For any initial data $(y_0, z_0, w_0) \in H^1(\Omega)\times L^2(\Omega)\times L^2(\Gamma)$, the solution of the closed loop system (1.1) - (1.3) and (1.5)-(1.6) satisfy $(y(t), y_t(t), y_t\big|_\Gamma(t)) \to (\chi, 0, 0)$ in $H^1(\Omega)\times L^2(\Omega)\times L^2(\Gamma)$ as $t \to \infty$, where

$$\chi = \left(\int_\Omega a\, dx\right)^{-1} \left(\int_\Omega (ay_0 + z_0)\, dx\right). \qquad (1.7)$$

2 The Wave Equation with Static Boundary Conditions

2.1 Preliminaries and well-posedness of the problem

In this subsection we study the existence and uniqueness of the solutions of the closed loop system (1.1)- (1.2) and (1.4). Let us consider the state space

$$\Upsilon = H^1(\Omega)\times L^2(\Omega)$$

equipped with the inner product

$$\langle (y,z),(\tilde{y},\tilde{z})\rangle_{\Upsilon} = \int_{\Omega}\left(\sum_{i,j=1}^{n} a_{ij}\partial_i y\,\partial_j \tilde{y} + z\tilde{z}\right)dx + \varepsilon\left(\int_{\Omega}(z+ay)\,dx\right)\left(\int_{\Omega}(\tilde{z}+a\tilde{y})\,dx\right), \quad (2.1) \qquad (3.1)$$

where $\varepsilon > 0$ is a constant to be determined. This inner product is inspired from the approach of [13] introduced for the boundary feedback case. The first result is stated in the following proposition.

Proposition 2.1. The state space $\Upsilon = H^1(\Omega)\times L^2(\Omega)$ 'endowed with the inner product (2.1) is a Hilbert space provided that ε is small enough.

Proof. It is sufficient to show that the norm $\|\cdot\|_{\Upsilon}$ induced by the inner product (2.1) is equivalent to the usual one $\|\cdot\|_{H^1(\Omega)\times L^2(\Omega)}$; that is, proving the existence of two positive constants K and $\tilde{K}$ such that

$$K\|(y,z)\|^2_{H^1(\Omega)\times L^2(\Omega)} \le \|(y,z)\|^2_{\Upsilon} \le \tilde{K}\|(y,z)\|^2_{H^1(\Omega)\times L^2(\Omega)}. \qquad (2.2)$$

On one hand,

$$\|(y,z)\|^2_{H^1(\Omega)\times L^2(\Omega)} = \int_{\Omega}\left(\sum_{i,j=1}^{n} a_{i,j}\partial_i y\,\partial_j y\right)dx + \int_{\Omega} z^2 dx + \varepsilon\left(\int_{\Omega}(z+ay)dx\right)^2$$

Applying Hölder's inequality and Young's inequality, we get

$$\|(y,z)\|^2_{H^1(\Omega)\times L^2(\Omega)} \le \frac{1}{2}\int_{\Omega}\sum_{i,j=1}^{n}\sup_{x\in\Omega}|a_{ij}(x)|\left((\partial_i y)^2 + (\partial_j y)^2\right)dx + \int_{\Omega} z^2 dx$$
$$+\varepsilon\, vol(\Omega)\int_{\Omega}(|z|+|a||y|)^2 dx.$$

Let $a_1 = \max_{i,j}\sup_{x\in\Omega}|a_{ij}(x)|$, and using the fact that $2|a||y||z| \le \|a\|_{\infty}(y^2+z^2)$, we get

$$\|(y,z)\|^2_{H^1(\Omega)\times L^2(\Omega)} \le \frac{1}{2}a_1\left(n\int_{\Omega}|\nabla y|^2 dx + n\int_{\Omega}|\nabla y|^2 dx\right) + \int_{\Omega} z^2 dx$$
$$+\varepsilon\, vol(\Omega)\int_{\Omega}(z^2 + \|a\|^2_{\infty} y^2 + \|a\|_{\infty} y^2 + \|a\|_{\infty} z^2)\,dx$$
$$= na_1\int_{\Omega}|\nabla y|^2 dx + \varepsilon\, vol(\Omega)\|a\|_{\infty}(\|a\|_{\infty}+1)\int_{\Omega} y^2 dx$$
$$+\left(1+\varepsilon\, vol(\Omega)(\|a\|_{\infty}+1)\right)\int_{\Omega} z^2\, dx.$$

Let $\delta_0 = \varepsilon\, vol(\Omega)\|a\|_{\infty}(\|a\|_{\infty}+1)$ and $\beta_0 = 1+\varepsilon\, vol(\Omega)(\|a\|_{\infty}+1)$ and $K = \max\{na_1, \alpha_0, \beta_0\}$. Then

$$\|(y,z)\|^2_{\Upsilon} \le K\|(y,z)\|^2_{H^1(\Omega)\times L^2(\Omega)} \qquad (2.3)$$

On the other hand, using the coercivity of (a_{ij})

$$\|(y,z)\|^2_{\Upsilon} \ge \alpha_0\sum_{i=1}^{n}\int_{\Omega}(\partial_i y)^2\, dx + \int_{\Omega} z^2 dx + \varepsilon\left[\left(\int_{\Omega} z\,dx\right)^2 + \left(\int_{\Omega} ay\,dx\right)^2 + 2\left(\int_{\Omega} z\,dx\right)\left(\int_{\Omega} ay\,dx\right)\right].$$

$$2\varepsilon\left(\int_\Omega z\,dx\right)\left(\int_\Omega ay\,dx\right)\geq-\varepsilon\left[\delta\left(\int_\Omega ay\,dx\right)^2+\frac{1}{\delta}\left(\int_\Omega z\,dx\right)^2\right] \quad ,\forall\delta\succ 0.$$

Then

$$\|(y,z)\|_\Upsilon^2\geq\alpha_0\int_\Omega|\nabla y|^2\,dx+\int_\Omega z^2dx+\varepsilon(1-\delta)\left(\int_\Omega ay\,dx\right)^2+\varepsilon(1-\frac{1}{\delta})\left(\int_\Omega z\,dx\right)^2.$$

Using generalized Poincare′'s inequality [3], we can prove that there exists a positive constant c_0 such that

$$\int_\Omega y^2\,dx\leq c_0\left[\int_\Omega|\nabla y|^2+\left(\int_\Omega ay\,dx\right)^2\right], \qquad \forall y\in H^1(\Omega) \tag{2.4}$$

which implies that

$$\left(\int_\Omega ay\,dx\right)^2\geq\frac{1}{c_0}\int_\Omega y^2dx-\int_\Omega|\nabla y|^2\,dx. \tag{2.5}$$

Now, for $0<\delta<1$

$$\|(y,z)\|_\Upsilon^2\geq\left(\alpha_0-\varepsilon(1-\delta)\right)\int_\Omega|\nabla y|^2\,dx+\frac{\varepsilon(1-\delta)}{c_0}\int_\Omega y^2\,dx+\left(1+\varepsilon\left(1-\frac{1}{\delta}\right)vol(\Omega)\right)\int_\Omega z^2dx.$$

We choose $\varepsilon>0$ and $0<\delta<1$ such that the coefficients of $\int_\Omega|\nabla y|^2\,dx, \int_\Omega y^2dx$ and $\int_\Omega z^2dx$ are positive; that is

$\alpha_0-\varepsilon(1-\delta)>0$, which implies that $\varepsilon<\dfrac{\alpha_0}{1-\delta}$.

$1+\varepsilon(1-\dfrac{1}{\delta})vol(\Omega)>0$, then $\varepsilon<\dfrac{1}{\left(\dfrac{1}{\delta}-1\right)vol(\Omega)}$.

Because $0<\delta<1$ and $\alpha_0>0$, it is sufficient to choose $\varepsilon>0$ such that

$$\varepsilon<\min\left\{\frac{\alpha_0}{1-\delta},\frac{1}{\left(\frac{1}{\delta}-1\right)vol(\Omega)}\right\}.$$

On the other hand, $c_0>0$, so $\dfrac{\varepsilon(1-\delta)}{c_0}>0$.

Finally,

$$\|(y,z)\|_\Upsilon^2\geq K\|(y,z)\|_{H^1(\Omega)\times L^2(\Omega)}^2, \tag{2.6}$$

where $K=\min\left\{\alpha_0-\varepsilon(1-\delta),\dfrac{\varepsilon(1-\delta)}{c_0},1+\varepsilon\left(1-\dfrac{1}{\delta}\right)vol(\Omega)\right\}$.

From (2.3) and (2.6) we get

ISSN 2347-1921

$$K\|(y,z)\|^2_{H^1(\Omega)\times L^2(\Omega)} \le \|(y,z)\|^2_{\Upsilon} \le \tilde{K}\|(y,z)\|^2_{H^1(\Omega)\times L^2(\Omega)}.$$

Therefore, the state space $\Upsilon = H^1(\Omega)\times L^2(\Omega)$ endowed with the inner product (2.1) is a Hilbert space.

We turn now to the formulation of the closed-loop system (1.1)- (1.2) and (1.4) in an abstract form in Υ. Let $z(t) = y_t(t)$ and $\Phi(t) = (y(t), z(t))$. Then, the closed loop system (1.1)-(1.2) and (1.4) can be written as

$$\begin{cases} \Phi_t(t) + \mathrm{T}\Phi(t) = 0 \\ \Phi(0) = \Phi_0 = (y(0), z(0)) = (y_0, z_0) \end{cases} \qquad (2.7)$$

where T is an unbounded linear operator defined by:

$$\mathrm{T}(y,z) = (-z, -Ay + az), \qquad \forall (y,z)\in D(\mathrm{T}) \qquad (2.8)$$

and

$$\begin{aligned} D(\mathrm{T}) &= \left\{(y,z)\in H^1(\Omega)\times L^2(\Omega) : \mathrm{T}(y,z)\in H^1(\Omega)\times L^2(\Omega) \text{ and } \partial_A y = 0 \text{ on } \Gamma\right\} \\ &= \left\{\begin{matrix}(y,z)\in H^1(\Omega)\times L^2(\Omega) : (-z, -Ay+az)\in H^1(\Omega)\times L^2(\Omega) \\ \text{and } \partial_A y = 0 \text{ on } \Gamma\end{matrix}\right\} \\ &= \left\{\begin{matrix}(y,z)\in H^1(\Omega)\times L^2(\Omega) : -z\in H^1(\Omega), -Ay+az\in L^2(\Omega) \\ \text{and } \partial_A y = 0 \text{ on } \Gamma\end{matrix}\right\} \\ &= \left\{(y,z)\in H^2(\Omega)\times H^1(\Omega),\ \partial_A y = 0 \text{ on } \Gamma\right\}. \end{aligned}$$

By using variational formulation and Lax-Milgram theorem [3], we conclude that T is a maximal monotone operator.

2.2 Stabilization of the problem

In this subsection, we prove a stability result which is similar to the one obtained in [13]

for the boundary feedback case.

Definition 2.2. The ω-limit set is

$$\omega(y_0, z_0) = \left\{\begin{matrix}(\omega_1,\omega_2)\in\Upsilon : \exists\{t_n\} \text{ increasing sequence of positive numbers;} \\ \lim_{n\to\infty}\left\|\left(y(t_n), z(t_n)\right) - (\omega_1,\omega_2)\right\| = 0\end{matrix}\right\}.$$

Theorem 2.2. For any initial data $\Phi_0 = (y_0, z_0)\in\Upsilon,$ the solution $\Phi(t) = (y(t), Z(t)) \to (\chi, 0)$ in Υ as $t\to\infty$, where $\chi = \left(\int_\Omega a\,dx\right)^{-1}\left(\int_\Omega (ay_0 + z_0)\,dx\right)$; that is, $\lim_{t\to\infty}\|(y(t), z(t)) - (\chi, 0)\|^2_\Upsilon = 0$.

Proof. Applying LaSalle's principle [24], we have:

i) $\omega(y_0, z_0) \ne \phi,\ \forall (y_0, z_0)\in\Upsilon$ and it is compact set.

ii) $\omega(y_0, z_0)$ is invariant under the semi-group $S(t)$.

iii) Let $(y(t), z(t)) = S(t)(y_0, z_0)$ be a solution of (2.7), then $\lim_{t\to\infty}(y(t), z(t))\in\omega(y_0, z_0)$.

iv) $\omega(y_0, z_0) \subset D(\mathrm{T})$

v) $t \to \|S(t)\omega\|_{\Upsilon}^2$ is a constant function for any $(\omega_1, \omega_2) \in \omega(y_0, z_0)$.

We want to prove that $(y(t), z(t)) \to (\chi, 0)$, as t goes to ∞.

From (iii), it is sufficient to prove that $\omega(y_0, z_0)$ contains only elements of the form $(\chi, 0)$.

Let $\omega_0 \in \omega(y_0, z_0)$, we prove that $\omega = (\chi, 0)$ we have

$\frac{d}{dt}\left(\|S(t)\omega_0\|_{\Upsilon}^2\right) = 0 \Rightarrow \left\langle \frac{d}{dt}(S(t)\omega_0), S(t)\omega_0 \right\rangle_{\Upsilon} = 0 \Rightarrow \left\langle \frac{d}{dt}\omega(t), \omega(t) \right\rangle_{\Upsilon} = 0$, where $\omega(t) = (y(t), z(t))$ is the solution of (2.7) corresponding to ω_0.

$\langle \mathrm{T}\omega(t), \omega(t) \rangle_{\Upsilon} = \int_{\Omega} az^2 dx = 0$. But $a(x) \ge a_0 > 0$,thus $z = 0 \text{ on } \Omega$.

Because $y_t = z = 0$, then y is a constant with respect to t. Then $y_{tt} = 0$ and $Ay = 0$.

Therefore, using Green's formula

$-\int_{\Omega} Ay\, y\, dx = -\int_{\Gamma} \partial_A y\, y\, d\sigma + \int_{\Omega} \sum_{i,j=1}^{n} a_{ij} \partial_j y \partial_i y\, dx = \int_{\Omega} \sum_{i,j=1}^{n} a_{ij} \partial_j y \partial_i y\, dx = 0$. But,

$\int_{\Omega} \sum_{i,j=1}^{n} a_{ij} \partial_i y \partial_j y\, dx \ge \alpha_0 \int_{\Omega} \sum_{i=1}^{n} (\partial_i y)^2 dx$, where $\alpha_0 > 0$, therefore y is a constant with respect to x.

Finally, $y = \chi$ where χ is a constant.

Hence the $\omega - \lim$ set contains only elements of the form $(\chi, 0)$, where χ is a constant, and we find $\lim_{t \to \infty} (y(t), z(t)) = (\chi, 0)$.

Now, we have to find the expression of χ. Because

$y_{tt}(x,t) - Ay(x,t) + a(x)\, y_t(x,t) = 0$ in $\Omega \times (0, \infty)$ and $\partial_A y = 0$ on Γ, then

$\left(\int_{\Omega} \left(y_t(x,t) + a(x) y(x,t)\right) dx\right)' = \int_{\Omega} Ay\, dx = \int_{\Gamma} \partial_A y = 0$, therefore

$\int_{\Omega} \left(y_t(x,t) + a(x) y(x,t)\right) dx$ is a constant function.

Thus,

$$\int_{\Omega} \left(y_t(x,t) + a(x) y(x,t)\right) dx = \int_{\Omega} \left(y_t(x,0) + a(x) y(x,0)\right) dx = \int_{\Omega} \left(z_0 + a y_0\right) dx, \quad \forall t \in (0, \infty)$$

By passing to the limit where t goes to ∞, and using the fact that

$\lim_{t \to \infty} (y(t), z(t)) = (\chi, 0)$, we get $\int_{\Omega} \left(0 + a(x)\chi\right) dx = \int_{\Omega} \left(z_0 + a y_0\right) dx$, this implies that

$\chi \int_{\Omega} a\, dx = \int_{\Omega} (z_0 + a y_0) dx$, so $\chi = \left(\int_{\Omega} a\, dx\right)^{-1} \int_{\Omega} (z_0 + a y_0) dx$. ■

3 The Wave Equation With Dynamic Boundary Conditions

3.1 Preliminaries and well-posedness of the problem

In this subsection we study the existence and uniqueness of the solutions of the closed-loop system (1.1)- (1.3) and (1.4)-(1.5). Let us consider the state space

$$\Upsilon_d = H^1(\Omega)\times L^2(\Omega)\times L^2(\Gamma),$$

equipped with the inner product

$$\langle (y,z,w),(\tilde{y},\tilde{z},\tilde{w})\rangle_{\Upsilon_d} = \int_\Omega \left(\sum_{i,j=1}^{n} a_{ij}\partial_i y\,\partial_j \tilde{y} + z\tilde{z} \right) dx + \int_\Gamma m\,w\,\tilde{w}\,d\sigma$$
$$+\mu\left(\int_\Omega (z+ay)\,dx + \int_\Gamma m\,w\,d\sigma \right)\left(\int_\Omega (\tilde{z}+a\tilde{y})\,dx + \int_\Gamma m\,\tilde{w}\,d\sigma \right), \tag{3.1}$$

Where $\mu \succ 0$ is a constant to be determined. This inner product is inspired from the

approach of [13] introduced for the boundary feedback case. The first result is stated in the following proposition.

Proposition 3.1. The state space $\Upsilon_d = H^1(\Omega)\times L^2(\Omega)\times L^2(\Gamma)$, endowed with the inner product (3.1) is a Hilbert space provided that μ is small enough.

Proof. It is sufficient to show that the norm $\|.\|_{\Upsilon_d}$ induced by the inner product (3.1) is equivalent to the usual one $\|\cdot\|_{H^1(\Omega)\times L^2(\Omega)\times L^2(\Gamma)}$; that is, we prove the existence of two positive constants K and $\tilde{K}$ such that

$$K\|(y,z,w)\|^2_{H^1(\Omega)\times L^2(\Omega)\times L^2(\Gamma)} \le \|(y,z,w)\|^2_{\Upsilon_d} \le \tilde{K}\|(y,z,w)\|^2_{H^1(\Omega)\times L^2(\Omega)\times L^2(\Gamma)}. \tag{3.2}$$

On one hand,

$$\|(y,z,w)\|^2_{\Upsilon_d} = \int_\Omega \left(\sum_{i,j=1}^{n} a_{ij}\partial_i y\,\partial_j y \right) dx + \int_\Omega z^2 dx + \int_\Gamma m\,w^2\,d\sigma + \mu\left(\int_\Omega (z+ay)dx + \int_\Gamma m\,w\,d\sigma \right)^2$$

Applying Hölder's inequality and Young's inequality, we get

$$\|(y,z)\|^2_{\Upsilon_d} \le \frac{1}{2}\int_\Omega \sum_{i,j=1}^{n} \sup_{x\in\Omega}|a_{ij}(x)|\left((\partial_i y)^2 + (\partial_j y)^2\right) dx + \int_\Omega z^2 dx + \|m\|_\infty \int_\Gamma w^2 d\sigma$$
$$+4\mu\left(\int_\Omega z\,dx\right)^2 + 4\mu\left(\int_\Omega ay\,dx\right)^2 + 4\mu\left(\int_\Gamma mw\,d\sigma\right)^2$$

Let $a_1 = \max_{i,j}\sup_{x\in\Omega}|a_{ij}(x)|$, we have

$$\|(y,z,w)\|^2_{\Upsilon_d} \le na_1\int_\Omega |\nabla y|^2\,dx + \int_\Omega z^2 dx + \|m\|_\infty \int_\Gamma w^2\,d\sigma$$
$$+4\mu\,vol(\Omega)\int_\Omega z^2 dx + 4\mu\|a\|^2_\infty\,vol(\Omega)\int_\Omega y^2 dx + 4\mu\|m\|^2_\infty\,vol(\Gamma)\int_\Gamma w^2 d\sigma.$$

Therefore,

$$\|(y,z,w)\|_{\Upsilon_d}^2 \le na_1\int_\Omega |\nabla y|^2\,dx + 4\mu\|a\|_\infty^2\,vol(\Omega)\int_\Omega y^2dx + (1+4\mu\,vol(\Omega))\int_\Omega z^2dx$$
$$+(\|m\|_\infty + 4\mu\|m\|_\infty^2\,vol(\Gamma))\int_\Gamma w^2 d\sigma$$

Let $\delta_0 = 4\mu\,vol(\Omega)\|a\|_\infty^2$, $\beta_0 = 1+4\mu\,vol(\Omega)$, $\psi_0 = \|m\|_\infty + 4\mu\|m\|_\infty^2\,vol(\Gamma)$ and $K = \max\{na_1, \delta_0, \beta_0, \psi_0\}$.

Consequently,

$$\|(y,z,w)\|_{\Upsilon_d}^2 \le K\,\|(y,z,w)\|_{H^1(\Omega)\times L^2(\Omega)\times L^2(\Gamma)}^2 . \qquad (3.3)$$

On the other hand, we have

$$\|(y,z,w)\|_{\Upsilon_d}^2 \ge \alpha_0\sum_{i=1}^n\int_\Omega(\partial_i y)^2dx + \int_\Omega z^2dx + m_0\int_\Gamma w^2d\sigma + \mu\left(\int_\Omega(z+ay)dx + \int_\Gamma mwd\sigma\right)^2$$
$$\ge \alpha_0\int_\Omega|\nabla y|^2dx + \int_\Omega z^2dx + m_0\int_\Gamma w^2d\sigma + \mu\left(\int_\Omega aydx\right)^2$$
$$+\mu\left(\int_\Omega zdx + \int_\Gamma mwd\sigma\right)^2 + 2\mu\left(\int_\Omega aydx\right)\left(\int_\Omega zdx + \int_\Gamma mwd\sigma\right).$$

(3.4)

Because

$$2\mu\left(\int_\Omega aydx\right)\left(\int_\Omega zdx + \int_\Gamma mwd\sigma\right) \ge -\mu\left[\delta\left(\int_\Omega aydx\right)^2 + \frac{1}{\delta}\left(\int_\Omega zdx + \int_\Gamma mwd\sigma\right)^2\right], \quad (3.5)$$

for any $\delta > 0$. Then, (3.4) and (3.5) imply that

$$\|(y,z,w)\|_{\Upsilon_d}^2 \ge \alpha_0\int_\Omega|\nabla y|^2dx + \int_\Omega z^2dx + m_0\int_\Gamma w^2d\sigma + \mu(1-\delta)\left(\int_\Omega aydx\right)^2$$
$$+\mu\left(1-\frac{1}{\delta}\right)\left(\int_\Omega zdx + \int_\Gamma mwd\sigma\right)^2 . \qquad (3.6)$$

Therefore, for $0<\delta<1$ (so $1-\delta>0$ and $1-\frac{1}{\delta}<0$)

$$\mu\left(1-\frac{1}{\delta}\right)\left(\int_\Omega zdx + \int_\Gamma mwd\sigma\right)^2 \ge 2\mu\left(1-\frac{1}{\delta}\right)\left(\int_\Omega zdx\right)^2 + 2\mu\left(1-\frac{1}{\delta}\right)\left(\int_\Gamma mwd\sigma\right)^2$$
$$\ge 2\mu\left(1-\frac{1}{\delta}\right)vol(\Omega)\int_\Omega z^2dx + 2\mu\left(1-\frac{1}{\delta}\right)\|m\|_\infty^2\,vol(\Gamma)\int_\Gamma w^2d\sigma,$$

hence, using (3.6) and (2.5) (given in Subsection 2.2) we get

$$\|(y,z,w)\|_{\Upsilon_d}^2 \ge \left(\alpha_0 - \mu(1-\delta)\right)\int_\Omega|\nabla y|^2dx + \left(1+2\mu\left(1-\frac{1}{\delta}\right)vol(\Omega)\right)\int_\Omega z^2dx$$
$$+\left(m_0 + 2\mu\left(1-\frac{1}{\delta}\right)\|m\|_\infty^2\,vol(\Gamma)\right)\int_\Gamma w^2d\sigma + \frac{\mu(1-\delta)}{c_0}\int_\Omega y^2dx. \qquad (3.7)$$

We choose $\mu > 0$ and $0 < \delta < 1$ such that the coefficients of $\int_\Omega |\nabla y|^2 dx$, $\int_\Omega y^2 dx$, $\int_\Omega z^2 dx$ and $\int_\Gamma w^2 d\sigma$ are positive; that is

$\alpha_0 - \mu(1-\delta) > 0$, which implies that $\mu < \dfrac{\alpha_0}{1-\delta}$.

$1 + 2\mu\left(1 - \dfrac{1}{\delta}\right) vol(\Omega) > 0$, then $\mu < \dfrac{1}{2\left(\dfrac{1}{\delta} - 1\right) vol(\Omega)}$.

$m_0 + 2\mu\left(1 - \dfrac{1}{\delta}\right)\|m\|_\infty^2 \, vol(\Gamma) > 0$, then $\mu < \dfrac{m_0}{2\mu\left(\dfrac{1}{\delta} - 1\right)\|m\|_\infty^2 \, vol(\Gamma)}$.

Because $0 < \delta < 1$, $\alpha_0 > 0$ and $m_0 > 0$, it is sufficient to choose $\mu > 0$ such that

$$0 < \mu < \min\left\{\frac{\alpha_0}{1-\delta}, \frac{1}{2\left(\frac{1}{\delta} - 1\right) vol(\Omega)}, \frac{m_0}{2\left(\frac{1}{\delta} - 1\right)\|m\|_\infty^2 \, vol(\Gamma)}\right\}.$$

On the other hand, $c_0 > 0$, so $\dfrac{\mu(1-\delta)}{c_0} > 0$.

Finally,

$$\|(y,z,w)\|_{\Upsilon_d}^2 \geq K \|(y,z,w)\|_{H^1(\Omega)\times L^2(\Omega)\times L^2(\Gamma)}^2, \qquad (3.8)$$

Where $K = \min\left\{\alpha_0 - \mu(1-\delta), \dfrac{\mu(1-\delta)}{c_0}, 1 + 2\mu\left(1 - \dfrac{1}{\delta}\right) vol(\Omega), m_0 + 2\mu\left(1 - \dfrac{1}{\delta}\right)\|m\|_\infty^2 \, vol(\Gamma)\right\}$. From (3.3) and (3.8), we get that

$$K \|(y,z,w)\|_{H^1(\Omega)\times L^2(\Omega)\times L^2(\Gamma)}^2 \leq \|(y,z,w)\|_{\Upsilon_d}^2 \leq \tilde{K} \|(y,z,w)\|_{H^1(\Omega)\times L^2(\Omega)\times L^2(\Gamma)}^2.$$

So, the state space $\Upsilon_d = H^1(\Omega) \times L^2(\Omega) \times L^2(\Gamma)$; endowed with the inner product (3.1) is a Hilbert space.

We turn now to the formulation of the closed-loop system (1.1)- (1.3) and (1.4)-(1.5) in an abstract form in Υ_d. Let $z(t) = y_t(t)$, $z(t) = y_t(t)$, $w(t) = y_t(t)|_\Gamma$ and $\Phi(t) = (y(t), z(t), w(t))$. Then, the closed loop system can be written as

$$\begin{cases} \Phi_t(t) + T_d \Phi(t) = 0 \\ \Phi(0) = \Phi_0 = (y(0), z(0), w(0)) = (y_0, z_0, w_0) \end{cases} \qquad (3.9)$$

Where T_d is an unbounded linear operator defined by:

$$T_d(y,z,w) = (-z, -Ay + az, \frac{1}{m(x)}\partial_A y), \qquad \forall (y,z,w) \in D(T_d) \qquad (3.10)$$

and

ISSN 2347-1921

$$
\begin{aligned}
D(\mathrm{T}_d) &= \{(y,z,w)\in \Upsilon_d : \mathrm{T}_d(y,z,w)\in \Upsilon_d \text{ and } w=z \text{ on } \Gamma\} \\
&= \left\{(y,z,w)\in \Upsilon_d : (-z,-Ay+az,\frac{1}{m(x)}\partial_A y)\in \Upsilon_d \text{ and } w=z \text{ on } \Gamma\right\} \\
&= \left\{\begin{array}{l}(y,z,w)\in \Upsilon_d : z\in H^1(\Omega), -Ay+az\in L^2(\Omega), \dfrac{1}{m(x)}\partial_A y\in L^2(\Gamma) \\ \text{and } w=z \text{ on } \Gamma\end{array}\right\} \quad (3.11) \\
&= \{(y,z,w)\in H^2(\Omega)\times H^1(\Omega)\times L^2(\Gamma),\ w=z \text{ on } \Gamma\}.
\end{aligned}
$$

By using variational formulation and Lax-Milgram theorem [3], we conclude that T_d is a maximal monotone operator.

3.2 Stabilization of the problem

In this subsection, we prove a stability result which is similar to the one obtained in [13] for the boundary feedback case.

Definition 3.2. The ω-limit set is

$$
\omega(y_0,z_0,w_0)=\left\{\begin{array}{l}(\omega_1,\omega_2,\omega_3)\in \mathrm{Y}_d : \exists\{t_n\} \text{an increasing sequence of positive numbers;} \\ \lim\limits_{n\to\infty}\left\|\big(y(t_n),z(t_n),w(t_n)\big)-(\omega_1,\omega_2,\omega_3)\right\|_{\mathrm{Y}_d}=0\end{array}\right\}.
$$

Theorem 3.3. For any initial data $\Phi_0=(y_0,z_0,w_0)\in \Upsilon_d$, the solution $\Phi(t)=(y(t),z(t),w(t))\to(\chi,0,0)$ in Υ_d as $t\to+\infty$, where

$$
\chi=\left(\int_\Omega a\,dx\right)^{-1}\int_\Omega (ay_0+z_0)\,dx\,;
$$

that is,

$$
\lim_{t\to\infty}\|(y(t),z(t),w(t))-(\chi,0,0)\|^2_{\mathrm{Y}_d}=0.
$$

Proof. Applying LaSalle's principle [24], we have:

i) $\omega(y_0,z_0,w_0)\neq\phi,\ \ \forall(y_0,z_0,w_0)\in \mathrm{Y}_d$ and it is compact set.

ii) $\omega(y_0,z_0,w_0)$ is invariant under the semi-group $S(t)$ $\big(S(t)\omega(y_0,z_0,w_0)=\omega(y_0,z_0,w_0)\big)$.

iii) Let $(y(t),z(t),w(t))=S(t)(y_0,z_0,w_0)$ be a solution of (3.9), then $\lim\limits_{t\to\infty}(y(t),z(t),w(t))\in\omega(y_0,z_0,w_0)$.

iv) $\omega(y_0,z_0,w_0)\subset D(\mathrm{T}_d)$.

v) $t\to\|S(t)\omega\|^2_{\mathrm{Y}_d}$ is a constant function for any $\omega\in\omega(y_0,z_0,w_o)$.

We will prove that $\big(y(t),z(t),w(t)\big)$ converges to $(\chi,0,0)$ as t goes to ∞. From (iii), it is sufficient to prove that $\omega(y_0,z_0,w_0)$ contains only elements of the form $(\chi,0,0)$. Let $\omega_0\in\omega(y_0,z_0,w_0)$ we prove that $\omega_0=(\chi,0,0)$. We have

$$
\frac{d}{dt}\Big(\|S(t)\omega_0\|^2_{\mathrm{Y}_d}\Big)=0\Rightarrow\left\langle\frac{d}{dt}\big(S(t)\omega_0\big),S(t)\omega_0\right\rangle_{\mathrm{Y}_d}=0\Rightarrow\left\langle\frac{d}{dt}\omega(t),\omega(t)\right\rangle_{\mathrm{Y}_d}=0,
$$

where $\omega(t)=(y(t),z(t),w(t))$ is the solution of (3.9) corresponding to ω_0.

$\langle\mathrm{T}_d\omega(t),\omega(t)\rangle_{\mathrm{Y}_d}=\int_\Omega az^2dx=0$. But, $a(x)\geq a_0>0$, so $z=0$ on Ω.

Because $y_t = z = 0$, then y is a constant with respect to t. Therefore, $w = z|_\Gamma = 0$, so

$y_{tt}|_\Gamma = 0$. Then $Ay = y_t = 0$ and $\partial_A y|_\Gamma = 0$.

Therefore, using Green's formula,

$$-\int_\Omega Ay\,y\,dx = -\int_\Gamma \partial_A y\,y\,d\sigma + \int_\Omega \sum_{i,j=1}^{n} a_{ij}\partial_j y \partial_i y\,dx = \int_\Omega \sum_{i,j=1}^{n} a_{ij}\partial_j y \partial_i y\,dx = 0.$$ But,

$\int_\Omega \sum_{i,j=1}^{n} a_{ij}\partial_j y \partial_i y\,dx \geq \alpha_0 \int_\Omega \sum_{i=1}^{n} (\partial_i y)^2 dx$, where $\alpha_0 > 0$, therefore y is a constant with respect to x.

Finally $y = \chi$, where χ is a constant.

Hence, the ω-limit set contains only elements of the form $(\chi, 0, 0)$, where χ is a constant, and we find $\lim_{t\to\infty}(y(t), z(t), w(t)) = (\chi, 0, 0)$.

Now, we have to find the expression of χ. Because

$y_{tt}(x,t) - Ay(x,t) + a(x)y_t(x,t) = 0$ in $\Omega \times (0,\infty)$ and $\partial_A y = 0$ on $\Gamma \times (0,\infty)$, then

$\left(\int_\Omega (y_t(x,t) + a(x)y(x,t))dx\right)' = \int_\Omega Ay\,dx = \int_\Gamma \partial_A y\,d\sigma = 0$, therefore $\int_\Omega (y_t(x,t) + a(x)y(x,t))dx$

is a constant function.

Thus,

$\int_\Omega (y_t(x,t) + a(x)y(x,t))dx = \int_\Omega (y_t(x,0) + a(x)y(x,0))dx = \int_\Omega (z_0 + ay_0)\,dx \quad \forall t \in (0,\infty)$. By passing to the limit where t goes to ∞, and using the

fact that $\lim_{t\to\infty}(y(t), z(t)) = (\chi, 0)$, we get $\int_\Omega (0 + a(x)\chi)dx = \int_\Omega (z_0 + ay_0)dx$, this implies that

$$\chi \int_\Omega a\,dx = \int_\Omega (z_0 + ay_0)dx,$$

so

$$\chi = \left(\int_\Omega a\,dx\right)^{-1} \int_\Omega (z_0 + ay_0)dx\$. \qquad \blacksquare$$

4 Applications To Other Systems

The method presented in the previous two sections can be applied for a large class of distributed systems (where the classical energy defines only a semi-norm in the state space) to prove that the solution exists and converges to an equilibrium point (when the time goes to infinity). This equilibrium point can be determined explicitly in term of the parameters of the considered systems. We give here some particular applications to Petrovsky system, coupled wave-wave equations and elasticity systems. For more details concerning these systems, see [15]-[23] and the references therein. The proof of the obtained stability results of this chapter is inspired from the approach introduced in [13] for the case of boundary feedback.

4.1 Petrovsky system.

Let Ω be a bounded open connected set in $\Re^n$ having a smooth boundary $\Gamma = \partial\Omega$ of class C^4.

We consider the following Petrovsky system with static boundary conditions:

$$\begin{cases} y_{tt}(x,t)+\Delta^2 y(x,t)+a(x)y_t(x,t)=0 & \text{in}\,\Omega\times(0,\infty) \\ \partial_\nu y(x,t)=0 & \text{on}\,\Gamma\times(0,\infty) \\ \partial_\nu \Delta y(x,t)=0 & \text{on}\,\Gamma\times(0,\infty) \\ y(x,0)=y_0(x), y_t(x,0)=z_0(x) & \text{in}\,\Omega, \end{cases} \quad (4.1)$$

Where (y_0,z_0) is a given initial data in $V\times L^2(\Omega)$ where $V=\{\varphi\in H^2(\Omega);\partial_\nu\varphi=0\,\text{on}\,\Gamma\}$, $a\in L^\infty(\Omega)$, such that there exists $a_0>0$ satisfying $a(x)\geq a_0\ \forall x\in\Omega$, and $\nu=(\nu_1,\nu_2,...,\nu_n)$ is the unit normal of Γ pointing towards the exterior of Ω.

Consider the state space

$$\mathrm{Y}_p=V\times L^2(\Omega)$$

Equipped with the inner product

$$\langle (y,z),(\tilde{y},\tilde{z})\rangle_{\mathrm{Y}_p}=\int_\Omega(\Delta y\Delta\tilde{y}+z\tilde{z})dx+\varepsilon\left(\int_\Omega(z+ay)dx\right)\left(\int_\Omega(\tilde{z}+a\tilde{y})dx\right). \quad (4.2)$$

where $\varepsilon>0$ is a constant to be determined.

Using LaSalle's principle and following the arguments used before, we obtain that, for any initial data Y_p the solution of the system (4.1) satisfy: $(y(t),y_t(t))\to(\chi,0)$ in Y_p as $t\to\infty$, where

$$\chi=\left(\int_\Omega a\,dx\right)^{-1}\left(\int_\Omega(ay_0+z_0)\,dx\right).$$

Remark 4.1 One can consider dynamical boundary conditions; that is

$$\begin{cases} y_{tt}(x,t)+\Delta^2 y(x,t)+a(x)y_t(x,t)=0 & \text{on}\,\Omega\times(0,\infty) \\ \partial_\nu y(x,t)=0 & \text{on}\,\Gamma\times(0,\infty) \\ -m(x)y_{tt}(x,t)+\partial_\nu\Delta y(x,t)=0 & \text{on}\,\Gamma\times(0,\infty) \\ y(x,0)=y_0(x), y_t(x,0)=z_0(x) & \text{in}\,\Omega \\ y_t|_\Gamma(x,0)=w & \text{on}\,\Gamma \end{cases} \quad (4.3)$$

where a and m are defined by (1.4) and (1.5) respectively.

We consider the state space:

$$\mathrm{Y}_{pd}=V\times L^2(\Omega)\times L^2(\Gamma),$$

equipped with the inner product

$$\begin{aligned}\langle (y,z,w),(\tilde{y},\tilde{z},\tilde{w})\rangle_{\mathrm{Y}_d}&=\int_\Omega\Delta y\Delta\tilde{y}dx+\int_\Omega z\tilde{z}dx+\int_\Gamma mw\tilde{w}d\sigma \\ &+\mu\left(\int_\Omega(z+ay)dx+\int_\Gamma mwd\sigma\right)\left(\int_\Omega(\tilde{z}+a\tilde{y})dx+\int_\Gamma m\tilde{w}d\sigma\right).\end{aligned} \quad (4.4)$$

where $\mu>0$ is a constant to be determined.

We obtain that, $(y(t), y_t(t), y_t|_\Gamma(t)) \to (\chi, 0, 0)$ in Y_{pd} as $t \to \infty$, where

$$\chi = \left(\int_\Omega a\,dx\right)^{-1}\left(\int_\Omega (ay_0 + z_0)\,dx\right)$$

4.2 Coupled wave-wave equations

Let Ω be a bounded open connected set in $\Re^n$ having a smooth boundary $\Gamma = \partial\Omega$ of class C^2.

We consider the following coupled wave-wave system with static boundary conditions:

$$\begin{cases} y_{tt}(x,t) - Ay(x,t) + a_1 y_t(x,t) + bu_{tt} = 0 & \text{in } \Omega \times (0,\infty) \\ u_{tt}(x,t) - Bu(x,t) + a_2 u_t(x,t) + by_{tt} = 0 & \text{in } \Omega \times (0,\infty) \\ \partial_A y(x,t) = \partial_B u(x,t) = 0 & \text{on } \Gamma \times (0,\infty) \\ y(x,0) = y_0(x), y_t(x,0) = z_0(x) & \text{in } \Omega \\ u(x,0) = u_0(x), u_t(x,0) = v_0(x) & \text{in } \Omega, \end{cases} \quad (4.5)$$

Where $(y_0, \mathrm{u}_0, \mathrm{z}_0, \mathrm{v}_0)$ is a given initial data in $(H^1(\Omega))^2 \times (L^2(\Omega))^2$, $A = \sum_{i,j=1}^n \partial_i(a_{ij}\partial_j)$, $B = \sum_{i,j=1}^n \partial_i(b_{ij}\partial_j)$ and $a_{ij}, b_{ij} \in C^1(\bar{\Omega})$ such that there exist $a_0, b_0 > 0$, satisfying $a_{ij} = a_{ji}, b_{ij} = b_{ji}\ \forall i, j = 1, 2, ..., n$.

$\sum_{i,j=1}^n a_{ij}\varepsilon_i\varepsilon_j \geq a_0 \sum_{i=1}^n \varepsilon_i^2, \sum_{i=1}^n b_{ij}\varepsilon_i\varepsilon_j \geq b_0 \sum_{i=1}^n \varepsilon_i^2 \quad \forall(\varepsilon_1, \varepsilon_2, ..., \varepsilon_n) \in \Re^n$.

Moreover, there exist two positive constants $a_{1,0}$ and $a_{2,0}$ such that

$a_1 \in L^\infty(\Omega); a_1(x) \geq a_{1,0}, \forall x \in \Omega$, $a_2 \in L^\infty(\Omega); a_2(x) \geq a_{2,0}, \forall x \in \Omega$, $a_2 \in L^\infty(\Omega); a_2(x) \geq a_{2,0}, \forall x \in \Omega$ and $b \in L^\infty(\Omega)$ satisfies $\|b\|_\infty < 1$. For more details concerning these systems, see [20],[23] and the references therein.

We consider the state space

$$Y_w = (H^1(\Omega))^2 \times (L^2(\Omega))^2$$

equipped with the inner product

$$\begin{aligned} \left\langle (y,u,y_t,u_t),(\tilde{y},\tilde{u},\tilde{y}_t,\tilde{u}_t)\right\rangle_{Y_w} &= \int_\Omega \left(\sum_{i,j=1}^n a_{ij}\partial_i y \partial_j \tilde{y} + \sum_{i,j=1}^n b_{ij}\partial_i u \partial_j \tilde{u} \right) dx \\ &+ \int_\Omega \left(y_t\tilde{y}_t + u_t\tilde{u}_t + b(y_t\tilde{u}_t + \tilde{y}_t u_t)\right) dx \\ &+ \varepsilon \left(\int_\Omega (y_t + bu_t)dx + \int_\Omega a_1 y\,dx \right)\left(\int_\Omega (\tilde{y}_t + b\tilde{u}_t)dx + \int_\Omega a_1\tilde{y}\,dx \right) \\ &+ \varepsilon \left(\int_\Omega (by_t + u_t)dx + \int_\Omega a_2 u\,dx \right)\left(\int_\Omega (b\tilde{y}_t + \tilde{u}_t)dx + \int_\Omega a_2\tilde{u}\,dx \right), \end{aligned}$$

(4.6)

where $\varepsilon > 0$ is a constant to be determined.

Using LaSalle's principle and following the arguments used before, we obtain that, for any initial data Υ_w the solution of the system (4.5) satisfy: $(y(t),u(t),z(t),v(t))\to(\chi_1,\chi_2,0,0)$ in Υ_w as $t\to\infty$ where

$$\chi_1\left(\int_\Omega a_1\,dx\right)+\chi_2\left(\int_\Omega a_2\,dx\right)=\left[\int_\Omega(1+b)(z_0+v_0)dx+\int_\Omega(a_1y_0+a_2u_0)dx\right]$$

If $A=B$, $a_1=a_2$ and $(y_0,z_0)=(u_0,v_0)$ then it follows from the symmetry that

$$\chi_1=\chi_2=\left(\int_\Omega a_1\,dx\right)^{-1}\left(\int_\Omega(1+b)z_0\,dx+\int_\Omega a_1y_0\,dx\right).$$

Remark 4.2

(i) One can consider dynamical boundary conditions for both equations

$$\begin{cases} y_{tt}(x,t)-Ay(x,t)+a_1y_t(x,t)+bu_{tt}=0 & \text{in }\Omega\times(0,\infty)\\ u_{tt}(x,t)-Bu(x,t)+a_2u_t(x,t)+by_{tt}=0 & \text{in }\Omega\times(0,\infty)\\ m(x)y_{tt}(x,t)+\partial_A y=0 & \text{on }\Gamma\times(0,\infty)\\ M(x)u_{tt}(x,t)+\partial_B u=0 & \text{on }\Gamma\times(0,\infty)\\ y(x,0)=y_0(x),\ y_t(x,0)=z_0(x) & \text{in }\Omega\\ u(x,0)=u_0(x),\ u_t(x,0)=v_0(x) & \text{in }\Omega\\ y_t\big|_\Gamma(x,0)=w_0^0(x),\ u_t\big|_\Gamma(x,0)=w_1^0(x) & \text{on }\Gamma \end{cases}$$

or static boundary condition for one equation and dynamical boundary condition for the other equation and we obtain in both cases the same results as for (4.5) with the constants χ_1 and χ_2 defined above.

(ii) We can also consider static or dynamical boundary conditions only for y and the homogeneous Dirichlet ones for u or the reverse). In this case, we get $\left(y(t),u(t),y_t(t),u_t(t)\right)\to(\chi,0,0,0)$ where

$$\chi=\left(\int_\Omega a_1\,dx\right)^{-1}\left(\int_\Omega(1+b)z_0dx+\int_\Omega a_1y_0\,dx\right).$$

(iii) Similar results can obtained for a coupled Petrovsky-Petrovsky or wave-Petrovsky systems with static or dynamical boundary conditions.

4.3 Elasticity systems

Let Ω be a bounded open connected set in $\Re^n$ having a smooth boundary $\Gamma=\partial\Omega$ of class C^2.

We consider the following elasticity system:

$$\begin{cases} y_{itt}(x,t)-\sum_{j=1}^{n}\sigma_{ij,j}(y)(x,t)+a_i(x)y_{it}(x,t)=0 & \text{in }\Omega\times(0,\infty),\forall i=1,2,\dots,n\\ \sum_{j=1}^{n}\sigma_{ij}(y)\nu_j=0 & \text{on }\Gamma\times(0,\infty),\forall i=1,2,\dots,n\\ y_i(x,0)=y_i^0(x),\ y_{it}(x,0)=z_i^0(x) & \text{in }\Omega,\forall i=1,2,\dots,n \end{cases}\quad(4.7)$$

Where $\left(y^0,z^0\right)=\left((y_1^0,\dots,y_n^0),(z_1^0,\dots,z_n^0)\right)$ is a given initial data $\Upsilon_e=\left(H^1(\Omega)\right)^n\times\left(L^2(\Omega)\right)^n$.

Here $y=(y_1,...,y_n):\Omega\to\Re^n$ is the solution of (4.7), $a_i(x)\in L^\infty(\Omega)$ such that there exists $a_{i0}>0$ satisfying $a_i(x)\geq a_{i,0}>0\,\forall x\in\Omega,\forall i=1,2,...,n$.

Moreover $\sigma_{ij,j}(y)=\dfrac{\partial\sigma_{ij}(y)}{\partial x_j},\sigma_{ij}(y)=\sum_{k,l=1}^{n}a_{ijkl}\varepsilon_{kl}(y),\ \varepsilon_{ij}(y)=\frac{1}{2}(y_{i,j}+y_{j,i})$, $y_{i,j}=\dfrac{\partial y_i}{\partial x_j}$, $y_{j,i}=\dfrac{\partial y_j}{\partial x_i}$ and $a_{ijkl}\in C^1(\bar{\Omega})$ such that there exists $a_0>0$ satisfying $a_{ijkl}=a_{klij}=a_{jikl},\ \forall i,j,k,l=1,2,...,n$ and $\sum_{i,j,k,l=1}^{n}a_{ijkl}\varepsilon_{ij}\varepsilon_{kl}\geq a_0\sum_{i,j=1}^{n}\varepsilon_{ij}\varepsilon_{ij}$,for all symmetric tensor ε_{ij}. For more details concerning these systems, see [15]-[18] and the references therein.

We consider the state space

$$\mathrm{Y}_e=\left(H^1(\Omega)\right)^n\times\left(L^2(\Omega)\right)^n$$

equipped with the inner product, for any $y=(y_1,...,y_n)$, $z=(z_1,...,z_n)$, $\tilde{y}=(\tilde{y}_1,...,\tilde{y}_n)$ and $\tilde{z}=(\tilde{z}_1,...,\tilde{z}_n)$,

$$\begin{aligned}\left\langle(y,z),(\tilde{y},\tilde{z})\right\rangle_{\mathrm{Y}_e}&=\int_\Omega\left(\sum_{i,j=1}^{n}\sigma_{ij}(y)\varepsilon_{ij}(\tilde{y})\right)dx+\int_\Omega(\sum_{i=1}^{n}z_i\tilde{z}_i)dx\\&+\varepsilon\sum_{i=1}^{n}\left[\left(\int_\Omega(z_i+a_iy_i)dx\right)\left(\int_\Omega(\tilde{z}_i+a_i\tilde{y}_i)\,dx\right)\right],\end{aligned}\tag{4.8}$$

where ε is a constant to be determined. Using LaSalle's principle and following the arguments used before, we obtain that, for any initial data $\Phi_0=(y^0,z^0)\in\mathrm{Y}_e$ the solution of the system (4.7) satisfies: $(y(t),z(t))\to(\chi,0)$ in Y_e as $t\to\infty$ where

$$\chi=(\chi_1,\chi_2,...,\chi_n)\text{ and }\sum_{i=1}^{n}\chi_i\left(\int_\Omega a_i(x)dx\right)=\int_\Omega\sum_{i=1}^{n}(z_i^0+a_{i,0}y_i^0)dx$$

If $a_i=a_j,y_i^0=y_j^0$, and $z_i^0=z_j^0,\forall i,j=1,2,...,n,$ then by symmetry of (4.7) we have

$$\chi_i=\left(\int_\Omega a_i\,dx\right)^{-1}\int_\Omega\sum_{i=1}^{n}\left(z_i^0+a_i(x)y_i^0\right)dx.$$

Remark 4.3

We can consider static conditions for y_i, $\mathrm{i}=1,2,...,\mathrm{r}$, dynamical boundary conditions for y_i, $i=r+1,...,p$, and the homogeneous Dirichlet ones for y_i, $i=p+1,...,n$, where $0\leq r\leq p\leq n$; that is

$$\begin{cases}\sum_{j=1}^{n}\sigma_{ij}(y)\nu_j=0, & \text{on}\,\Gamma\times(0,\infty),\forall i=1,...,r,\\ \sum_{j=1}^{n}\sigma_{ij}(y)\nu_j+m_iy_{itt}=0, & \text{on}\,\Gamma\times(0,\infty),\forall i=r+1,...,p,\\ y_i=0 & \text{on}\,\Gamma\times(0,\infty),\forall i=p+1,...,n.\end{cases}$$

In this case, $(y(t),y_t(t))\to(\chi,0)\,\text{as}\,t\to\infty$, where $\chi=(\chi_1,\chi_2,...,\chi_n)$.

$\chi_i=0\,\text{for}\,i=p+1,...,n,$and

$$\sum_{i=1}^{p} \chi_i \left(\int_{\Omega} a_i(x)\, dx \right) = \int_{\Omega} \sum_{i=1}^{p} \left(z_i^0 + a_i(x) y_i^0 \right) dx.$$

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