ISSN 2347-1921

FREQUENCY FUNCTIONS FOR VARIOUS CHARACTERISTICS OF THE STARS

J.A.Basabrain

Department of Statistics College of Science for Girls, King Abdulaziz University, Jeddah, Saudi Arabia

jbasabrain@yahoo.com

Abstract In this paper, two fundamental theorems of stellar statistics are given. Applications of the theorems for the frequency functions of the various characteristics of the stars are developed

Keywords Stellar statistics; frequency functions of the stars; integral equations

Council for Innovative Research

Peer Review Research Publishing System

Journal: JOURNAL OF ADVANCES IN MATHEMATICS

Vol .10, No.4

www.cirjam.com , editorjam@gmail.com

1.Introduction

Modern observational astronomy has been characterized by an enormous growth of data stimulated by the advent of new technologies in telescopes, detectors and computations. This new astronomical data gives rise to innumerable statistical problems (Feigelson and Babu 1992,Sharaf andMominkhan 2014).Also, a

new statistical method for cosmic distances determination was developed by Sharaf et al. (2003). The method depends on the assumption that the members of the group scatter around a mean absolute magnitude in Gaussian distribution. The mean apparent magnitude of the members is then expressed by frequency function to correct for observational incompleteness at the faint end. The problem reduces to the solution of a highly transcendental equation for a given magnitude parameter (α).

Due to the importance role of the stellar statistic and its consequent results in establishing stellar frequency functions, the present paper is devoted to find the basic equations for the frequency functions of the various characteristics of the stars

2. Two fundamental Theorems of Stellar Statistics

2.1 The first theorem

Consider n(say) of absolute characteristics $X_1, X_2, \cdots X_n$ of a star. Assume $x_1, x_2, \cdots x_n$ be the apparent characteristics associated with absolute characteristics $X_1, X_2, \cdots X_n$.Assume that r is the distance of the star from the Sun(usually in parsec). The apparent characteristics $x_1, x_2, \cdots x_n$ are functions of the absolute characteristics $X_1, X_2, \cdots X_n$ and the distance r that is:

$$x_j = f_j(X_j, r) \; ; \; j = 1, 2, \cdots, n.$$

Inversely

$$X_j = F(x_j, r) \; ; j = 1, 2, \cdots, n$$

Let $\Phi(X_1, X_2, \cdots X_n)$ be the frequency function for the absolute characteristics $X_1, X_2, \cdots X_n$,

let $b(x_1, x_2, \cdots, x_n) dx_1 \, dx_2, \cdots, dx_n$ denotes the number of stars with apparent characteristics between $(x_1, x_1 + dx_1)$, $(x_2, x_2 + dx_2), \cdots, (x_n, x_n + dx_n)$ in a small region of the sky subtends a solid angle S(see Fig. 1) at a distance $r + dr, r$,and $D(r)$ the density function, then

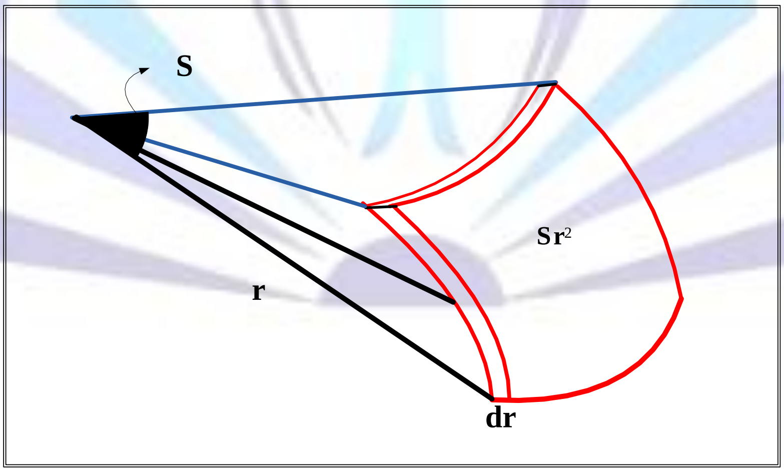

Fig.1: The element of volume on the distance interval $(r, r + dr)$ within a field subtending solid angle S is $Sr^2 dr$

$$b(x_1, x_2, \cdots, x_n) = S \int_0^\infty r^2 \, D(r) \Phi(X_1, X_2, \cdots, X_n) \frac{\partial X_1}{\partial x_1} \frac{\partial X_2}{\partial x_2}, \cdots, \frac{\partial X_n}{\partial x_n} dr \qquad (1)$$

Equation (1) is the first theorem.

3.2 The second theorem

ISSN 2347-1921

If p($x_1, x_2, \cdots x_n$) is the average parallax for the stars having apparent characteristics $x_1, x_2, \cdots x_n$. Since the parallax p of a star at a distance r parsec is $p = 1/r$ (p in second of arc),then

$$b(x_1, x_2, \cdots, x_n) p(x_1, x_2, \cdots, x_n) = S \int_0^\infty r\, D(r) \Phi(X_1, X_2, \cdots, X_n) \frac{\partial X_1}{\partial x_1} \frac{\partial X_2}{\partial x_2}, \cdots, \frac{\partial X_n}{\partial x_n} dr \quad (2)$$

Equation (2-19) is the second theorem

2.3 The solution of equations

If the functions $b(x_1, x_2, \cdots, x_n)$ and $p(x_1, x_2, \cdots, x_n)$ can be derived from observations, the function $D(r)$ and $\Phi(X_1, X_2, \cdots X_n)$ can theoretically be found from the solutions of the integral Equations(1) and (2) .

2.4 Special cases

First: Function of one absolute characteristic

Let the absolute characteristic and the apparent characteristic of a star are respectively be X and x. In this case the two fundamental theorems become

$$b(x) = S \int_0^\infty r^2 D(r) \Phi(X) \frac{\partial X}{\partial x} dr, \quad (3)$$

$$p(x) b(x) = S \int_0^\infty r\, D(r) \Phi(X) \frac{\partial X}{\partial x} dr. \quad (4)$$

Second: Function of two absolute characteristics

Let the two absolute characteristics and the two apparent characteristics of a star are respectively (X_1, X_2) and (x_1, x_2) .In this case the two fundamental theorems become

$$b(x_1, x_2) = S \int_0^\infty r^2 D(r)\, \Phi(X_1, X_2) \frac{\partial X_1}{\partial x_1} \frac{\partial X_2}{\partial x_2} dr, \; . \quad (5)$$

$$b(x_1, x_2) p(x_1, x_2) = S \int_0^\infty r\, D(r)\, \Phi(X_1, X_2) \frac{\partial X_1}{\partial x_1} \frac{\partial X_2}{\partial x_2} dr.. \quad (6)$$

▲Example of function of one absolute characteristic

The input: $X = x_1 \times r^m ,\ D(r) = \frac{1}{r^n},\ \Phi(X) = e^{-X}$

The output :

$$b \square \frac{2\, S\, x_1^{\frac{\square m \square n \square 3}{m}} \square \left[\frac{2 \left(m \square n \square 3 \right)}{m} \right]}{m}$$

$$a \square \frac{2\, S\, x_1^{\frac{\square m \square n \square 2}{m}} \square \left[\frac{2 \left(m \square n \square 2 \right)}{m} \right]}{m}$$

▲ Example of function of two absolute characteristics

The input: $X = \{x_1 \times r^m, x_2 \times r^p\}, D(r) = \frac{1}{r^n}, \Phi(X) = \exp[-\sqrt{x_1 x_2 \times r^{m+p}}$

The output :

▲Example of function of three absolute characteristics

```
X□ □x1 □r, x2 □r, x3 □r□
DE□r□ □ 1 □r^n
□□X□ □ Sin□X□□1□□□ □Cos□X□□2□□□ □X□□3□□
```

The input: $X=\{r\,x_1, r\,x_2, r\,x_3\}, D(r)=\frac{1}{r^n}, \Phi(X)=r\,x_3\sin(r\,x_1)\cos(r\,x_2)$

The output :

$$b\square\frac{1}{2}\,S\,x_3\left(\square\frac{\left(\left(x_1\square x_2\right)^2\right)^{n\square 2}}{\left(x_1\square x_2\right)^7}\square\frac{\left(\left(x_1\square x_2\right)^2\right)^{n\square 2}}{\left(x_1\square x_2\right)^7}\right)\cos\left(\frac{\square n}{2}\right)\square\left(7\square n\right)$$

$$a\square\frac{1}{2}\,S\,x_3\left(\frac{\left(\left(x_1\square x_2\right)^2\right)^{\frac{n\square 1}{2}}}{\left(x_1\square x_2\right)^7}\square\frac{\left(\left(x_1\square x_2\right)^2\right)^{\frac{n\square 1}{2}}}{\left(x_1\square x_2\right)^7}\right)\sin\left(\frac{\square n}{2}\right)\square\left(6\square n\right)$$

4. Applications

4.1Apparent and absolute luminosities

Let L and ℓ denote the absolute and apparent luminosities of a star. We define the former as the luminosity the star would appear to have if it were at unit distance $(r=1)$ .Then, since the apparent brightness of a star varies inversely as the square of the distance,

$$L=\ell r^2 . \tag{7}$$

In Equations (3) and (4), X and x are to be replaced by L and ℓ respectively . Also

$$\frac{\partial L}{\partial \ell}=r^2 , \tag{8}$$

hence

$$b(\ell)=S\int_0^\infty r^4\,D(r)\Phi(\ell r^2)dr , \tag{9}$$

$$a(\ell)\equiv b(\ell)p(\ell)=S\int_0^\infty r^3\,D(r)\Phi(\ell r^2)dr . \tag{10}$$

In practice we associate a function Φ with each of the various spectral divisions (or subdivisions) in the Hertzsprung – Russel diagram , distinguishing between giants and the stars of the main sequence.

4.2Apparent and absolute magnitudes

The absolute magnitude , M of a star is given in terms of the apparent magnitude m and parallax p by

$$M=m+5+5\log p, \tag{11}$$

M is thus defined in terms of the standard distance of 10 parsecs .We write , for convenience ,

$$M_1=M-5, \tag{12}$$

so that M_1 is defined in terms of the standard distance of 1 parsec , and

$$M_1=m+5\log p. \tag{13}$$

In this formula the base of the logarithm is 10 .

We shall refer to M_1 in this connection as the *modified absolute magnitude*. Also, with r measured in parsecs, we have $p = 1/r$ and

$$M_1 = m - 5\log r. \tag{14}$$

In Equations (3) and (4), X and x are to be replace by M_1 and m respectively . Also

$$\frac{\partial M_1}{\partial m} = 1,$$

hence,

$$b(m) = S\int_0^\infty r^2 D(r)\Phi(m - 5\log r)dr \tag{15}$$

and

$$b(m)p(m) = S\int_0^\infty r D(r)\Phi(m - 5\log r)dr \tag{16}$$

In these formulae , $b(m)$ is the number and $p(m)$ is the mean parallax of stars of apparent magnitude m . Another forms of these equations will be derived as follows.

Write

$$\rho = -5\log r \Rightarrow \tag{17}$$

$$r = 10^{-\rho/5} \Rightarrow \ln r = -\left(\frac{\rho}{5}\right)\ln 10 \Rightarrow r = \exp\left\{-\frac{\rho}{5}\ln 10\right\}.$$

Let

$$c = \frac{\ln 10}{5} = 0.4605 \Rightarrow \tag{18}$$

$$r = \exp\{-c\rho\}\ . \tag{19}$$

Consequently

$$dr = -c\exp\{-c\rho\}d\rho\ ,$$

$$r = 0 \quad \Rightarrow \quad \rho = \infty, \tag{20}$$

$$r = \infty \quad \Rightarrow \quad \rho = -\infty.$$

Thus

$$b(m) = cS\int_{-\infty}^{\infty} \exp\{-3c\rho\}D\left(e^{-c\rho}\right)\Phi(m+\rho)d\rho\,, \tag{21}$$

$$b(m)p(m) = cS\int_{-\infty}^{\infty} \exp\{-2c\rho\}\ D\left(e^{-c\rho}\right)\Phi(m+\rho)d\rho\,, \tag{22}$$

or, on setting

$$\Delta(\rho) \equiv cS\ \exp\{-3c\rho\}D\left(e^{-c\rho}\right), \tag{23}$$

$$b(m)=\int_{-\infty}^{\infty}\Delta(\rho)\Phi(m+\rho)d\rho\,, \quad (24)$$

$$a(m)\equiv b(m)p(m)=\int_{-\infty}^{\infty}e^{c\rho}\Delta(\rho)\Phi(m+\rho)d\rho\,. \quad (25)$$

4.3Linear velocity and proper motion

Let T denote the transverse linear velocity of a star in a given region and μ the corresponding total annual proper motion. Then

$$T=\mu r\,. \quad (26)$$

With r measured in parsecs and μ in seconds of arc, T will be measured in terms of the unit k or 4.74 km./sec. Here T and μ correspond to the characteristics X and x . Let Ψ(T) be the frequency function of the linear velocities. Then by Equations (3) and (4) we get;

$$b(\mu)=S\int_{0}^{\infty}r^{3}D(r)\Psi(\mu r)dr\,, \quad (27)$$

$$a(\mu)\equiv b(\mu)\cdot p(\mu)=S\int_{0}^{\infty}r^{2}D(r)\Phi(\mu r)\,dr\,. \quad (28)$$

in which $b(\mu)$ and $p(\mu)$ are respectively the number and mean parallax of stars with proper motion μ . Let

$$r=\exp\{\rho\}\quad ;\quad \mu=\exp\{\alpha\}\,. \quad (29)$$

Then Equation (27) becomes:

$$b(\mu)\equiv b_1(\alpha)=S\int_{-\infty}^{\infty}\exp\{4\rho\}\,D\left(e^{\rho}\right)\Psi\left(e^{\rho+\alpha}\right)d\rho \quad (30)$$

and setting:

$$S\exp\{4\rho\}D\left(e^{\rho}\right)=\Delta_1(\rho), \quad (31)$$

$$\Psi\left(e^{\rho+\alpha}\right)=\Psi_1(\rho+\alpha). \quad (32)$$

Thus,

$$b(\mu)\equiv b_1(\alpha)=\int_{-\infty}^{\infty}\Delta_1(\rho)\Psi_1(\rho+\alpha)\,d\rho\,. \quad (33)$$

Similarly we get

$$a(\mu)\equiv a_1(\mu)=b(\mu)\cdot p(\mu)=\int_{-\infty}^{\infty}e^{-\rho}\,\Delta_1(\rho)\Psi_1(\rho+\alpha)d\rho\,. \quad (34)$$

Formulae (33) and (34) are integral Equations from which the functions $\Delta_1(\rho)$ and $\Psi_1(\rho+\alpha)$ can be determined when the functions $b(\mu)$ and $a(\mu)$ have been obtained from observations.

4.4The number and mean parallax of stars of magnitude m and proper motion μ

Denote , as before, the frequency function of the modified absolute magnitudes by $\Phi(M_1)$ and the frequency function of the linear velocity by Ψ(T).Let $b(m,\mu)dm\,d\mu$ be the number of stars with apparent magnitudes between m and $m+dm$ and proper motion between μ and $\mu+d\mu$. Then, assuming that there is no correlation between M_1 and T. Consequently, according to Equations (5) and (6) with

$$X_1 = M_1 \quad ; \quad X_2 = T \quad ; \quad x_1 = m \quad ; \quad x_2 = \mu$$

we have:

$$b(m,\mu) = S\int_0^\infty r^2\, D(r)\Phi(M_1)\Psi(T)\, \frac{\partial M_1}{\partial m}\frac{\partial T}{\partial \mu}\, dr\,, \tag{35}$$

$$b(m,\mu)\, p(m,\mu) = S\int_0^\infty r\, D(r)\Phi(M_1)\Psi(T)\, \frac{\partial M_1}{\partial m}\frac{\partial T}{\partial \mu}\, dr\,. \tag{36}$$

Since

$$M_1 = m - 5\log r \quad ; \quad T = \mu\, r\,, \tag{37-1}$$

then

$$\frac{\partial M_1}{\partial m} = 1 \quad ; \quad \frac{\partial T}{\partial \mu} = r\,. \tag{37-2}$$

Thus

$$b(m,\mu) = S\int_0^\infty r^3\, D(r)\Phi(m - 5\log r)\Psi(\mu\, r)\, dr \tag{38}$$

$$a(m,\mu) \equiv b(m,\mu)\, p(m,\mu) = S\int_0^\infty r^2\, D(r)\Phi(m - 5\log r)\Psi(\mu\, r)\, dr\,. \tag{39}$$

Setting

$$r = \exp\{-c\rho\}\,,$$

where c is given by Equation (18) and:

$$c\, S \exp\{-\ 4\, c\, \rho\}\, D(e^{-c\rho})\, \Psi(\mu\, e^{-c\rho}) = \Delta_1(\mu,\rho)\,, \tag{40}$$

then Equations (38)and (39) become:

$$b(m,\mu) = \int_{-\infty}^{\infty} \Delta_1(\mu,\rho)\, \Phi(m+\rho)\, d\rho \quad , \tag{41}$$

$$b(m,\mu)\, p(m,\mu) = \int_{-\infty}^{\infty} e^{c\rho}\, \Delta_1(\mu,\rho)\, \Phi(m+\rho)\, d\rho\,. \tag{42}$$

4.5 The total apparent brightness in any region of the sky

Considering stars of a single spectral type, the mean absolute luminosity, $\overline{L}$, is given by:

$$\overline{L} = \int_0^\infty L\, \Phi(L) dL\,. \tag{43}$$

If we assume that the mean absolute luminosity of the stars in the element of volume, $S\, r^2\, dr$, of the cone is given by Equation (43), these stars will have an apparent luminosity $\overline{L}/r^2$ on the average. As the number of stars in the volume element is $S\, D(r) r^2 dr$, the total apparent luminosity arising from these stars is $\overline{L}\, S\, D(r) dr$.

We then have the total apparent luminosity, λ , of the region

$$\lambda = \bar{L}\, S \int_0^{\infty} D(r) dr \,. \tag{44}$$

Since

$$\ell_1 / \ell_2 = 10^{-0.4(m_1 - m_2)} \,, \tag{45}$$

then, assuming the zero apparent magnitude corresponds to $\ell = 1$, i.e. $m_2 = 0$ for $\ell_2 = 1$ then from Equation (45)

$$\ell_1 = 10^{-0.4\, m_1} \,,$$

that is:

$$m_1 = -2.5 \log \ell_1 \,. \tag{46}$$

Consequently, the total apparent luminosity, λ , for the given region and given spectral type is equivalent to the brightness of a single star of apparent magnitude $-2.5 \log \lambda$. The stars of the other spectral types furnish similar results.

4.6 The mean proper motion of stars of apparent magnitude m

From the above,, the number of stars with a given assigned apparent magnitude m

and with proper motion between μ and $\mu + d\mu$ is proportional to $b(m, \mu) d\mu$ and, if $\mu(m)$ denotes the mean proper motion of all stars of apparent magnitude m, then, we have:

$$\mu(m) = \frac{\int_0^{\infty} \mu\, b(m, \mu) d\mu}{\int_0^{\infty} b(m, \mu) d\mu} \,. \tag{47}$$

Thus , we have :

$$\int_0^{\infty} \mu\, b(m, \mu) d\mu = S \int_0^{\infty} \mu\, d\mu \left\{ \int_0^{\infty} r^3\, D(r)\, \Phi(m - 5 \log r)\, \Psi(\mu\, r) dr \right\} , \tag{48}$$

$$\int_0^{\infty} b(m, \mu) d\mu = S \int_0^{\infty} d\mu \left\{ \int_0^{\infty} r^3\, D(r)\, \Phi(m - 5 \log r)\, \Psi(\mu\, r) dr \right\} . \tag{49}$$

Inverting the order of the integrations of the right hand sides of Equations (48) and (49) we get for $\mu(m)$ the expression

$$\mu(m) = \frac{\int_0^{\infty} r^3\, D(r)\, \Phi(m - 5 \log r) \left\{ \int_0^{\infty} \Psi(\mu\, r) \mu\, d\mu \right\} dr}{\int_0^{\infty} r^3\, D(r)\, \Phi(m - 5 \log r) \left\{ \int_0^{\infty} \Psi(\mu\, r)\, d\mu \right\} dr} \,. \tag{50}$$

On the assumption that the frequency function of the linear velocities T is independent of r ,then

$$\mu(m) = \frac{\int_0^{\infty} r\, D(r)\ \ \Phi(m - 5 \log r)\ dr \int_0^{\infty} T\, \Psi(T)\, dT}{\int_0^{\infty} r^2\, D(r)\, \Phi(m - 5 \log r)\ \ dr \int_0^{\infty} \Psi(T)\, dT} \,.$$

Since

ISSN 2347-1921

$$\int_0^{\infty} \Psi(T)\,dT = 1 \quad ; \quad \int_0^{\infty} T\,\Psi(T)\,dT = \overline{T},$$

where $\overline{T}$ is the mean transverse linear velocity. Since

$$\mu(m) = p(m)\cdot\overline{T}. \tag{51}$$

As Equation (51) holds for each small range of magnitude, we have:

$$\overline{\mu} = \overline{p}\cdot\overline{T}, \tag{52}$$

where $\overline{\mu}$ and $\overline{p}$ denote the mean proper motion and parallax of stars within any given magnitude range.

From Equation (51) we have:

$$b(m).\mu(m) = b(m).p(m).\overline{T}.$$

Hence from Equations (51) and (52) we get:

$$b(m)\,.\mu(m) = \overline{T}\int_{-\infty}^{\infty} \exp\{c\,\rho\}\ \Delta(\rho)\Phi(m+\rho)d\rho, \tag{53}$$

$$b(m) = \int_{-\infty}^{\infty} \Delta(\rho)\Phi(m+\rho)d\rho. \tag{54}$$

If the functions b(m) and $\mu(m)$ are obtained from observations and if we suppose that $\overline{T}$ can be found for the given region , say from a representative number of stars of known parallax , the formulae (53) and (54) are two integral equations from which the functions $\Delta(\rho)$ and $\Phi(m+\rho)$ can theoretically be obtained.

In concluding the present paper, two fundamental theorems of stellar statistics were given. Applications of the theorems for the frequency functions of the various characteristics of the stars are developed. Of these functions are , **1**-the apparent and absolute luminosities,2-apparent and absolute magnitudes,,3-Linear velocity and proper motion,, 4-The number and mean parallax of stars of magnitude m and proper motion $\boldsymbol{\mu}$,

5- The total apparent brightness in any region of the sky 6- The mean proper motion

of stars of apparent magnitude m

References

1. Feigelson,E.D. and Babu,G.,J.(Editors):1992, *Statistical Challenges in Modern Astronomy*,Spriger- Verlag,Berlin
2. Sharaf,M.A. and Mominkhan,Z:20014,"An Error Control Method to Determined the stellar Density Function in a Region of the Sky" ,Applied Mathematics **5**,2077
3. Sharaf,M.A:2006,"Computations of the Cosmic Distance Equation" Appl.Math.Comput.174,1269-1278