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BEST CO-APPROXIMATION AND BEST SIMULTANEOUS CO-APPROXIMATION IN INTUITIONISTIC FUZZY NORMED LINEAR SPACES

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ABSTRACT

The main purpose of this paper is to study the t-best co-approximation and t-best simultaneous co-approximation in intuitionistic fuzzy normed spaces. We develop the theory of *t*-best co-approximation and *t*-best simultaneous co-approximation in quotient spaces. This new concept is employed by us to improve various characterisations of *t*-co-proximinal and *t*-co-Chebyshev sets.

Keywords

t-norm, t-conorm; intuitionistic fuzzy normed linear space ; open(closed) ball and bounded set inIntuitionistic fuzzy normed linear space .

1. INTRODUCTION

The theory of a fuzzy sets was firstly introduced by Zadeh [14] in 1965 and thereafter several authors applied it to different branches of pure and applied mathematics. On the other hand , the notion of fuzzyness has a wide application in many areas of science and engineering .

Katsaras [5]in 1984, first introduced the notion of fuzzy norm on a linear space . The concept of a fuzzy norm on a linear space by assigning a fuzzy real number to each element of the linear spaces introduced by Felbin [4] in 1992.

In 1986 ,Atanassov [2] introduced the concept of intuitionistic fuzzy sets. Park[8] first introduced the concept of intuitionistic fuzzy metric space and Saadati and Park [9] introduced the concept of intuitionistic fuzzy normed space, while the notion of intuitionistic fuzzy n-normed linear space was introduced by S. Vijayabalaji , N. Thillaigovindan and Y. Bae [13].

In 2011 ,Abrishami Moghaddam and Sistani [1], firstly introduced the concept of the set of all t-best co-approximation on fuzzy normed spaces.Surender Reddy [12] in 2012 discussed the concept of the t-Best Co-approximation in fuzzy anti-2-normed linear spaces. J. Kavikumar , N. S. Manian and M.B.K. Moorthy [7] introduced the concept of Best Co-approximation and Best Simultaneous Co-approximation in Fuzzy Normed Spaces .

In this paper we study the set of all t-best co-approximation and t-best simultaneous co-approximation in intuitionistic fuzzy normed linear spaces and we develop the theory of t-best co-approximation and t-best simultaneous co-approximation in quotient spaces. This new concept is employed us to improve various characterizations of t-co-proximinal and t-co-Chebyshevsets.

2. PRELIMINARIES

Definition 2.1.[11]: A binary operation $*: [0,1] \times [0,1] \to [0,1]$ is said to be a continuous t-norm if the following axioms are satisfied :

(a) $*$ is $associative$ and $commutative$.

(b) $*$ is $continuous$.

(c) $a * 1 = a$ for all $a \in [0,1]$.

(d) $a * b \le c * d$ $whenever$ $a \le c$ and $b \le d$ for $each$ $a, b, c, d \in [0,1]$.

Definition 2.2.[11]: A binary operation $\lozenge: [0,1] \times [0,1] \to [0,1]$ is said to be a continuous t-conorm if the following axioms are satisfied :

(a) $\lozenge$ is $associative$ and $commutative$

(b) $\lozenge$ is $continuous$

(c) $a \lozenge 0 = a$ for all $a \in [0,1]$

(d) $a \lozenge b \le a \lozenge c$ $whenever$ $a \le c$ and $b \le d$ for $each$ $a, b, c, d \in [0,1]$.

Remark 2.3[11]:

(1) For any $r_1, r_2 \in (0,1)$ $with$ $r_1 > r_2$, there exists $r_3, r_4 \in (0,1)$ $such$ $that$ $r_1 * r_3 \ge r_2$ and $r_1 \ge r_2 \lozenge r_4$.

(2)For any $r_5 \in (0,1)$, $there$ $exists$ $r_6, r_7 \in (0,1)$ $such$ $that$ $r_6 * r_6 \ge r_5$

and $r_7 \lozenge r_7 \le r_5$.

Definition 2.4.[9]: The 5-tuple $(X, \mu, \upsilon, *, \lozenge)$ is said to be an intuitionistic fuzzy normed linear space (IFNLS) if X be a linear space over the field F (R or ₵) ,$*$ is a continuous t-norm , $\lozenge$ is a continuous t-conorm , and μ , υ fuzzy sets on $X \times (0, \infty)$ satisfy the following conditions for every $x, y \in X$ and $s, t > 0$:

(1) $\mu(x,t) + \upsilon(x,t) \le 1$

(2) $\mu(x,t) > 0$

(3) $\mu(x,t) = 1$ if and $only$ if $x = 0$

(4) $\mu(\alpha x, t) = \mu\left(x, \frac{t}{|\alpha|}\right)$ for $each$ $\alpha \neq 0$

(5) $\mu(x + y, t + s) \ge \mu(x,t) * \mu(y,s)$

(6) $\lim_{t \to \infty} \mu(x,t) = 1$

(7) $\upsilon(x,t) < 1$

(8) $\upsilon(x,t) = 0$ if and $only$ if $x = 0$

(9) $\upsilon(\alpha x,t) = \upsilon\left(x,\frac{t}{|\alpha|}\right)$ $for\ each\ \alpha \neq 0$

(10) $\upsilon(x+y,t+s) \leq \upsilon(x,t) \lozenge \upsilon(y,s)$

(11) $\lim_{t\to\infty} \upsilon(x,t) = 0$

Lemma 2.5.[9]: Let $(X,\mu,\upsilon,*,\lozenge)$ be an intuitionistic fuzzy normed linear space then :

(i) $\mu(x,t)$ and $\upsilon(x,t)$ are non-decreasing and non-increasing with respect to t, respectively .

(ii) $\mu(x-y,t) = \mu(y-x,t)$ and $\upsilon(x-y,t) = \upsilon(y-x,t)$ for every $t > 0$ and $x,y \in X$.

Example2.6. [9]: Let $(X,\|.\|)$ be a normed linear space , and let $a * b = min\{a,b\}$ and $a \lozenge b = max\{a,b\}$ for all $a,b \in [0,1]$, For all $x \in X$ and $t > 0$, $\mu(x,t) = \frac{t}{t+\|x\|}$ and $\upsilon(x,t) = \frac{\|x\|}{t+\|x\|}$

Then $(X,\mu,\upsilon,*,\lozenge)$ is an intuitionistic fuzzy normed linear space .

Definition 2.7.[9]: Let $(X,\mu,\upsilon,*,\lozenge)$ be an intuitionistic fuzzy normed linear space . We define an open ball $B(x,r,t)$ with the center $x \in X$ and the radius $0 < r < 1$, as $B(x,r,t) = \{y \in X : \mu(x-y,t) > 1-r,\ \upsilon(x-y,t) < r\}$ for every $t > 0$ also a subset $A \subseteq X$ is called open if for each $x \in A$, there exist $t > 0$ and $0 < r < 1$ such that $B(x,r,t) \subseteq A$. Let $\tau_{(\mu,\upsilon)}$ denote the family of all open subset of X. $\tau_{(\mu,\upsilon)}$ is called the topology induced by intuitionistic fuzzy norm .

Definition 2.8.[9]: Let $(X,\mu,\upsilon,*,\lozenge)$ is an intuitionistic fuzzy normed linear space . For $t > 0$, we define closed ball $B[x,r,t]$ with center $x \in X$ and radius $0 < r < 1$ as $B[x,r,t] = \{y \in X : \mu(x-y,t) \geq 1-r,\ \upsilon(x-y,t) \leq r\}$.

Definition2.9.[9]: Let$(X,\mu,\upsilon,*,\lozenge)$ be IFNLS , A subset G of X is called intuitionistic fuzzy-bounded set (IF-bounded) if there exists $t > 0$ and $0 < r < 1$ such that $\mu(x,t) \geq 1-r$ and $\upsilon(x,t) \leq r$ for all $x \in G$.

3. t-BEST CO-APPROXIMATION IN INTUITIONISTIC FUZZY NORMED LINEAR SPACES

Definition 3.1: Let $(X,\mu,\upsilon,*,\lozenge)$ be IFNLS and G be a nonempty subset of X. An element $g_0 \in G$ is called an intuitionistic fuzzy -t-best co-approximation to x from G (IF-t-best co-approximation) if for $t > 0$, $\mu(g_0-g,t) \geq \mu(x-g,t)$ and $\upsilon(g_0-g,t) \leq \upsilon(x-g,t)$ for all $g \in G$. The set of all IF-t-best co-approximation to x from G will be denoted by $IF - R_G^t(x)$.

Remark 3.2: The set $IF - R_G^t(x)$ of all IF-t-best co-approximation to x from G can be written as :

$IF - R_G^t(x) = \{g_0 \in G : \mu(g_0-g,t) \geq \mu(x-g,t)$ and $\upsilon(g_0-g,t) \leq \upsilon(x-g,t)$ for all $g \in G\}$

Definition 3.3: Let $(X,\mu,\upsilon,*,\lozenge)$ be IFNLS and G be a nonempty subset of . The set of the t-co-metric complement, Define as: for $t > 0$ $\{x \in X: \mu(g,t) \geq \mu(g-x,t),\ \upsilon(g,t) \leq \upsilon(g-x,t),\ \forall g \in G\} = IF - (R_G^t)^{-1}(\{0\})$ will be denoted by $IF - \tilde{G}$.

Proposition 3.4: Let $(X,\mu,\upsilon,*,\lozenge)$ be IFNLS and G be a subspace of X, Then for all $x \in X$, $g_0 \in IF - R_G^t(x)$ if and only if $x - g_0 \in IF - \tilde{G}$ for $t > 0$.

Proof :(⟹) Suppose that $g_0 \in IF - R_G^t(x)$, $x \in X$

$\Rightarrow \mu(g_0-g,t) \geq \mu(x-g,t)$ and $\upsilon(g_0-g,t) \leq \upsilon(x-g,t)$, $\forall g \in G$

Let $g_1 = g + g_0$, $\forall g \in G$ $\Rightarrow$ $g_1 \in G$

$\Rightarrow \mu(g_0-g_1,t) \geq \mu(x-g_1,t)$ and $\upsilon(g_0-g_1,t) \leq \upsilon(x-g_1,t)$

Since $\mu(g_1-g_0,t) = \mu(g_0-g_1,t)$

$\Rightarrow \mu(g+g_0-g_0,t) \geq \mu(x-g-g_0,t)$

$\Rightarrow \mu(g,t) \geq \mu((x-g_0)-g,t) = \mu(g-(x-g_0))$

$\Rightarrow \mu(g,t) \geq \mu(g-(x-g_0),t)$

similarly , we get

$\upsilon(g,t) \leq \upsilon(g-(x-g_0),t)$, $\forall g \in G$

$\Rightarrow x - g_0 \in IF - \tilde{G}$.

(⟸) Assume $x - g_0 \in IF - \tilde{G}$

$\Rightarrow \mu(g,t) \geq \mu(g-(x-g_0),t)$ and $\upsilon(g,t) \leq \upsilon(g-(x-g_0),t)$, $\forall g \in G$

Let $g_1 = g - g_0 \Rightarrow g_1 \in G$

$\mu(g_1, t) \geq \mu(g_1 - (x - g_0), t)$ and $\upsilon(g_1, t) \leq \upsilon(g_1 - (x - g_0), t)$

$\Rightarrow \mu(g - g_0, t) \geq \mu(g - g_0 - (x - g_0), t)$ and $\upsilon(g - g_0, t) \leq \upsilon(g - g_0 - (x - g_0), t)$

$\Rightarrow \mu(g - g_0, t) \geq \mu(g - x, t)$ and $\upsilon(g - g_0, t) \leq \upsilon(g - x, t), \forall g \in G$

Hence $g_0 \in IF - R_G^t(x)$ *if and only if* $x - g_0 \in IF - \tilde{G}$.■

Definition 3.5: Let $(X, \mu, \upsilon, *, \Diamond)$ *be an* IFNLS *and* G be a nonempty subset of X . If for every $x \in X$ has at least one IF-t-best co-approximation in G , then G is called an intuitionistic fuzzy-t-co-proximinal set (IF-t- co-proximinal set) .

Definition 3.6: Let $(X, \mu, \upsilon, *, \Diamond)$ *be an* IFNLS *and* G be a nonempty subset of X . If for every $x \in X$ has exactly one IF-t-best co-approximation in G , then G is called an intuitionistic fuzzy -t-co-Chebyshev set (IF -t-co-Chebyshev set) .

Definition 3.7: Let $(X, \mu, \upsilon, *, \Diamond)$ be an IFNLS. A subset G is said to be convex set if $(1 - \lambda)x + \lambda g_0 \in G$ whenever $g_0 \in G$, $x \in X$ and $0 < \lambda < 1$.

Theorem 3.8: Let $(X, \mu, \upsilon, *, \Diamond)$ be an IFNLS and G is a nonempty subset of X , if $g_0 \in IF - R_G^t(x)$ and $(1 - \lambda)x + \lambda g_0 \in G$ *for* $0 < \lambda < 1$, $t > 0$, then $(1 - \lambda)x + \lambda g_0 \in IF - R_G^t(x)$.

Proof : Let $g_0 \in IF - R_G^t(x)$ *and* $(1 - \lambda)x + \lambda g_0 \in G$ *for* $0 < \lambda < 1$, $t > 0$

$\Rightarrow \mu(g_0 - g, t) \geq \mu(x - g, t)$ and $\upsilon(g_0 - g, t) \leq \upsilon(x - g, t), \forall g \in G \; \ldots. (1)$

Therefore , for a given $t > 0$, take the natural number n such that $t > \frac{1}{n}$

By assumption and definition 2.4. , we have

$\mu([(1 - \lambda)x + \lambda g_0] - g, t)$

$= \mu([(1 - \lambda)x - \lambda g + \lambda g + \lambda g_0] - g, t)$

$= \mu((1 - \lambda)x - (1 - \lambda)g + \lambda(g_0 - g), t)$

$= \mu((1 - \lambda)(x - g) + \lambda(g_0 - g), t)$

$\geq \mu\left(x - g, \frac{t}{2(1 - \lambda)}\right) * \mu\left(g_0 - g, \frac{t}{2\lambda}\right)$

$\geq \mu\left(x - g, \frac{1}{2(1 - \lambda)n}\right) * \mu\left(x - g, \frac{1}{2\lambda n}\right) = \lim_{n \to \infty} \mu\left(x - g, \frac{1}{2\lambda n}\right) = \mu(x - g, t)$ [*since* (1) *and* $t > \frac{1}{n}$]

and for a given $t > 0$,take the natural number n such that $t < \frac{1}{n}$

$\upsilon([(1 - \lambda)x + \lambda g_0] - g, t)$

$= \upsilon([(1 - \lambda)x - \lambda g + \lambda g + \lambda g_0] - g, t)$

$= \upsilon((1 - \lambda)x - (1 - \lambda)g + \lambda(g_0 - g), t)$

$= \upsilon((1 - \lambda)(x - g) + \lambda(g_0 - g), t)$

$\leq \upsilon\left(x - g, \frac{t}{2(1 - \lambda)}\right) \Diamond \upsilon\left(g_0 - g, \frac{t}{2\lambda}\right)$

$\leq \upsilon\left(x - g, \frac{1}{2(1 - \lambda)n}\right) \Diamond \upsilon\left(x - g, \frac{1}{2\lambda n}\right) = \lim_{n \to \infty} \upsilon\left(x - g, \frac{1}{2\lambda n}\right) = \upsilon(x - g, t)$ [*since* (1) *and* $t < \frac{1}{n}$]

Thus $(1 - \lambda)x + \lambda g_0 \in IF - R_G^t(x)$.■

Corollary 3.9: Let $(X, \mu, \upsilon, *, \Diamond)$ be an IFNLS. If G is convex subset of X, then $IF - R_G^t(x)$ is convex subset of X .

Proof : Let G is convex subset of X and $g_0 \in IF - R_G^t(x)$, *for every* $x \in X$ *and* $0 < \lambda < 1$

Since G is convex subset of X

By theorem 3.8 , we get

$(1 - \lambda)x + \lambda g_0 \in IF - R_G^t(x)$

Hence $IF - R_G^t(x)$ is convex subset of X . ■

Theorem 3.10: Let $(X,\ \mu,\ \upsilon,*,\lozenge)$ be an IFNLS and G be a subspace of X, $Then$:

$(i)\ IF - R^{|\alpha|t}_{\alpha G}(\alpha x)\ = IF - \alpha R^t_G(x)\ \ for\ every\ \ x \in X\ ,\ t > 0\ \ and\ \ \alpha \in R/\{0\}.$

$(ii)\ IF - R^t_{G+y}(x+y)\ = IF - R^t_G(x) + y\ \ for\ every\ \ x, y \in X\ ,\ t > 0$.

proof : $(i)\ \ g_0 \in IF - R^{|\alpha|t}_{\alpha G}(\alpha x)$

$\Rightarrow g_0 \in \alpha G\ \ ,\mu(g_0 - g, |\alpha|t) \geq \mu\,(\alpha x - g, |\alpha|t)\ \ and\ \ \upsilon(g_0 - g, |\alpha|t) \leq \upsilon\,(\alpha x - g, |\alpha|t), \forall g \in \alpha G$

$\Leftrightarrow\ \mu\left(\frac{1}{\alpha}g_0 - \frac{1}{\alpha}g\ ,t\right) \geq \mu\left(x - \frac{1}{\alpha}g\ ,t\right)\ \ and\ \ \upsilon\left(\frac{1}{\alpha}g_0 - \frac{1}{\alpha}g\ ,t\right) \leq \upsilon\left(x - \frac{1}{\alpha}g\ ,t\right), \forall\ \frac{1}{\alpha}g \in G$

$\Leftrightarrow \mu\left(\frac{1}{\alpha}g_0 - g_1\,,t\right) \geq \mu(x - g_1\ ,\ t)$ and $\upsilon\left(\frac{1}{\alpha}g_0 - g_1\,,t\right) \leq \upsilon(x - g_1\ ,\ t)$, $\forall g_1 = \frac{1}{\alpha}\ g \in G$

$\Leftrightarrow \frac{1}{\alpha}g_0 \in IF - R^t_G(x) \Leftrightarrow g_0 \in IF - \alpha R^t_G(x)$

Hence $IF - R^{|\alpha|t}_{\alpha G}(\alpha x)\ = IF - \alpha R^t_G(x)$

$(ii)\ \ g_0 \in IF - R^t_{G+y}(x+y)$

$\Leftrightarrow \mu(g_0 - (g+y), t\,) \geq \mu\,((x+y) - (g+y), t)\ \ and\ \ \upsilon(g_0 - (g+y), t) \leq \upsilon\big((x+y) - (g+y), t\big),\ \ \forall g + y \in G + y$

$\Leftrightarrow \mu\big((g_0 - y) - g\ ,t\big) \geq \mu(x - g\ ,t)\ \ and\ \ \upsilon\big((g_0 - y) - g\ ,t\big) \leq\ \upsilon(x - g\ ,t)\ \ ,\forall g \in G$

$\Leftrightarrow\ \ g_0 - y \in IF - R^t_G(x)$

$\Leftrightarrow g_0 \in IF - R^t_G(x) + y$

Hence $IF - R^t_{G+y}(x+y)\ =\ IF - R^t_G(x) + y$ ■

Corollary 3.11: Let $(X,\mu\,,\upsilon,*,\lozenge)$ be an IFNLS and G be a subspace of X ,Then the following statements are hold :

$(i)\ G$ is IF-t-co- proximinal set (resp. IF-t-co-Chebyshevset) if and only if $|\alpha|G$ is IF-$|\alpha|t$-co-proximinal set (resp. IF-$|\alpha|$t-co-Chebyshev set) for any scalar $\alpha \in R|\{0\}$.

$(ii)\ G$ is IF-t-co- proximinal set (resp. IF-t-co-Chebyshev set) if and only if $G + y$ is IF-t-co- proximinal set (resp. IF-t-co-Chebyshev set) for every $y \in X$.

proof : $(i)\ G$ is IF-t-co-proximinal $\Leftrightarrow\ IF - R^t_G(x) \neq \emptyset$

$by\ \ theorm\ 3.10$

$\Leftrightarrow IF - \alpha R^t_G(x) \neq \emptyset$

$\Leftrightarrow\ IF - R^{|\alpha|t}_{\alpha G}(\alpha x) \neq \emptyset$

Then $|\alpha|G$ is IF-t-co-proximinal set .

similarly , we get

$|\alpha|G$ is IF-t-co-Chebyshev set .

$(ii)\ \ G$ is IF-t-co-proximinal set $\Leftrightarrow IF - R^t_G(x) \neq \emptyset$

$\Leftrightarrow IF - R^t_G(x) + y \neq \emptyset\ \ \Leftrightarrow IF - R^t_{G+y}(x+y) \neq \emptyset$

Then $G + y$ is IF-t-co-proximinal set .

similarly, we get

$G + y$ is IF-t-co-Chebyshev set . ■

4. t-CO-PROXIMINALITY AND t-CO-CHEBYSHEVITY IN QUOTIENT SPACES

Definition 4.1.[3]: Let $(X,\mu,\ \upsilon,*,\lozenge)$ be an IFNLS and M is a closed subspace of X , for $t > 0$, we define $\emptyset(x + M\,,t) = sup\{\,\mu\,(x + y\,,t\,) :\ \ y \in M\}$

$\varphi(x + M\,,t) = inf\{\,\upsilon(x + y\,,t) : y \in M\}$ where $x + M = \{x + m : m \in M\}$.

Theorem 4.2. [3]: Let $(X,\mu\,,\upsilon,*,\lozenge)$ be an IFNLS and M is a closed subspace of X, $\emptyset(x + M\,,t)\ and\ \varphi(x + M\,,t)$ are defined in Definition **4.1** , and $X/M = \{x + M : x \in X\}$. Then $(X|M, \emptyset, \varphi, *, \lozenge)$ is an intuitionistic fuzzy normed linear space .

Theorem 4.3: Let $(X,\mu,\upsilon,*,\lozenge)$ be an IFNLS and M is a closed subspace of X and $G \supseteq M$ a subspace of X. If G is an IF-t-co-proximinal set of X, Then $G|M$ is an IF-t-co-proximinal set of $X|M$.

Proof : Let G is an IF-t-co-proximinal set of X.

$\Longrightarrow \exists\, g_0 \in G, x \in X$ such that $\mu(g_0 - g,t) \geq \mu(x-g,t)$ and $\upsilon(g_0 - g,t) \leq \upsilon(x-g,t)$, $\forall g \in G$

$\Longrightarrow \mu(g_0 - m + m - g, t) \geq \mu(x - m + m - g, t)$ and $\upsilon(g_0 - m + m - g, t) \leq \upsilon(x - m + m - g, t)$, $\forall m \in M$

$\Longrightarrow \mu\big((g_0+m)-(g+m),t\big) \geq \mu\big((x+m)-(g+m),t\big)$ and $\upsilon\big((g_0+m)-(g+m),t\big) \leq \upsilon\big((x+m)-(g+m),t\big)$

$,\forall g+M \in G|M$ and $x+M \in X|M$

$\Longrightarrow g_0 + M \in IF - R^t_{G|M}(x+M)$

$\Longrightarrow IF - R^t_{G|M}(x+M) \neq \emptyset$

$\Longrightarrow G|M$ is an IF-t-co-proximinal set of $X|M$. ■

Corollary 4.4: Let $(X,\mu,\upsilon,*,\lozenge)$ be an IFNLS and M is a closed subspace of X and $G \supseteq M$ a subspace of X. If $G|M$ is an IF-t-co-proximinal with $X|M$, Then G is an IF-t-co-proximinal with X.

Proof: Let $G|M$ is an IF-t-co-proximinal with $X|M$.

$\Longrightarrow IF - R^t_{G|M}(x+M) \neq \emptyset$

$Let\ g_0 + M \in IF - R^t_{G|M}(x+M)$

$\Longrightarrow \mu\big((g_0+m)-(g+m),t\big) \geq \mu\big((x+m)-(g+m),t\big)$ and $\upsilon\big((g_0+m)-(g+m),t\big) \leq \upsilon\big((x+m)-(g+m),t\big), \forall\, m \in M$

$\Longrightarrow \mu(g_0 - g,t) \geq \mu(x-g,t)$ and $\upsilon(g_0-g,t) \leq \upsilon(x-g,t)$, $\forall g \in G$

$\Longrightarrow g_0 \in \mathrm{IF} - R^t_G(x)$

$\Longrightarrow$ IF-$R^t_G(x) \neq \emptyset$

$\Longrightarrow G$ is IF-t-co-proximinal with X. ■

Theorem 4.5: Let $(X,\mu,\upsilon,*,\lozenge)$ be an IFNLS and M is a closed subspace of X and $G \supseteq M$ a subspace of X. If $G|M$ is an IF-t-co-Chebyshev with $X|M$, Then G is an IF-t-co-Chebyshev with X.

Proof: Let $G|M$ is IF-t-co-Chebyshev with $X|M$ and G has two distinct t-best co-approximation of $x \in X$ such as y_1, y_2 in X.

$\Longrightarrow \mu(y_1 - g,t) \geq \mu(x-g,t)$ and $\upsilon(y_1-g,t) \leq \upsilon(x-g,t), \forall g \in G$

also $\mu(y_2 - g,t) \geq \mu(x-g,t)$ and $\upsilon(y_2-g,t) \leq \upsilon(x-g,t)$, $\forall g \in G$

$\Longrightarrow \mu\big((y_1+m)-(g+m),t\big) \geq \mu\big((x+m)-(g+m),t\big)$ and $\upsilon\big((y_1+m)-(g+m),t\big) \leq \upsilon\big((x+m)-(g+m),t\big)$

$,\forall g \in G, m \in M$

also $\mu\big((y_2+m)-(g+m),t\big) \geq \mu\big((x+m)-(g+m),t\big)$ and $\upsilon\big((y_2+m)-(g+m),t\big) \leq \upsilon\big((x+m)-(g+m),t\big)$, $\forall g \in G, m \in M$

$\Longrightarrow y_1 + M, y_2 + M \in IF - R^t_{G|M}(x+M)$

since $y_1, \neq y_2 \Longrightarrow y_1 + M \neq y_2 + M$

$\Longrightarrow IF - R^t_{G|M}(x+M)$ is not $\mathrm{IF-t-co-Chebyshev}$, this contradiction(#).

$\Longrightarrow y_1 = y_2$

Then G is an IF-t-co-Chebyshev with X. ■

Definition 4.6.[6]: Let $(X,\mu,\upsilon,*,\lozenge)$ be an IFNLS and M is a closed subspace of X, for $t > 0$ and $x \in X$ the distance between x and M define as :

$d_\mu(x,M,t) = sup\{\mu(x-y,t) : y \in M\}$ and $d_\upsilon(x,M,t) = inf\{\upsilon(x-y,t) : y \in M\}$.

Theorem 4.7: Let M and G are two subspaces of $(X,\mu,\upsilon,*,\lozenge)$ such that $M \subset G$ and $x + G \in X/G$, $g_1 \in G$. If g_1 is IF-t-best co-approximation to x from G, Then $g_1 + M$ is an IF-t-best co-approximation to $x + M$ from the quotient space G/M.

Proof : Suppose that g_1 is IF-t-best co-approximation to x from G and $g_1 + M$ is not IF-t-best co-approximation to $x + M$ from the quotient space G/M .

$\Longrightarrow \exists\, \acute{g}_1 + M \in G|M$ such that $\mu(\acute{g}_1 + M - (g_1 + M), t) < \mu\,(x + M - (\acute{g}_1 + M), t)$ and

$\upsilon(\acute{g}_1 + M - (g_1 + M), t) > \upsilon(x + M - (\acute{g}_1 + M), t)$

$\Longrightarrow \mu(\acute{g}_1 - g_1 + M, t) < \mu\,(x - \acute{g}_1 + M, t)$ and $\upsilon(\acute{g}_1 - g_1 + M, t) > \upsilon(x - \acute{g}_1 + M, t)$

Since $d_\mu(x\,, M, t) = sup\{\mu\,(x - y\,, t) : y \in M\}$ and $d_\upsilon(x\,, M, t) = inf\{\upsilon(x - y\,, t) : y \in M\}$

$\Longrightarrow sup\{\mu(x - \acute{g}_1 + M\,, t)\} > sup\{\,\mu\,(\acute{g}_1 - g_1 + M\,, t)\}$ and $inf\{\upsilon(x - \acute{g}_1 + M\,, t)\} < inf\{\,\upsilon(\acute{g}_1 - g_1 + M\,, t)\}$

$\Longrightarrow d_\mu(x - \acute{g}_1, M, t) > d_\mu(\acute{g}_1 - g_1, M\,, t)$ and $d_\upsilon(x - \acute{g}_1, M, t) < d_\upsilon(\acute{g}_1 - g_1, M\,, t)$

this implies that there exists $g \in M$ such that

$$\mu(x - \acute{g}_1 - g\,, t) > d_\mu(\acute{g}_1 - g_1, M, t) > \mu(\acute{g}_1 - g_1 + g, t) \;\; and \;\; \upsilon(x - \acute{g}_1 - g\,, t) < d_\upsilon(\acute{g}_1 - g_1, M, t) < \upsilon(\acute{g}_1 - g_1 + g, t)$$

$\Longrightarrow \exists g + \acute{g}_1 \in G$ such that $\mu\big((g + \acute{g}_1) - g_1, t\big) < \mu(x - (g + \acute{g}_1)\,, t)$ and $\upsilon\big((g + \acute{g}_1) - g_1, t\big) > \upsilon(x - (g + \acute{g}_1)\,, t)$

$\Longrightarrow g_1$ is not an IF-t-best co-approximation to x from G , this contradiction with hypothesis(#) .

Then $g_1 + M$ is an IF-t-best co-approximation to $x + M$ from the quotient space G/M . ■

5.t-BEST SIMULTANEOUS CO-APPROXIMATION IN INTUITIONISTIC FUZZY NORMED LINEAR SPACES

Definition 5.1: Let $(X,\ \mu\,, \upsilon, *, \lozenge)$ be an IFNLS and G be a subset of X , M be IF-bounded subset in X . An element $g_0 \in G$ is called IF-t-best simultaneous co-approximation to M from G , if for $t > 0$,

$\mu(g_0 - g\,, t\,) \geq inf\{\mu\,(m - g, t) : m \in M\}$ and $\upsilon(g_0 - g, t) \leq sup\{\,\upsilon(m - g\,, t) : m \in M\}$ for all $g \in G$.

The set of all IF-t-best simultaneous co-approximation to M from G , will be denoted by $IF - S_G^t(M)$ and define as follows :

$$IF - S_G^t(M) = \{g_0 \in G : \ \mu(g_0 - g\,, t\,) \geq inf_{m\in M}\ \mu(m - g\,, t) \;\; and \;\; \upsilon(g_0 - g\,, t) \leq sup_{m\in M}\, \upsilon(m - g\,, t\,), \forall g \in G\,\}$$

Definition 5.2: Let G be a subset of $(X, \mu\ , \upsilon, *, \lozenge)$. It is called IF-t-best simultaneous co-proximinal subset of X , if for each IF-bounded set M in X , there exists at least one IF-t-best simultaneous co-approximation from G to M .

Definition 5.3: Let G be a subset of $(X,\ \mu\,, \upsilon, *, \lozenge)$. It is called IF-t-best simultaneous co-Chebyshev subset of X, if for each IF-bounded M in X there exists a unique IF- t- best simultaneous- co-approximation from G to M .

Theorem 5. 4: Let $(X, \mu\ , \upsilon, *, \lozenge)$ and G be a subset of X . If M is IF-bounded subset of X and $*, \lozenge$ satisfying the condition $a * b \geq a$, $a \lozenge b \leq a$, $\forall\, a, b \in [0,1]$, then $IF - S_G^t(M)$ is IF-bounded subset of X .

Proof : Let M is IF-bounded subset of X and $g_0 \in IF - S_G^t(M)$

$\Longrightarrow$ there exist $0 < r < 1$ $such\ that$ $\mu(x, t) \geq 1 - r$, $\upsilon(x, t) \leq r$, $\forall x \in M\,, t > 0$ and

$\mu(g_0 - g, t) \geq inf_{m\in M}\, \mu(m - g, t)$and $\upsilon(g_0 - g, t) \leq sup_{m\in M}\, \upsilon(m - g, t)$, $\forall g \in G$

$\Rightarrow$ for every $g \in G, m \in M$, $\mu(g_0, 3t)$= $\mu(g_0 - m + m, 3t) \geq \mu(g_0 - m, 2t) * \mu\,(m, t)$

$$\geq \mu(g_0 - g + g - m, 2t) * (1 - r)$$

$$\geq \mu(g_0 - g\,, t) * \mu\,(g - m, t) * (1 - r)$$

$$\geq inf_{m\in M}\ \mu(m - g, t) * \mu(m - g, t) * (1 - r)$$

$$\geq inf_{m\in M}\ \mu(m - g, t)\ * (1 - r)$$

$$\geq 1 - r_0 \quad for\ some \quad 0 < r_0 < 1$$

and $\upsilon(g_0, 3t) = \upsilon(g_0 - m + m\,, 3t)\ \leq\ \upsilon(g_0 - m\,, 2t)\ \lozenge\ \upsilon(m\,, t)$

$$\leq\ \upsilon(g_0 - g + g - m\,, 2t) \lozenge r$$

$$\leq\ \upsilon(g_0 - g, t) \lozenge \upsilon(g - m\,, t) \lozenge r$$

$$\leq sup_{m\in M}\ \ \upsilon(m - g\,, t)\ \lozenge\ \upsilon(m - g\,, t)\ \lozenge\ r$$

$$\leq sup_{m\in M}\ \ \upsilon(m - g\,, t)\ \ \lozenge\ r$$

$$\leq\ r_0 \quad \text{for some} \quad 0 < r_0 < 1$$

Then $IF - S_G^t(M)$ is an IF-bounded subset of X . ■

Theorem 5.5: Let $(X,\mu\ ,\upsilon,*,\Diamond) IFNLS\ and\ M$ is IF-bounded subset of X . If G is a convex subset of X and $*,\Diamond$ satisfying the condition $a*b \geq a\ ,\ a \Diamond b \leq a\ ,\ \forall\ a,b \in [0,1]$, then $IF - S_G^t(M)$ is a convex subset of X .

Proof : Suppose that G is a convex subset of X

$\Rightarrow (1-\lambda)x + \lambda g_0 \in G$ for every $g_0 \in G\ ,\ x \in X\ and\ \ 0<\lambda<1$

Therefore , $for\ a\ given\ \ t>0\ ,\ take\ n \in N\ \ such\ that\ \ t > \frac{1}{n}\ \ ,\ \ we\ get$

$\mu([(1-\lambda)m\ +\ \lambda g_0] - g\ ,t)$

$= \mu([(1-\lambda)m - \lambda g + \lambda g + \lambda g_0] - g\ ,t)$

$= \mu\big((1-\lambda)m - (1-\lambda)g + \lambda(g_0 - g)\ ,t\big)$

$= \mu\big((1-\lambda)(m-g) + \lambda(g_0 - g)\ ,t\big)$

$\geq \mu\left(m-g\ ,\frac{t}{2(1-\lambda)}\right) * \mu\left(g_0 - g\ ,\frac{t}{2\lambda}\right)$

$\geq \mu\left(m-g\ ,\frac{1}{2(1-\lambda)n}\right) * inf_{m\in M}\ \mu\left(m-g,\frac{1}{2\lambda n}\right) = \lim_{n\to\infty}\ inf_{m\in M}\ \mu\left(m-g,\frac{t}{2\lambda n}\right) = inf\ _{m\in M}\ \mu\ (m-g,t)$

and $for\ a\ given\ \ t>0\ ,\ take\ n \in N\ \ such\ that\ \ t < \frac{1}{n}\ \ ,\ \ we\ get$

$\upsilon([(1-\lambda)m\ +\ \lambda g_0] - g\ ,t)$

$= \upsilon([(1-\lambda)m - \lambda g + \lambda g + \lambda g_0] - g\ ,t)$

$= \upsilon\big((1-\lambda)m - (1-\lambda)g + \lambda(g_0 - g)\ ,t\big)$

$= \upsilon\big((1-\lambda)(m-g) + \lambda(g_0 - g)\ ,t\big)$

$\leq\ \upsilon\left(m-g\ ,\frac{t}{2(1-\lambda)}\right) \Diamond\ \upsilon\left(g_0 - g,\frac{t}{2\lambda}\right)$

$\leq\ \upsilon\left(m-g\ ,\frac{1}{2(1-\lambda)n}\right) \Diamond\ \ sup_{m\in M}\ \upsilon\left(m-g,\frac{1}{2\lambda n}\right) = \lim_{n\to\infty}\ sup_{m\in M}\ \upsilon\left(m-g,\frac{1}{2\lambda n}\right) = sup\ _{m\in M}\ \upsilon\ (m-g,t)$

$\Rightarrow (1-\lambda)m + \lambda g_0 \in IF - S_G^t(M)$

Then $IF - S_G^t(M)\ \ is\ a\ convex\ subset\ of\ X$. ■

Theorem 5.6 : Let G is a subset of $(X,\mu\ ,\upsilon,*,\Diamond)$ and M is IF-bounded in X , Then the following assertions are hold for $t>0$:

(1) $IF - S_{G+x}^t(x+M) = IF - S_G^t(M) + x\ \ \ ,\ \ \forall\ x \in X$.

(2) $IF - S_{\alpha G}^{|\alpha| t}(\alpha M) = IF -\ \alpha\, S_G^t(M)\ ,\ \forall\ \ \alpha \in R$.

Proof : (1) let $g_0 \in IF - S_G^t(M) + x$

$\Rightarrow$ Forevery $g_1 \in G$, $\mu\,(g_1 - (g_0 + x)\ ,t) = \mu(g_1 - x - g_0\ ,t) \geq\ \ inf_{m\in M}\ \mu(m - (g_1 - x)\ ,t) =\ inf_{m\in M}\ \mu(m + x - g_1\ ,t) \Rightarrow \mu(g_1 - (g_0 + x)\ ,t)\ \ \geq\ \ inf_{m\in M}\ \mu\,(m + x - g_1\ ,t)$

similarly , we get $\upsilon(g_1 - (g_0 + x)\ ,t)\ \ \leq\ \ sup_{m\in M}\ \ \upsilon(m + x - g_1\ ,t)$

Then $g_0 + x \in IF - S_{G+x}^t(M+x)$

Let $g_0 + x \in IF - S_{G+x}^t(M+x)$

$\Rightarrow$For every $g_1 + x \in G + x$, $\mu(g_1 - g_0\ ,t) = \mu(g_1 + x - (g_0 + x)\ ,t) \geq inf_{m\in M}\ \ \mu\,(m + x - (g_1 + x),t)$

$= inf_{m\in M}\ \mu\,(m - g_1\ ,t)$

$\Rightarrow \mu(g_1 - g_0\ ,t) \geq inf_{m\in M}\ \mu\,(m - g_1\ ,t), \forall\ g_1 \in G$

similarly , we get

$\upsilon(g_1 - g_0, t) \leq sup_{m\in M}\ \ \upsilon(m - g_1\ ,t), \forall\ g_1 \in G$

Then $IF - S_{G+x}^t(M+x) = IF - S_G^t(M) + x\ ,\ \forall\ x \in X$.

(2) $IF - S_{\alpha G}^{|\alpha| t}(\alpha M) = IF - \alpha S_G^t(M)\ ,\ \ \forall\ \alpha \in R$

Proof : clearly equality holds for $\alpha = 0$

Let $\alpha \neq 0$,$g_0 \in IF - S_{\alpha G}^{|\alpha| t}(\alpha M)$ if and only if

$g_0 \in \alpha\, G$ such that $\mu(g_0 - g, |\alpha|t) \geq inf_{m\in M}\ \mu(\alpha m - g\,, |\alpha|t)$ *and*

$\upsilon(g_0 - g, |\alpha|t) \leq sup_{m\in M}\ \upsilon(m - g\,, |\alpha|t)\,, \forall g \in G$

$\Leftrightarrow \mu\left(\frac{1}{\alpha} g_0 - \frac{1}{\alpha} g\,,\ t\right) \geq\ inf_{m\in M}\ \mu\left(m - \frac{1}{\alpha} g\,,\ t\right)$ and $\upsilon\left(\frac{1}{\alpha} g_0 - \frac{1}{\alpha} g\,,\ t\right) \leq\ sup_{m\in M}\ \upsilon\left(m - \frac{1}{\alpha} g\,,\ t\right)\ , \forall \frac{1}{\alpha} g \in G$

$\Leftrightarrow\ \mu\left(\frac{1}{\alpha} g_0 - g_1\,,\ t\right) \geq\ inf_{m\in M}\ \mu\,(m - g_1\,,\ t\,)$and $\upsilon\left(\frac{1}{\alpha} g_0 - g_1\,,\ t\right) \leq\ sup_{m\in M}\ \upsilon(m - g_1\,,\ t\,)\ , \forall g_1 = \frac{1}{\alpha} g \in G$

$\Leftrightarrow \frac{1}{\alpha} g_0\ \in IF - S_G^t(M)\ \Leftrightarrow\ g_0 \in IF - \alpha\, S_G^t(M)$

Then $IF - S_{\alpha G}^{|\alpha|t}(\alpha M)\ = IF - \alpha\, S_G^t(M)$■

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