# PERIODIC SOLUTIONS OF A MODEL OF LOTKA-VOLTERRA WITH VARIABLE STRUCTURE AND IMPULSES

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## ABSTRACT

We consider a generalized version of the classical Lotka Volterra model with differential equations. The version has a variable structure (discontinuous right hand side) and the solutions are subjected to the discrete impulsive effects. The moments of right hand side discontinuity and the moments of impulsive effects coincide and they are specific for each solution. Using the Brouwer fixed point theorem, sufficient conditions for the existence of periodic solution are found.
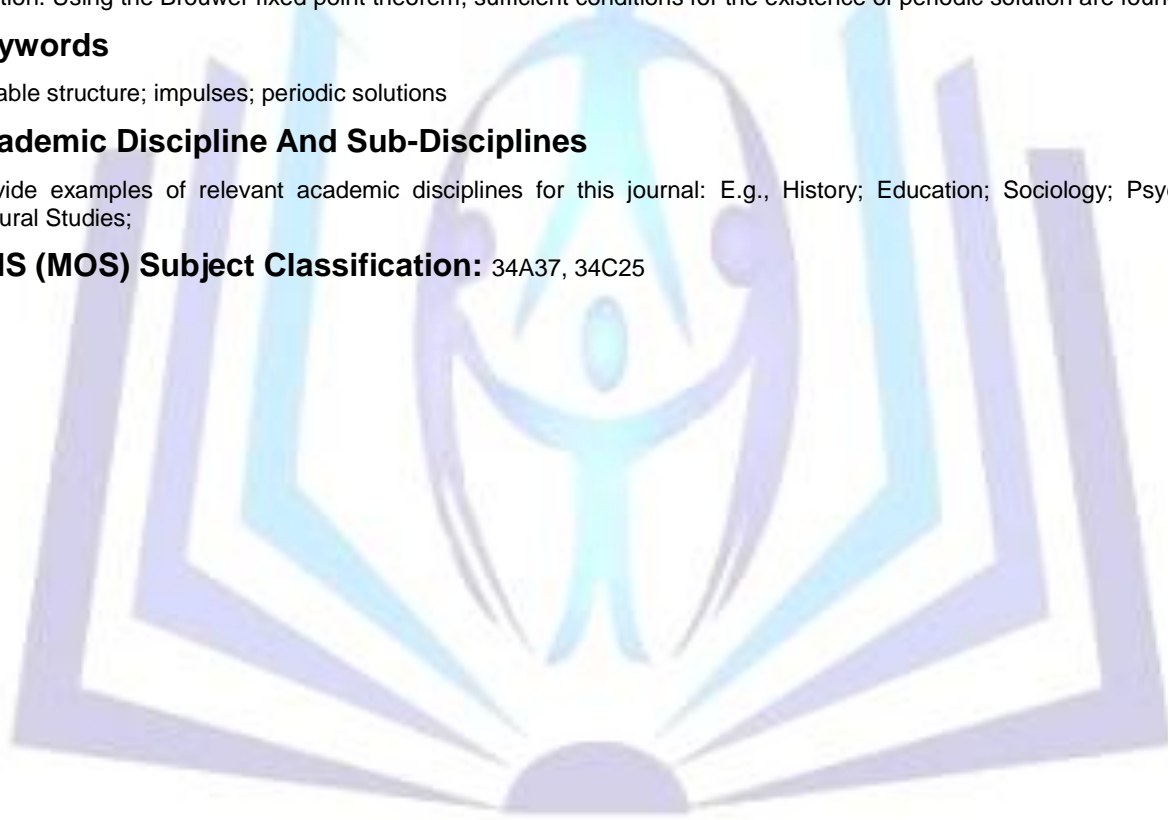
## Keywords

variable structure; impulses; periodic solutions

## Academic Discipline And Sub-Disciplines

Provide examples of relevant academic disciplines for this journal: E.g., History; Education; Sociology; Psychology; Cultural Studies;

## AMS (MOS) Subject Classification: 34A37, 34C25

# 1 INTRODUCTION

We assume that the predator-prey community is subjected to the external effects. These effects are discrete in time and relate to:

1. Impulsive changes in the biomass amount of both species (forming part of the community). Mathematically this is expressed through a jump-like change of the state (phase or solution) of the modeling system;

2. Instantaneous rate of change of community development speed. Usually this effect is associated with a jump-like change system parameters. Generally, this effect is expressed by changing the structure of the modeling.

We shall assume that these jump-like changes (both the state and also the speed of community development) are performed simultaneously. The moments of external effects are variable and depend on the quantities of biomass, i.e. on the specific solution of modeling system. The above-described community has different variations of mathematical modeling. In general, these options depend on:

1. The nature of interactions between the species in the periods outside of the moments of external influences (reflected in the type of right hand side of modeling system);

2. Impulsive moments and the way they are set (these are the moments that the external effects take place);

3. Impulsive effects (the quantities of withdrawal or adding biomass to the predator and prey);

4. Changing the structure of the modeling system (values which change the parameters of the community development), etc.

Impulsive differential equations are suitable mathematical tool for processes modeling with short-term changes in their phase state. This type of equations is developed intensively because of their serious applications. Concrete results relating to the Lotka-Volterra models, subjected to pulse effects, are contained in [5], [6], [9], [10] and [11].

Various aspects of the qualitative theory of impulsive differential equations with variable structure and impulses are discussed in [1], [2], [4], [7] and [8].

This paper contains the application of the above mentioned type of differential equations. More precisely, conditions associated with the elements of system, ensuring the existence of periodic solutions are found.

# 2 DESCRIPTION OF THE MODEL

Consider the following initial value problem for systems of differential equations with variable structure and moments of impulsive effects

$$\frac{dm}{dt} = f_i^1(m, M) = m\left(r_i^1 - q_i^1 M\right), \ \left(m(t), M(t)\right) \notin \Phi_i \Leftrightarrow t_{i-1} < t < t_i \ ; \tag{1}$$

$$\frac{dM}{dt} = f_i^2(m, M) = -M\left(r_i^2 - q_i^2 m\right), \ \left(m(t), M(t)\right) \notin \Phi_i \Leftrightarrow t_{i-1} < t < t_i \ ; \tag{2}$$

$$m(t_i + 0) = g_i\left(m(t_i)\right), \ \left(m(t_i), M(t_i)\right) \in \Phi_i \ ; \tag{3}$$

$$M(t_i + 0) = M_{i+1}^{00}, \ \left(m(t_i), M(t_i)\right) \in \Phi_i, \ i = 1, 2, \dots \ ; \tag{4}$$

$$m(0) = m_0, \quad M(0) = M_0, \tag{5}$$

where:

- $m = m(t) > 0$ and $M = M(t) > 0$ are the prey and predator biomasses at the moment $t \geq 0$;

- $r_i^1 = const > 0$ and $r_i^2 = const > 0$ are specific growth factors, relevant to the first species (prey) and second (predator), respectively;

- $q_i^1 = const > 0$ and $q_i^2 = const > 0$ are the coefficients indicating interspecies competition. In the common case, they are different for the prey and predator;

- $\left(m_i^{00}, M_i^{00}\right) = \left(\dfrac{r_i^2}{q_i^2}, \dfrac{r_i^1}{q_i^1}\right);$

- $\Phi_1, \Phi_2, \ldots$ are switching sets, $\Phi_i = \left\{ (m, M); \ m > m_i^{00}, \ M = M_i^{00} \right\}$, $i = 1, 2, \ldots$;

- $g_i : R^+ \to R^+$;

- $t_1, t_2, \ldots$ are switching moments;

- $m_0 > 0$ and $M_0 > 0$ are the prey and predator biomasses at the initial moment $t = 0$.

**Remark 1.** The solution $\left( m(t; m_0, M_0), M(t; m_0, M_0) \right)$ of the initial value problem (1)-(5) is defined as follows:

1.1. For $0 = t_0 \leq t < t_1$:

1.1.1. $\left( m(t; m_0, M_0), M(t; m_0, M_0) \right)$ coincides with the solution of the system (1), (2) for $i = 1$ with initial condition (5);

1.1.2. $\left( m(t; m_0, M_0), M(t; m_0, M_0) \right) \notin \Phi_1$;

1.2. For $t = t_1$:

1.2.1. $\left( m(t_1; m_0, M_0), M(t_1; m_0, M_0) \right) \in \Phi_1$;

1.2.2. $m(t_1 + 0; m_0, M_0) = g_1 \left( m(t_1; m_0, M_0) \right) = m_1^+$, $M(t_1 + 0; m_0, M_0) = M_2^{00} = M_1^+$;

2.1. For $t_1 < t < t_2$:

2.1.1. $\left( m(t; m_0, M_0), M(t; m_0, M_0) \right)$ coincides with the solution of system (1), (2) for $i = 2$ with initial condition $m(t_1) = m_1^+$, $M(t_1) = M_1^+$;

2.1.2. $\left( m(t; m_0, M_0), M(t; m_0, M_0) \right) \notin \Phi_2$;

2.2. For $t = t_2$:

2.2.1. $\left( m(t_2; m_0, M_0), M(t_2; m_0, M_0) \right) \in \Phi_2$;

2.2.2. $m(t_2 + 0; m_0, M_0) = g_2 \left( m(t_2; m_0, M_0) \right) = m_2^+$, $M(t_2 + 0; m_0, M_0) = M_3^{00} = M_2^+$;

and so on.

Further, we use the notations $m_i^+ = g_i \left( m(t_i; m_0, M_0) \right)$, $M_i^+ = M_{i+1}^{00}$, $i = 1, 2, \ldots$.

For $i = 1, 2, \ldots$, consider the corresponding initial value problems

$$\frac{dm}{dt} = f_i^1(m, M); \tag{6}$$

$$\frac{dm}{dt} = f_i^1(m, M); \tag{7}$$

$$m(t_{0i}) = m_{0i}, \ M(t_{0i}) = M_{0i}. \tag{8}$$

Denote their solutions by $\left( m_i(t; t_{0i}; m_{0i}, M_{0i}), M_i(t; t_{0i}; m_{0i}, M_{0i}) \right)$, where the initial points $(t_{0i}, m_{0i}, M_{0i})$ $\in R^+ \times R^+ \times R^+$, $i = 1, 2, \ldots$.

**Remark 2.** It is known that for $i = 1, 2, \ldots$, the system (6), (7) (without impulses) has the following properties:

1. Unstable (saddle) stationary point $(0,0)$. It is filled $f_i^1(0,0) = f_i^2(0,0) = 0$;

2. Stable stationary point $\left(m_i^{00}, M_i^{00}\right) = \left( r_i^2 \Big/ q_i^2 , \ r_i^1 \Big/ q_i^1 \right)$. It is filled $f_i^1\left(m_i^{00}, M_i^{00}\right) = f_i^2\left(m_i^{00}, M_i^{00}\right) = 0$;

3. First integral of the form

$$V_i(m,M) = q_i^1 M + q_i^2 m - r_i^1 \ln M - r_i^2 \ln m + r_i^1 \left( \ln r_i^1 \Big/ q_i^1 - 1 \right) + r_i^2 \left( \ln r_i^2 \Big/ q_i^2 - 1 \right);$$

4. For each constant $c \geq 0$, implicitly given curve

$$\gamma_i^c = \left\{ (m,M); \ V_i(m,M) = c \right\}$$

   is a trajectory of system (1), (2). This trajectory is closed;

5. For each constant $c \geq 0$, the set

$$G_i^c = \left\{ (m,M); \ V_i(m,M) \leq c \right\}$$

   is a connected domain, situated in $R^+ \times R^+$, which possesses contour $\partial G_i^c = \gamma_i^c$;

6. If $0 < c_1 < c_2$, then $\gamma_i^{c_1} \subset G_i^{c_2}$;

7. Let $c$ and $C$ be arbitrary constants such that $0 < c < C$. We assume that for $i = 1, 2, ...,$ the phase space of system (1), (2) is defined as $G_i^{c,C} = \overline{G_i^C \setminus G_i^c}$, i.e.

$$G_i^{c,C} = \left\{ (m,M); \ c \leq V_i(m,M) \leq C \right\};$$

8. The switching sets of system (1)-(4) satisfy the inclusions $\Phi_i \subset G_i^{c,C}$. It is true

$$\Phi_i = \Phi_i^{c,C} = G_i^{c,C} \cap \left\{ (m,M); \ m > m_i^{00}, \ M = M_i^{00} \right\}.$$

9. If the point $\left(m_{0i}, M_{0i}\right) \in R^+ \times R^+$, then the solution $\left( m_i\left(t; t_{0i}; m_{0i}, M_{0i}\right), \ M_i\left(t; t_{0i}; m_{0i}, M_{0i}\right) \right)$ of the initial value problem (6), (7), (8) is continuable up to $\infty$;

10. If the initial point $\left(m_{0i}, M_{0i}\right) \in G_i^{c,C}$, then the solution of (6), (7), (8) for each $t \geq t_{0i}$ also belongs to $G_i^{c,C}$;

11. Because the set $G_i^{c,C}$ is closed and bounded and the functions $f_i^1$ and $f_i^2$ are continuous on $G_i^{c,C}$, then there exist positive constants $M_{f_i^1}$ and $M_{f_i^2}$, such that

$$\left( \forall (m,M) \in G_i^{c,C} \right) \Rightarrow \left| f_i^1(m,M) \right| \leq M_{f_i^1}, \ \left| f_i^2(m,M) \right| \leq M_{f_i^2}.$$

12. It is fulfilled

$$\left( m(t; m_0, M_0), M(t; m_0, M_0) \right) \tag{9}$$

$$= \begin{cases} \left( m_1\left(t; t_0; m_0^+, M_0^+\right), M_1\left(t; t_0; m_0^+, M_0^+\right) \right), \ 0 = t_0 \leq t \leq t_1; \\ \left( m_i\left(t; t_{i-1}; m_{i-1}^+, M_{i-1}^+\right), M_i\left(t; t_{i-1}; m_{i-1}^+, M_{i-1}^+\right) \right), \ t_{i-1} < t \leq t_i, \ i = 2, 3, ..., \end{cases}$$

   where $\left(m_0^+, M_0^+\right) = \left(m_0, M_0\right)$;

13. The following inequalities are valid

$$\left( m(t; m_0, M_0), M(t; m_0, M_0) \right)$$

$$= \left( m_i\left(t; t_{i-1}; m_{i-1}^+, M_{i-1}^+\right), M_i\left(t; t_{i-1}; m_{i-1}^+, M_{i-1}^+\right)\right) < M_i^{00}, \quad t_{i-1} < t < t_i, \ i = 1, 2, \dots .$$

**Remark 3.** Further, we denote by $m_i^{c,\max}$ and $m_i^{C,\max}$ the bigger solutions of the equations

$$V_i\left(m, M_i^{00}\right) = V_i\left(m, \frac{r_i^1}{q_i^1}\right) \ = \ c \ \Leftrightarrow \ m - \frac{r_i^2}{q_i^2} \ln m = \frac{r_i^2}{q_i^2}\left(1 - \ln \frac{r_i^2}{q_i^2}\right) + \frac{c}{q_i^2}$$

and

$$V_i\left(m, M_i^{00}\right) = V_i\left(m, \frac{r_i^1}{q_i^1}\right) \ = \ C \ \Leftrightarrow \ m - \frac{r_i^2}{q_i^2} \ln m = \frac{r_i^2}{q_i^2}\left(1 - \ln \frac{r_i^2}{q_i^2}\right) + \frac{C}{q_i^2},$$

respectively. The constants $m_i^{c,\min}$ and $m_i^{C,\min}$ are smaller solutions of the above two equations. We introduce the following sets

$$\Phi_i^{c,C,\max} = \left\{(m, M); \ m_i^{c,\max} \le m \le m_i^{C,\max}, \ M = M_i^{00}\right\} \subset G_i^{c,C}, \quad i = 1, 2, \dots$$

and

$$\Phi_i^{c,C,\min} = \left\{(m, M); \ m_i^{C,\min} \le m \le m_i^{c,\min}, \ M = M_i^{00}\right\} \subset G_i^{c,C}, \quad i = 1, 2, \dots .$$

It is clear that the switching sets

$$\Phi_i = \Phi_i^{c,C} = \Phi_i^{c,C,\max}, \quad i = 1, 2, \dots .$$

**Remark 4.** It can be seen immediately that the sets $\Phi_i^{c,C,\min}$ and $\Phi_i^{c,C,\max}$ are closed intervals located in $R^+ \times R^+$. This means that they are compact, connected and convex sets.

**Remark 5.** Pay attention that for $M(t) = M_i^{00}$ and $m(t) > m_i^{00}$, the right hand side of (1) becomes zero. Therefore, we have $\dfrac{dm}{dt} = 0$. It can be shown that just then the victim's biomass is maximum. Consequently, the withdrawal of biomass from the victim in these moments $(t_1, t_2, \dots)$ is justified.

## 3 EXISTENCE OF PERIODIC SOLUTIONS

Introduce the following conditions:

H1. There exists $k_0 \in N$ such that

$$r_{k_0+i}^1 = r_i^1; \quad r_{k_0+i}^2 = r_i^2; \quad q_{k_0+i}^1 = q_i^1; \quad q_{k_0+i}^2 = q_i^2, \quad i = 1, 2, \dots .$$

H2. The functions $g_i \in C\left[\Phi_i^{c,C,\max}, R^+\right]$, $g_i\left(\Phi_i^{c,C,\max}\right) \subset \left\{m; \ m_{i+1}^{C,\min} < m < m_{i+1}^{c,\min}\right\}$ and for each $m \in \Phi_{k_0}^{c,C,\max}$, it is satisfied $g_{k_0+i}(m) = g_i(m), \ i = 1, 2, \dots .$

**Remark 6.** From condition H1, we obtain the equalities:

$$m_{k_0+i}^{00} = m_i^{00}; \quad M_{k_0+i}^{00} = M_i^{00}; \quad V_{k_0+i}(m, M) = V_i(m, M), \ (m, M) \in R^+ \times R^+;$$

$$\gamma_{k_0+i}^c = \gamma_i^c, \ c = const > 0; \quad G_{k_0+i}^{c,C} = G_i^{c,C}, \ c = const, \ C = const, \ 0 < c < C;$$

$$m_{k_0+i}^{c,\min} = m_i^{c,\min}; \quad m_{k_0+i}^{C,\min} = m_i^{C,\min}; \quad m_{k_0+i}^{c,\max} = m_i^{c,\max}; \quad m_{k_0+i}^{C,\max} = m_i^{C,\max};$$

$$\Phi_{k_0+i}^{c,C,\min} = \Phi_i^{c,C,\min}; \quad \Phi_{k_0+i}^{c,C,\max} = \Phi_i^{c,C,\max}.$$

**Theorem 1.** *Let the conditions H1 and H2 hold.*

*Then the system (1)-(4) has at least one periodic solution with initial point* $\left(m_0, M_0\right) \in \Phi_1^{c,C,\min}$ *and period*

$$T = t_{k_0} = t_{k_0}\left(m_0, M_0\right).$$

**Proof.** For convenience, the proof will be presented in several parts.

**Part 1.** Let the initial point $\left(m_0, M_0\right) \in \Phi_1^{c,C,\min} \subset G_1^{c,C}$. The solution of initial value problem (1)-(5) is defined for $0 = t_0 \le t \le T$. In fact, for $0 = t_0 \le t \le t_1$, the solution of considered problem with variable structure and impulses coincides with the solution of initial value problem (6), (7), (8), where $i=1$ and $\left(m_{01}, M_{01}\right) = \left(m_0, M_0\right)$ (see equality (9)). From condition H2, it is seen that the point $\left(m_1^+, M_1^+\right) \in \Phi_2^{c,C,\min} \subset G_2^{c,C}$. For $t_{i-1} < t \le t_i$, the solution of (1)-(5) coincides with the solution of initial value problem (6), (7), (8), where $i = 2,3,...,k_0$ and the initial point $\left(m_{0i}, M_{0i}\right) = \left(m_i^+, M_i^+\right) \in \Phi_{i+1}^{c,C,\min} \subset G_{i+1}^{c,C}$. In other words, for $0 < t \le T$, we have

$$\left(m\left(t; m_0, M_0\right), M\left(t; m_0, M_0\right)\right)$$

$$= \left(m_i\left(t; t_{i-1}; m_{i-1}^+, M_{i-1}^+\right), M_i\left(t; t_{i-1}; m_{i-1}^+, M_{i-1}^+\right)\right), \quad t_{i-1} < t \le t_i, \quad i = 1,2,...,k_0.$$

**Part 2.** Consider the mapping $\varphi_1 : \Phi_1^{c,C,\min} \to \Phi_1^{c,C,\max}$, defined as follows

$$\left(\forall\left(m_0^+, M_0^+\right) \in \Phi_1^{c,C,\min}\right) \Rightarrow \varphi_1\left(m_0^+, M_0^+\right) = \left(m_1\left(t_1; 0, m_0^+, M_0^+\right), M_1\left(t_1; 0, m_0^+, M_0^+\right)\right)$$

$$= \left(m_1\left(t_1; 0, m_0^+, M_0^+\right), M_1^{00}\right) \in \Phi_1^{c,C,\max}.$$

We will show that the mapping $\varphi_1$ is continuous on $\Phi_1^{c,C,\min}$. For this purpose, it is sufficient to establish that $m_1\left(t_1; 0, m_0^+, M_0^+\right)$ depends continuously on the initial point $\left(m_0^+, M_0^+\right)$. Let a point $\left(m_0^{*+}, M_0^{*+}\right) \in \Phi_1^{c,C,\min}$ and a constant $t_1^* > 0$ be such that:

-   $\left(m_1\left(t; 0, m_0^{*+}, M_0^{*+}\right), M_1\left(t; 0, m_0^{*+}, M_0^{*+}\right)\right) \notin \Phi_1^{c,C,\max}, \ 0 \le t < t_1^*$;

-   $\left(m_1\left(t_1^*; 0, m_0^{*+}, M_0^{*+}\right), M_1\left(t_1^*; 0, m_0^{*+}, M_0^{*+}\right)\right) \in \Phi_1^{c,C,\max}$.

For concreteness, we assume that $t_1^* \le t_1$ is fulfilled. The remaining case is considered analogously. According to the property 13 of Remark 2, it follows that for each sufficiently small constant $\Delta > 0$, we have

$$M_1\left(t_1 - \Delta; 0, m_0^{*+}, M_0^{*+}\right) < M_1^{00}; \quad M_1\left(t_1 + \Delta; 0, m_0^{*+}, M_0^{*+}\right) > M_1^{00}.$$

From the theorem of continuous dependence (see Theorem 7.1, § 7, Ch. I, [4]), it follows that

$$\left(\exists \delta_1 = const > 0\right) : \left(\forall\left(m_0^{*+}, M_0^{*+}\right) \in \Phi_1^{c,C,\min}, \ \left\|\left(m_0^{*+}, M_0^{*+}\right) - \left(m_0^+, M_0^+\right)\right\| < \delta_1\right) \tag{10}$$

$$\Rightarrow M_1\left(t_1 - \Delta; 0, m_0^{*+}, M_0^{*+}\right) < M_1^{00}; \quad M_1\left(t_1 + \Delta; 0, m_0^{*+}, M_0^{*+}\right) > M_1^{00}; \tag{11}$$

$$m_1\left(t; 0, m_0^{*+}, M_0^{*+}\right) > m_1^{00}, \quad t_1 - \Delta \le t \le t_1 + \Delta;$$

$$\left|m_1\left(t_1 - \Delta; 0, m_0^{*+}, M_0^{*+}\right) - m_1\left(t_1 - \Delta; 0, m_0^+, M_0^+\right)\right| < \Delta; \tag{12}$$

$$\left|m_1\left(t_1 + \Delta; 0, m_0^{*+}, M_0^{*+}\right) - m_1\left(t_1 + \Delta; 0, m_0^+, M_0^+\right)\right| < \Delta. \tag{13}$$

From (11), taking into account the assumption (more precisely $t_1^* \le t_1$), we reach the following inequalities $t_1 - \Delta < t_1^* \le t_1 < t_1 + \Delta$. According to the property 11 of Remark 2, we have:

$$\left| m_1\left(t_1;0,m_0^+,M_0^+\right) - m_1\left(t_1-\Delta;0,m_0^+,M_0^+\right) \right| \le \int_{t_1-\Delta}^{t_1} \left| f_1^1\left(m_1\left(\tau;0,m_0^+,M_0^+\right)\right) \right| d\tau \le M_{f_1^1}\Delta ;$$

$$\left| m_1\left(t_1^*;0,m_0^{*+},M_0^{*+}\right) - m_1\left(t_1-\Delta;0,m_0^{*+},M_0^{*+}\right) \right| \le \int_{t_1-\Delta}^{t_1^*} \left| f_1^1\left(m_1\left(\tau;0,m_0^{*+},M_0^{*+}\right)\right) \right| d\tau \le M_{f_1^1}\Delta.$$

From the above two inequalities and (12), we find

$$\left| m_1\left(t_1;0,m_0^{*+},M_0^{*+}\right) - m_1\left(t_1;0,m_0^+,M_0^+\right) \right| \qquad (14)$$

$$\le \left| m_1\left(t_1-\Delta;0,m_0^{*+},M_0^{*+}\right) - m_1\left(t_1-\Delta;0,m_0^+,M_0^+\right) \right|$$

$$\le \left| m_1\left(t_1;0,m_0^{*+},M_0^{*+}\right) - m_1\left(t_1-\Delta;0,m_0^{*+},M_0^{*+}\right) \right|$$

$$+ \left| m_1\left(t_1;0,m_0^+,M_0^+\right) - m_1\left(t_1-\Delta;0,m_0^+,M_0^+\right) \right|$$

$$\le \left(1 + 2M_{f_1^1}\right)\Delta.$$

From (10) and (14), we conclude that

$$\left(\forall \varepsilon > 0\right)\left(\exists \delta = const > 0\right): \left(\forall\left(m_0^{*+},M_0^{*+}\right) \in \Phi_1^{c,C,\min}, \ \left\|\left(m_0^{*+},M_0^{*+}\right) - \left(m_0^+,M_0^+\right)\right\| < \delta\right)$$

$$\Rightarrow \left| m_1\left(t_1;0,m_0^{*+},M_0^{*+}\right) - m_1\left(t_1;0,m_0^+,M_0^+\right) \right| < \varepsilon.$$

Thus, we have proved that the mapping $\varphi_1$ is continuous.

**Part 3.** Consider the mapping $\psi_1 : \Phi_1^{c,C,\max} \to \Phi_2^{c,C,\min}$ defined as follows

$$\left(\forall\left(m_1,M_1\right) \in \Phi_1^{c,C,\max}\right) \Rightarrow \psi_1\left(m_1,M_1\right) = \left(g_1\left(m_1,M_1\right),M_2^{00}\right) \in \Phi_2^{c,C,\min}.$$

According to condition H2, the mapping $\psi_1$ is continuous.

**Part 4.** Consider the mapping $\varphi_i : \Phi_i^{c,C,\min} \to \Phi_i^{c,C,\max}$, $i = 2,3,...,k_0$, which is defined by means of the equalities

$$\left(\forall\left(m_{i-1}^+,M_{i-1}^+\right) \in \Phi_i^{c,C,\min}\right) \Rightarrow \varphi_i\left(m_{i-1}^+,M_{i-1}^+\right) = \left(m_i\left(t_i;0,m_{i-1}^+,M_{i-1}^+\right), M_i\left(t_i;0,m_{i-1}^+,M_{i-1}^+\right)\right)$$

$$= \left(m_i\left(t_i;0,m_{i-1}^+,M_{i-1}^+\right), M_i^{00}\right) \in \Phi_i^{c,C,\max}.$$

Similarly as in Part 2, it can be established that this mapping is continuous on its domain.

**Part 5.** Let the mapping $\psi_i : \Phi_i^{c,C,\max} \to \Phi_{i+1}^{c,C,\min}$, $i = 2,3,...,k_0$, be defined as follows

$$\left(\forall\left(m_i,M_i\right) \in \Phi_i^{c,C,\max}\right) \Rightarrow \psi_i\left(m_i,M_i\right) = \left(g_i\left(m_i,M_i\right),M_{i+1}^{00}\right) \in \Phi_{i+1}^{c,C,\min}.$$

According to condition H2, the mapping $\psi_i$ is continuous.

**Part 6.** Let us introduce the mapping $F : \Phi_1^{c,C,\min} \to \Phi_{k_0+1}^{c,C,\min}$, where

$$\left(\forall\left(m_0^+,M_0^+\right) \in \Phi_1^{c,C,\min}\right)$$

$$\Rightarrow F\left(m_0^+, M_0^+\right) = \psi_{k_0}\left(\varphi_{k_0}\left(\psi_{k_0-1}\left(\varphi_{k_0-1}\left(...\left(\psi_1\left(\varphi_1\left(m_0^+, M_0^+\right)\right)\right)...\right)\right)\right)\right).$$

From condition H1, taking in mind Remark 6, we find that $\Phi_{k_0+1}^{c,C,\min} = \Phi_1^{c,C,\min}$, i.e. $F : \Phi_1^{c,C,\min} \to \Phi_1^{c,C,\min}$. From previous parts (with numbers 2, 3, 4 and 5) of the proof, it follows that $F \in C\left[\Phi_1^{c,C,\min}, \Phi_1^{c,C,\min}\right]$. From Remark 4, we have that $\Phi_1^{c,C,\min}$ is compact, connected and convex set. Using the Brouwer fixed point theorem, we reach the conclusion that

$$\left(\exists\left(m_0, M_0\right) \in \Phi_1^{c,C,\min}\right): \ F\left(m_0, M_0\right) = \left(m_0, M_0\right).$$

**Part 7.** Taking into account the definitions of functions $\varphi_i$ and $\psi_i$, $i = 1, 2, ..., k_0$, we conclude that

$$F : \left(m_0, M_0\right) \to \left(m\left(t_{k_0} + 0; m_0, M_0\right), M\left(t_{k_0} + 0; m_0, M_0\right)\right)$$

and therefore,

$$\left(m\left(t_{k_0} + 0; m_0, M_0\right), M\left(t_{k_0} + 0; m_0, M_0\right)\right) = \left(m_0, M_0\right).$$

Finally, from conditions H1 and H2, we obtain that $\left(m\left(t; m_0, M_0\right), M\left(t; m_0, M_0\right)\right)$ is a periodic solution of the initial value problem (1)-(5). The period of solution is $T = t_{k_0}$.

The theorem is proved.

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