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PRIMARY DECOMPOSITION OF A 0-CONNECTED NILPOTENT SPACE

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ABSTRACT

It is well known that the different stages of the Cartan-Whitehead decomposition of a 0-connected space can be obtained as the Adams cocompletion of the space with respect to suitable sets of morphisms. In thispaper Cartan-Whitehead decomposition is obtained for a nilpotent space, in terms of Adams cocompletion, using the primary homotopy theory developed by Neisendonfer.

Indexing terms/Keywords

Category of fractions; Calculus of right fractions; Adams cocompletion; Primary homotopy theory; Nilpotent space; Cartan-Whitehead decomposition.

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1. INTRODUCTION

Deleanu, Frei and Hilton have developed the notion of generalized Adams completion in a categorical context; they have also studied the dual notion, namely the Adams cocompletion of an object in a category [9]. It is to be emphasized that many algebraic and geometrical constructions in Algebraic Topology, Differential Topology, Differentiable Manifolds, Algebra, Analysis, Topology, etc., can be viewed as Adams completions or cocompletions of objects in suitable categories, with respect to carefully chosen sets of morphisms. Behera and Nanda [3] have shown that the different stages of the Cartan-Whitehead decomposition of a 0-connected space are the Adams cocompletion of a space with respect to suitable sets of morphisms. In [12], Neisendonfer has studied the primary homotopic theory in an exhaustive manner.. The central idea of this note is to study how Cartan-Whitehead decomposition of a 0-connected nilpotent space is characterized in terms of its Adams cocompletions; it is done using the primary homotopy theory developed by Neisendonfer.

Let $\mathcal{C}$ be an arbitrary category and S a set of morphisms of $\mathcal{C}$. Let $\mathcal{C}[S^{-1}]$ denote the category of fractions of $\mathcal{C}$with respect$S$ and$F\colon \mathcal{C} \to \mathcal{C}[S^{-1}]$be the canonical functor. Let $\mathcal{S}$ denote the category of sets and functions. Then for a given object Y of $\mathcal{C}$,

$$\mathcal{C}[S^{-1}](Y,-) : \mathcal{C} \to \mathcal{S}$$

defines a covariant functor. If this functor is representable by an object Y_S of $\mathcal{C}$, that is, if

$$\mathcal{C}[S^{-1}](Y,-) \cong \mathcal{C}(Y_S,-)$$

then Y_S is called the (generalized) Adams cocompletion of Y with respect to the set of morphisms S or simply the S-cocompletion of$Y$. We shall often refer to Y_S as the cocompletion of Y [9].

Given aset S of morphisms of $\mathcal{C}$, the saturation of S, denoted as $\bar{S}$ is the set of all morphisms u in $\mathcal{C}$ such that $F(u)$ is an isomorphism in $\mathcal{C}[S^{-1}]$. Furthermore,S is said to be saturated if $S = \bar{S}$[4,9].

Deleanu, Frei and Hilton have shown that if the set of morphisms $S$is saturated then the Adams cocompletion of a space is characterized by a certain couniversal property ([9],dual of Theorem 1.2). In most of the applications, however, the set of morphisms S is not saturated. There is a stronger version of Deleanu, Frei and Hilton's characterization of Adams cocompletion in terms of couniversal property as described below.

Theorem1.1. ([4], dual of Theorem 1.2) Let S be a set of morphisms of $\mathcal{C}$ admitting a calculus of right fractions. Then an object Y_S of $\mathcal{C}$ is the S-cocompletion of the object Y with respect to S if and only if there exists a morphism $e : Y_S \to Y$ in $\bar{S}$ which is couniversal with respect to morphisms in$S$: given a morphism $s : Z \to Y$ in S there exists a unique morphism $t : Y_S \to Z$ in $\bar{S}$ such that $st = e$. In other words, the following diagram is commutative :

$$\begin{array}{ccc} Y_S & \xrightarrow{e} & Y \\ t\downarrow & \nearrow s & \\ Z & & \end{array}$$

Also the above theorem turns out to be essentially the dual of Theorem 1.2 [9] if we assume S to be saturated; hence the proposition can be proved by recasting the dual of the proof of the Theorem 1.2 [9] with minor changes. The details are omitted.

The following Theorem (dual of Theorem 1.3, [4]) shows that under certain conditions the morphisms $e\colon Y_S \to Y$ always belongs to S.

Theorem1.2 . ([4], dual of Theorem 1.3) Let $S = S_1 \cap S_2$. be a set of morphisms in a category $\mathcal{C}$ admitting a calculus of right fractions. Let $e : Y_S \to Y$ be the canonical morphism as defined in Theorem 1.1, where Y_S is the S-cocompletion of Y. Assume further that S_1 and S_2 have the following properties :

(i) S_1 and S_2 are closed under composition.

(ii) $fg \in S_1$implies that$g \in S_1$.

(iii) $fg \in S_2$implies that$f \in S_2$.

Then $e \in S$.

2.The category $\mathcal{N}_0$.

Let S^m denote the m-dimensional sphere. Suppose $m \geq 2$, and let $k : S^m \to S^m$ denote a map of degree k. The space $S^{m-1} \cup_k e^m$ is denoted by $P^m(k)$ or $P^m(\mathbb{Z}/k\mathbb{Z})$. If $m \geq 2$, the m-th mod$k$homotopy group of X is $[P^m(k); X]$, denoted by $\pi_m(X; \mathbb{Z}/k\mathbb{Z})$ [12]. If $m \geq 3$, $\pi_m(X; \mathbb{Z}/k\mathbb{Z})$ is a group and $m \geq 4$, $\pi_m(X; \mathbb{Z}/k\mathbb{Z})$ is an abelian group [12].

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If $f : X \longrightarrow Y$ is a map, then there are induced maps $f_*: \pi_m(X;\mathbb{Z}/k\mathbb{Z}) \rightarrow \pi_m(Y;\mathbb{Z}/k\mathbb{Z})$ defined by $f_*[g] = [fg]$. If $m \geq 3$, f_* is a homomorphism and, if $m = 2$, f_* is compatible with the action of π_2[12].

For a group G, the lower central series

$$\cdots \subseteq \Gamma^{i+1}(G) \subseteq \Gamma^{i}(G) \cdots \subseteq \Gamma^{1}(G)$$

of G, is defined by the setting

$$\Gamma^1(G) = G, \quad \Gamma^{i+1}(G) = [G, \Gamma^i(G)], \quad i \geq 1.$$

G is said to be nilpotent if $\Gamma^j(G) = \{1\}$ for j sufficiently large [10].

A connected CW-complex X is said to nilpotent if $\pi_1(X)$ is nilpotent and operates nilpotently on $\pi_n(X)$ for every $n \geq 2$[10].

Let $\mathcal{N}_0$ denote the category of 0-connected based nilpotent spaces and based maps and let $\widetilde{\mathcal{N}}_0$ be the corresponding homotopy category. We assume that the underlying sets of the elements of $\widetilde{\mathcal{N}}_0$ are the elements of $\mathcal{U}$, where $\mathcal{U}$ is a fixed Grothendeick universe. We now fix suitable sets of morphisms of $\widetilde{\mathcal{N}}_0$.

Let S_n denote the set of all maps α in $\widetilde{\mathcal{N}}_0$ having the following property: $\alpha : X \rightarrow Y$ is in S_n if and only if $\alpha_* : \pi_m(X;\mathbb{Z}/k\mathbb{Z}) \rightarrow \pi_m(Y;\mathbb{Z}/k\mathbb{Z})$ is an isomorphism for $m > n$ and a monomorphism for $m = n$.

Proposition 2.1. S_n admits a calculus of right fractions.

Proof. Clearly S_n is closed under composition. We shall verify conditions (i) and (ii) of Theorem 1.3* [9]. Only condition (ii) is in question. For proving this condition it is enough to prove that every diagram

$$\begin{array}{ccc} & & A \\ & & \downarrow \alpha \\ C & \underset{\gamma}{\rightarrow} & B \end{array}$$

in $\widetilde{\mathcal{N}}_0$ with $\gamma \in S_n$, can be embedded in a weak pull-back diagram

$$\begin{array}{ccc} D & \overset{\delta}{\rightarrow} & A \\ \beta \downarrow & & \downarrow \alpha \\ C & \underset{\gamma}{\rightarrow} & B \end{array}$$

with $\delta \in S_n$. Suppose $\alpha = [f]$ and $\gamma = [s]$. We replace f and s by fibrations f' and s' respectively; we have $f = f'r : A \overset{r}{\rightarrow} P_f \overset{f'}{\rightarrow} B$ and $s = s't : C \overset{t}{\rightarrow} P_s \overset{s'}{\rightarrow} B$ where r and t are homotopy equivalences and P_f and P_s are mapping tracks of f and s respectively. Let $\bar{r}$ and $\bar{t}$ be the homotopy inverses of r and t respectively. Let D be the usual pull-back of f' and s' and $p : D \rightarrow P_f$, $q : D \rightarrow P_s$ be the respective projections. Let $\delta = [\bar{r}p]$ and $\beta = [\bar{t}q]$. Thus $\alpha\delta = [f][\bar{r}p] = [f\bar{r}p] = [f'r\bar{r}p] = [f'p] = [s'q] = [s't\bar{t}q] = [s\bar{t}q] = [s][\bar{t}q] = \gamma\beta$. Moreover, if $\alpha\mu = \gamma\lambda$, let $u : U \rightarrow A$, $v : U \rightarrow C$ be in the classes μ, λ respectively so that $fu \simeq sv$ or $f'ru \simeq sv$. Let $F : U \times I \rightarrow B$ be a homotopy with $F_0 = f'ru$ and $F_1 = sv$. Since f' is a fibration there exists a homotopy $G : U \times I \rightarrow P_f$ such that $f'G_t = F_t$ and $G_0 = ru$. Thus $f'G_1 = F_1 = sv = s'tv$. By the pull-back property of D there exists a map $k : U \rightarrow D$ such that $pk = G_1 \simeq ru$ and $qk = tv$. Thus if $\rho = [k]$, then $\delta\rho = [\bar{r}p][k] = [\bar{r}pk] = [\bar{r}ru] = [u] = \mu$ and $\beta\rho = [\bar{t}q][k] = [\bar{t}qk] = [\bar{t}tv] = [v] = \lambda$.

It remains to be shown that $\delta \in S_n$. We assume that the map $\alpha : A \rightarrow B$ is a fibration with fibre Q. We note that Q is also the fibre of $\beta : D \rightarrow C$. We have the following commutative diagram.

$$\begin{array}{ccccccccccc} \cdots \rightarrow & \pi_{m+1}(C;\mathbb{Z}/k\mathbb{Z}) & \rightarrow & \pi_m(Q;\mathbb{Z}/k\mathbb{Z}) & \rightarrow & \pi_m(D;\mathbb{Z}/k\mathbb{Z}) & \rightarrow & \pi_m(C;\mathbb{Z}/k\mathbb{Z}) & \rightarrow & \pi_{m-1}(Q;\mathbb{Z}/k\mathbb{Z}) & \rightarrow \cdots \\ & \gamma_* \downarrow & & \| & & \delta_* \downarrow & & \gamma_* \downarrow & & \| & \\ \cdots \rightarrow & \pi_{m+1}(B;\mathbb{Z}/k\mathbb{Z}) & \rightarrow & \pi_m(Q;\mathbb{Z}/k\mathbb{Z}) & \rightarrow & \pi_m(A;\mathbb{Z}/k\mathbb{Z}) & \rightarrow & \pi_m(B;\mathbb{Z}/k\mathbb{Z}) & \rightarrow & \pi_{m-1}(Q;\mathbb{Z}/k\mathbb{Z}) & \rightarrow \cdots \end{array}$$

By Five Lemma δ_* is an isomorphism for $m > n$ and a monomorphism for $m = n$, showing $\delta \in S_n$. This completes the proof of Proposition 2.1.

In fact, the set S_n admits astrong calculus of right fractions. A set S of morphisms of a small $\mathcal{V}$-category $\mathcal{C}$, $\mathcal{V}$ being a Grothendieck universe, admits a strong calculus of right fractions[14] if

(i) S admits a calculus of right fractions,

(ii) for any set $\{s_i : B_i \to A,\ i \in I,\ I$ is a $\mathcal{V}$-set$\}$, there exists a commutative completion $\{f_i : C \to B_i,\ i \in I\}$ such that $s_i f_i \in S$ for every $i \in I$.

Proposition 2.2. $S_n$admits a strong calculus of right fractions.

Proof. Let $\{s_i : Y_i \to X,\ i \in I\}$ be a given set of morphisms in $\widetilde{\mathcal{N}}_0$ with every $s_i \in S_n$ and $I \in \mathcal{U}$. We have a map from $X \to P^n X$, where $P^n X$ denotes the Postnikov decomposition of X ([10], proof of Proposition 1.1). Convert this into a fibration:$X_n \overset{u_n}{\to} X \to P^n X$,$X_n$ being its fibre. Considering the homotopy exact sequence of this fibration we get$\pi_m(X_n;\mathbb{Z}/k\mathbb{Z})$= 0 for $m \le n$ and $\pi_m(X_n;\mathbb{Z}/k\mathbb{Z}) \cong \pi_m(X;\mathbb{Z}/k\mathbb{Z})$for $m > n$. Thus $u_n \in S_n$. Since $\pi_1(X_n;\mathbb{Z}/k\mathbb{Z})$= 0 we have a map (lifting) $f_i : X_n \to Y_i$ such that $s_i f_i = u_n$and the proposition is proved.

Remark 2.3. We note that the map$u_n : X_n \to X$is independent of the index$i$.

Proposition 2.4. For a given object$X$of the category$\widetilde{\mathcal{N}}_0$, there exists asubset$S_X$of the set$\{ s : X' \to X \mid s \in S_n\}$such that $S_X$is an element of theuniverse $\mathcal{U}$and for each $s : X' \to X,\ s \in S_n$, there exists an$s' : X'' \to X$in$S_X$and a morphism $u : X'' \to X'$ in $\widetilde{\mathcal{N}}_0$such that $su = s'$.

Proof. For a given object X in $\widetilde{\mathcal{N}}_0$, let S_X denote the set of morphisms$S_X = \{s : Y \to X \mid s \in S_n,\ Y$ is an object of $\widetilde{\mathcal{N}}_0\}$. We assert that$S_X$ is an element of $\mathcal{U}$. For any object Y of $\widetilde{\mathcal{N}}_0$, let $S_{Y,X} = \{s : Y \to X,\ s \in S_n\}$. It is clear that$S_X = \cup_Y S_{Y,X}$ and $S_{Y,X} = S_n \cap Mor_{\widetilde{\mathcal{N}}_0}(Y,X)$. Since $\widetilde{\mathcal{N}}_0$ is a small $\mathcal{U}$-category, $Mor_{\widetilde{\mathcal{N}}_0}(Y,X)$ belongs to $\widetilde{\mathcal{N}}_0$ and so does $S_{Y,X}$, being a subset of $Mor_{\widetilde{\mathcal{N}}_0}(Y,X)$. Therefore, the set S_X, being a union of sets all belonging to $\mathcal{U}$ and indexed by the objects Y of $\widetilde{\mathcal{N}}_0$ (which is a subset of $\mathcal{U}$) is itself in $\mathcal{U}$.In view of Proposition 2.2 and Remark 2.3, there exists a lifting $f_s : X_n \to Y$ of u such that $sf_s = u_n$where $s \in S_X$ is arbitrary, u_n is the map as constructed in Proposition 2.2. This completes the proof of the Proposition 2.4.

Corollary 2.5.$u_n \in S_n$and with respect to any$s \in S_X$, $u_n$has couniversal property.

3. Existence of Adams cocompletion in $\widetilde{\mathcal{N}}_0$.

Since the category $\widetilde{\mathcal{N}}_0$as stated above is neither complete nor small, the dual of Theorem 2.6 [7] nan not be used to show the existence of Adams cocompletion of an object in the category$\widetilde{\mathcal{N}}_0$ with respect to the set of morphisms S_n.The following theorem shows that under certain conditions the Adams cocompletion of an object in the category $\widetilde{\mathcal{N}}_0$ always exists; the theorem is essentially the dual of Theorem 4.7 [1] and dual of Theorem 3.8 [2] (it is also a generalization of the dual of the Theorem in [7]).

Theorem 3.1.Let $\mathcal{U}$ be a fixed Grothendieck universe. Let $\tilde{\mathcal{C}}$ be the category defined as follows : the objects of $\tilde{\mathcal{C}}$ are connected based nilpotent spaces whose underlying sets are elements of $\mathcal{U}$; the morphisms of $\tilde{\mathcal{C}}$ are based homotopy classes of based-point preserving maps between such based nilpotent spaces. Let S be a family of morphisms of $\tilde{\mathcal{C}}$ admitting a calculus of right fractions and satisfying the following axioms of compatibility with products :

(P) If $s_i : X_i \to Y_i$lies in S for each $i \in I$, where the index set I is an element of $\mathcal{U}$, then

$$\prod_{i\in I} s_i : \prod_{i\in I} X_i \to \prod_{i\in I} Y_i$$

lies in S.

Assume that the family S and the object X of $\tilde{\mathcal{C}}$ satisfy the condition :

$(*)$There exists a subset S_X of the set $\{s : X' \to X \mid s \in S\}$such that S_X is an element of the universe $\mathcal{U}$ and for each $s : X' \to X,\ s \in S$, there exist an $s' : X'' \to X$ in S_X and a morphism $u : X'' \to X'$ of $\tilde{\mathcal{C}}$ rendering the following diagram is commutative :

$$\begin{array}{ccc} X' & \overset{s}{\to} & X \\ u\uparrow & \nearrow s' & \\ X'' & & \end{array}$$

Then the Adams cocompletion X_S of X does exist.

As remarked by Adams of page 34 of [2] this result remains valid if $\tilde{\mathcal{C}}$ is the homotopy category of 0-connected based nilpotent spaces (whose underlying sets belong to $\mathcal{U}$). It is to be emphasized that condition $(*)$ is essential in order to be able to apply E.H. Brown's representability theorem to prove this result.

From the Propositions 2.1, 2.2 and 2.4 and Remark2.3, we note that the conditions of Theorem 3.1 are satisfied and so by Theorem 1.2, we obtain the following theorem.

Theorem 3.2. Every object$X$of the category$\widetilde{\mathcal{N}}_0$ has an Adams cocompletion$X_{S_n}$with respect to the set of morphisms$S_n$and there exists a morphism$e_n : X_{S_n} \to X$ in$\bar{S}_n$which is couniversal with respect to morphisms inS_n.

Proposition 3.3. The morphism$e_n\colon X_{S_n} \to X$ as constructed in Theorem3.2is inS_n.

Proof. The proof follows from the dual of Theorem 1.3 [4]. We take $S_n^1 = \{\alpha : X \to Y \ in\ \widetilde{\mathcal{N}}_0 \big| \alpha_* : \pi_m(X;\mathbb{Z}/k\mathbb{Z}) \to \pi_m(Y;\mathbb{Z}/k\mathbb{Z})$is a monomorphism for $m \geq n\}$ and $S_n^2 = \{\alpha : X \to Y \ in\ \widetilde{\mathcal{N}}_0 \mid \alpha_* : \pi_m(X;\mathbb{Z}/k\mathbb{Z}) \to \pi_m(Y;\mathbb{Z}/k\mathbb{Z})$ is an epimorphism for $m \geq n+1\}$. Clearly (a) $S_n = S_n^1 \cap S_n^2$ and (b) S_n^1 and S_n^2 satisfy all conditions of Theorem 1.2; hence $e \in S_n$. This completes the proof of the Proposition 3.3.

4.A primary decomposition of a 0-connected based nilpotent space.

Now we obtain the primary decomposition of a 0-connected based nilpotent space with the help of the set of morphisms S_n as described below. In fact the different stages of the Cartan-Whitehead decomposition of a 0-connected nilpotent space are the Adams cocompletions of the space with respect to the sets of morphisms S_n. In the process, starting from a 0-connected based nilpotent space X we get a tower of spaces,

$$\cdots \to X_{n+1} \xrightarrow{\theta_{n+1}} X_n \to \cdots \to X_1 \xrightarrow{\theta_1} X_0$$

and the direct limit of this tower gives us a space which in some sense is the Cartan-Whitehead decomposition of X. First we prove the following proposition.

Proposition 4.1.X_n, as constructed in the proof of Proposition 2.2, is homotopically equivalent to X_{S_n}, as constructed in Theorem3.2.

Proof. By the couniversal property of $u_n : X_n \to X$ we have a map $s : X_n \to X_{S_n}$such that $e_n s = u_n$. By the couniversal property of $e_n : X_{S_n} \to X$ we have a map $t : X_{S_n} \to X_n$ such that $u_n t = e_n$. Thus$u_n = e_n s = u_n ts$implies that $ts = 1_{X_n}$and $e_n = u_n t = e_n st$implies that $st = 1_{X_{S_n}}$and the required homeomorphism between X_n and X_{S_n} is obtained. This completes the proos of Proposition 4.1.

Theorem 4.2. Let Xbe a 0-connected based nilpotent space. Then for $n \geq 3$, there exist 0-connected based nilpotent spaces X_n, maps$e_n\colon X_n \to X$ andfibrations $\theta_{n+1} : X_{n+1} \to X_n$such that

(a) $e_{n_*} : \pi_m(X_n;\mathbb{Z}/k\mathbb{Z}) \to \pi_m(X;\mathbb{Z}/k\mathbb{Z})$is an isomorphism form $> n$and$\pi_m(X_n;\mathbb{Z}/k\mathbb{Z}) = 0$form $\leq n$,

(b) $e_{n+1} = e_n \circ \theta_{n+1}$.

Proof. For each integer $n \geq 3$, let X_n be the S_n-completion of X and $e_n : X_n \to X$ be the canonical map as stated in Theorem 3.1. Since $e_n \in S_n \subset S_{n+1}$, it follows from the couniversal property of e_{n+1} that there exists a unique morphism $\theta_{n+1} : X_{n+1} \to X_n$ such that $e_{n+1} = e_n \circ \theta_{n+1}$, i.e., the following diagram is commutative:

$$\begin{array}{ccc} X_{n+1} & \xrightarrow{e_{n+1}} & X \\ \theta_{n+1} \downarrow & \nearrow e_n & \\ X_n & & \end{array}$$

The maps θ_n can of course be replaced by fibrations in the usual manner. Since $e_n \in S_n$, $e_{n_*} : \pi_m(X_n;\mathbb{Z}/k\mathbb{Z}) \to \pi_m(X;\mathbb{Z}/k\mathbb{Z})$is an isomorphism for $m > n$; it is already proved in Proposition 2.2 that $\pi_m(X_n;\mathbb{Z}/k\mathbb{Z}) = 0$ for $m \leq n$.

This completes the proof of the Theorem 4.2.

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