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Almost multipliers and some of their properties

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ABSTRACT

In this paper we introduce the notion of almost multiplier maps and study some properties of these maps on a class of normed algebras, namely stable one. Also we compare the algebra of linear almost multipliers with the algebra of bounded linear operators.

Indexing terms/Keywords

almost multipliers; stable normed algebras; almost additive maps.

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INTRODUCTION

The concept of a multiplier map was introduced by Helgason in [3] as a bounded continuous function g defined on the regular maximal ideal space $\Delta(A)$ such that $g\hat{A} \subseteq \hat{A}$, where $\hat{A}$ denote the Gelfand representation of the Banach algebra A. Successively the general theory of multipliers on faithful Banach algebra has been developed by Wang in [7] and Birtal in [2]. The study of multipliers has an important role in many areas of analysis as well as in probability, optimization theory, differential equations, theory of signal, acoustic, mathematical finance and economics. In this paper we introduce almost multipliers and study their properties on the stable normed algebras instead of faithful algebras. In [1], we compare the stable normed algebras with the faithful normed algebras.

suppose A is a normed algebra. We say that a mapping $T: A \to A$ is a left(resp. right) multiplier on A if $T(xy) = T(x)y\ (resp.\ T(xy) = xT(y))$, for all $x, y \in A$. The map T is a multiplier on A if $T(x)y = xT(y)$. We recall that a normed algebra A is left (resp. right)faithful if for all $x \in A,\ xA = 0\ (resp.\ Ax = 0)$, implies $x = 0$. The normed algebra A is faithful if it is left and right faithful. Obviously if A has a bounded approximate identity, then it is faithful.

Preliminary Definitions and Theorems

Definition 1. We say that the normed algebra A is a left (resp. right) stable normed algebra if for all $a \in A$ and $M > 0$, if we have $||ab|| \leq M||b||\ \ (resp.\ ||ba|| \leq M||b||\ )$ for all $b \in A$, then we can conclude $||a|| \leq M$. The normed algebra A is stable if it is both left and right stable.

Proposition 2. Every normed algebra with a bounded left (resp. right) approximate identity with bound 1, is a left (resp. right) stable normed algebra.

Proof. [1].

Corollary 3. Every unital normed algebra is stable.

Proposition 4. Every stable normed algebra A is faithful.

Proof. [1].

Definition 5. Suppose that A and B are normed algebras. The map $f: A \to B$, is an almost additive map if for $\varepsilon > 0$,

$$||f(x+y) - f(x) - f(y)|| \leq \varepsilon(||x|| + ||y||)$$

for all $x, y \in A$.

An almost linear map is a homogeneous almost additive map.

Definition 6. Let A be a normed algebra and let $\varepsilon > 0$, an almost left (resp. right) multiplier on A is a map $T: A \to A$ such that

$$||T(xy) - T(x)y|| \leq \varepsilon||x||\ ||y||\ (resp.\ ||T(xy) - xT(y)|| \leq \varepsilon||x||\ ||y||)$$

for all $x, y \in A$. We say that the map T is an almost multiplier on A if

$$||T(x)y - xT(y)|| \leq \varepsilon||x||\ ||y||$$

for all $x, y \in A$.

We denote $\mathcal{AM}(A)$ (resp. $\mathcal{AM}_l(A)$, $\mathcal{AM}_r(A)$) the collection of all almost multipliers (resp. almost left multipliers, almost right multipliers) on A.

Example 7. Every bounded linear operator on Banach algebra A is an almost multiplier, But there are linear almost multipliers which are not multiplier.

Miura, Hirasawa and Takashi in [5], introduce a non-linear almost multiplier map which is not multiplier:

Example 8. Fix $\varepsilon > o$ arbitrarily. By the continuity of function $t \mapsto e^{it}$, there corresponds a δ with $0 < \delta < 1$, such that $|t| < 2\pi(1-\delta)$ implies $|e^{it} - 1| < \varepsilon$. With this δ, the mapping $f: \mathbb{C} \to \mathbb{C}$ by

$$f(x) = \begin{cases} 0 \quad , & z = 0 \\ |z| e^{i\delta\theta}, & z \epsilon \mathbb{C} - \{0\} \end{cases}$$

where $\theta \in [0,2\pi)$ denotes the argument of z, is an almost multiplier which is not a multiplier.

Here we give another simple example.

Example 9. Define $f: R \to R$ by

$$f(x) = \begin{cases} x, & x \in \mathbb{Q} \\ 0, & x \notin \mathbb{Q} \end{cases}$$

then f is a non-linear almost multiplier which is not multiplier.

Theorem 10. Suppose that A is a faithful complex Banach algebra. If the map $T: A \to A$ satisfies $T(0) = 0$ and

$$||T(a)b - aT(b)|| \leq \varepsilon ||a||^p \; ||b||^p \quad (a, b \in E)$$

for some $\varepsilon > 0$ and $p \neq 1$, then T is a multiplier on A.

Proof. [5]

Almost multipliers

In this section we study the basic properties of the almost multipliers on the stable normed algebras.

Theorem 1. Let A be a normed algebra. Then

1) $\mathcal{AM}_l(A) \cap \mathcal{AM}_r(A) \subseteq \mathcal{AM}(A)$.

2) If A is stable, then $\mathcal{AM}_l(A) \cap \mathcal{AM}_r(A) = \mathcal{AM}(A)$.

3) If A is stable and commutative, then $\mathcal{AM}_l(A) = \mathcal{AM}_r(A) = \mathcal{AM}(A)$.

Proof. 1) Suppose ε and δ satisfy in the definition of T as an almost left and right multiplier.

$$\begin{aligned} ||T(x)y - xT(y)|| &\leq ||T(xy) - T(x)y|| + ||T(xy) - xT(y)|| \\ &\leq (\varepsilon + \delta)||x|| \; ||y||. \end{aligned}$$

So T is an almost multiplier.

2) To prove of this part it is enough to show $\mathcal{AM}(A) \subseteq \mathcal{AM}_l(A) \cap \mathcal{AM}_r(A)$. Let $T \in \mathcal{AM}(A)$ with corresponding scaler $\varepsilon > 0$. Now for all $x, y, z \in A$, $||zT(xy) - T(z)xy|| \le \varepsilon||z||\ ||x||\ ||y||$. Also we have

$$\begin{aligned} ||zT(x)y - T(z)xy|| &= ||(zT(x) - T(z)x)y|| \\ &\le ||zT(x) - T(z)x||\ ||y|| \le \varepsilon||z||\ ||x||\ ||y||. \end{aligned}$$

Then,

$$\begin{aligned} ||zT(xy) - zT(x)y|| &\le ||zT(xy) - T(z)xy|| + ||T(z)xy - zT(x)y|| \\ &\le 2\varepsilon||x||\ ||y||\ ||z||. \end{aligned}$$

for all $x, y, z \in A$. Since the normed space A is stable, we have

$$||T(xy) - T(x)y|| \le 2\varepsilon||x||\ ||y||.$$

Similarly we can conclude, $T \in \mathcal{AM}_r(A)$.

3) By 1,2 the proof of this part is clear.

Theorem 2. Let A be a commutative stable normed algebra. Every almost multiplier on A is almost additive.

Proof. For all $x, y, z \in A$ and $T \in \mathcal{AM}(A)$ with corresponding scaler $\varepsilon > 0$, we have

$$\begin{aligned} ||zT(x+y) - z(T(x) + T(y))|| &\le ||zT(x+y) - (T(z)x + T(z)y)|| \\ &\quad + ||T(z)x - zT(x)|| + ||T(z)y - zT(y)|| \\ &\le \varepsilon||z||\ ||x+y|| + \varepsilon||z||\ ||x|| + \varepsilon||z||\ ||y|| \\ &\le 2\varepsilon(||x|| + ||y||)||z||. \end{aligned}$$

Since A is a stable normed algebra, we have

$$||T(x+y) - T(x) - T(y)|| \le 2\varepsilon(||x|| + ||y||).$$

For the rest of this note, the next obvious lemma will be useful.

Lemma 3. Let A be a unital normed algebra with unit e, and T be an almost multiplier (resp. almost left or right multipliers) with corresponding scaler $\varepsilon > 0$, then

$$||T(a)|| - ||aT(e)|| \le ||T(a)e - aT(e)|| \le \varepsilon||a||.$$

and so, $||T(a)|| \le (\varepsilon + ||T(e)||)||a||$.

In the rest of the paper for an almost multiplier map T in a unital normed algebra, we define $||T||_0 = \varepsilon + ||T(e)||$.

Theorem 4. Let A be a commutative unital normed algebra. Then $\mathcal{AM}(A)$ with composition as multiplication is an algebra.

Proof. It is easy to see that they are linear spaces. At first we show that the composition of two almost multiplier f and g with corresponding ε and δ respectively, is an almost multiplier.

$$\begin{aligned} ||xf \circ g(y) - f \circ g(x)y|| &\le ||xf(g(y)) - f(x)g(y)|| \\ &\quad + ||f(x)g(y) - g(x)f(y)|| + ||g(x)f(y) - f(g(x))y|| \\ &\le \varepsilon||x||\ ||g(y)|| + ||f(x)||\ ||g(y)|| + ||g(x)||\ ||f(y)|| \\ &\quad + \varepsilon||g(x)||\ ||y|| \end{aligned}$$

So by the previous lemma,

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$$||xf\circ g(y) - f\circ g(x)y|| \le \varepsilon||g||_0\, ||x||\, ||y|| + 2||f||_0\, ||g||_0\, ||x||\, ||y|| + \varepsilon||g||_0\, ||x||\, ||y||$$
$$\le (2\varepsilon||g||_0 + 2||f||_0\, ||g||_0)||x||\, ||y||.$$

Proposition 1. Let A be a normed algebra with a bounded left (resp. right) approximate identity and $\mathcal{B}$ be the set of all linear almost left (resp. right) multipliers on A, Then $\mathcal{B} = B(A)$.

Proof. Obviously every element of $B(A)$ is almost left multiplier. We show that every linear almost left multiplier is continuous. Let $(a_n) \subseteq A$ be a null sequence of A. By the Cohen factorization theorem we can find $b \in A$ and null sequence(c_n) in A, such that $a_n = bc_n$. Hence $||T(a_n) - T(b)c_n|| = ||T(bc_n) - T(b)c_n|| \le \varepsilon||b||||c_n||$. So $(T(a_n))$ is a null sequence.

Corollary 2. Let A be a Banach algebra with a bounded two sided approximate identity with bounded 1 and $\mathcal{B}$ be the set of all linear almost multiplier on A, then $\mathcal{B} = B(A)$.

Proof. By proposition 2.2 and theorem 3.1 (2), it is clear.

In proposition 3.5 we show that every linear almost multiplier is continuous, but by example 2.10 we see that there are non-linear almost multiplier which are not necessarily continuous.

In the following theorems we consider homogeneous almost (left and right) multipliers.

Theorem 3. Let A be a commutative stable normed algebra. Then the map T is almost multiplier if and only if $||T(x^2) - xT(x)|| \le \varepsilon||x||^2$, for all $x \in A$ and some $\varepsilon > 0$.

Proof. It is enough to show that if $||T(x^2) - xT(x)|| \le \varepsilon||x||^2$, for all $x \in A$, then T is a almost multiplier.

For all $x, y \in A$, whit $||x|| \le 1, ||y|| \le 1$, we have

$$||T(x+y)^2 - (x+y)T(x+y)|| \le \varepsilon||x+y||^2 \le \varepsilon(||x|| + ||y||)^2 \le 4\varepsilon.$$

Because A is commutative and T is almost linear

$$\begin{aligned} ||T(x^2 + y^2 + 2xy) - (x+y)T(x+y)|| = {} & ||T(x^2 + y^2 + 2xy) - T(x^2) - T(y^2) \\ & -2T(xy) + T(x^2) + T(y^2) + 2T(xy) \\ & -(x+y)(T(x) + T(y)) \\ & +(x+y)(T(x) + T(y)) \\ & -(x+y)T(x+y)|| \le 4\varepsilon \end{aligned}$$

and

$$\begin{aligned} ||2T(xy) - xT(y) - yT(x)|| \le {} & 4\varepsilon + ||T(x^2 + y^2 + 2xy) - T(x^2) - T(y^2) \\ & -2T(xy)|| + ||T(x^2) - xT(x)|| \\ & +||T(y^2) - yT(y)|| \\ & +||(x+y)(T(x+y) - T(x) - T(y))|| \\ & \le 4\varepsilon + 3\varepsilon(||x||^2 + ||y||^2 + ||x||\, ||y||) \\ & +\varepsilon||x||^2 + \varepsilon||y||^2 + 2\varepsilon(||x|| + ||y||)(||x|| + ||y||) \\ & \le 26\varepsilon. \quad (1) \end{aligned}$$

By replacing xz instead of x in (1) for all $z \in A$, with $||z|| \le 1$,

$$||2T(xyz) - xzT(y) - yT(xz)|| \le 26\varepsilon. \quad (2)$$

We have

$$||2T(xyz) - xzT(y) - y(xT(z) + zT(x))/2|| \leq$$
$$||2T(xyz) - xzT(y) - yT(xz) + yT(xz) - y(xT(z) + zT(x))/2|| \leq$$
$$||2T(xyz) - xzT(y) - yT(xz) + y(2T(xz) - xT(z) - zT(x))/2|| \leq$$
$$||2T(xyz) - xzT(y) - yT(xz)|| + \frac{1}{2}||y||\ ||2T(xz) - xT(z) - zT(x)||$$
$$\leq 26\varepsilon + 13\varepsilon = 39\varepsilon.$$

So

$$||4T(xyz) - 2xzT(y) - xyT(z) - zyT(x)|| \leq 78\varepsilon. \qquad (3)$$

By replacing yz instead of y in (1),

$$||2T(xyz) - yzT(x) - xT(yz)|| \leq 26\varepsilon. \qquad (4)$$

Similarly (1) and (4) implies

$$||4T(xyz) - 2yzT(y) - xyT(z) - zxT(y)|| \leq 78\varepsilon. \qquad (5)$$

By (3),(5)

$$||2xzT(y) + xyT(z) + zyT(x) - 2yzT(x) - xyT(z) - zxT(y)|| \leq 78\varepsilon + 78\varepsilon$$
$$= 156\varepsilon.$$

Then $||z(xT(y) - T(x)y)|| \leq 156\varepsilon$, for all $x, y, z \in A$, with $||x|| \leq 1$, $||y|| \leq 1$ and $||z|| \leq 1$.

For all $x, y, z \in E$ we have

$$||\frac{z}{||z||}(\frac{x}{||x||}T(\frac{y}{||y||}) - T(\frac{x}{||x||})\frac{y}{||y||})|| \leq 156\varepsilon.$$

Then

$$||z(xT(y) - T(x)y)|| \leq 156\varepsilon||x||\ ||y||\ ||z||,$$

for all $x, y, z \in A$. Since A is stable, so we have

$$||xT(y) - T(x)y|| \leq 156\varepsilon||x||\ ||y||.$$

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