ON   α-KENMOTSU MANIFOLDS SATISFYING FLATNESS CONDITIONS

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ABSTRACT

The main interest of the present paper is to study α-Kenmotsu manifolds that satisfy some certain tensor conditions where α is a smooth function defined by dα∧η=0 on M^n. In particular, the flatness conditions of α-Kenmotsu manifolds are investigated. We conclude the paper with an example on α-Kenmotsu manifolds depending on α.

Indexing terms/Keywords

Kenmotsu manifolds; Special weakly Ricci-symmetric manifolds; Weyl conformal curvature tensor; Conharmonic curvature tensor; Projective curvature tensor.

Academic Discipline And Sub-Disciplines

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TYPE (METHOD/APPROACH)

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INTRODUCTION

Tanno classified connected almost contact metric manifolds whose automorphism groups possess the maximum dimension (see [16]). The sectional curvature of plane sections for such manifolds containing the vector field ξ is a constant which is called c. The author proved that these manifolds can be divided into three classes:

(1) Homogenous normal contact Riemannian manifolds with c > 0,

(2) Global Riemannian products of a line or a circle with Kaehlerian manifold whose constant holomorphic sectional curvature under the condition c = 0,

(3) A warped product space R×$_f$C for the case c < 0.

It is well known that the class (1) are characterized by admitting a Sasakian structure. Kenmotsu

defined a structure closely related to the warped product which is characterized the differential

geometric properties of the manifolds of class (3). The structure is known as Kenmotsu structure and

in general, these structures are not Sasakian (see [8]).

It is well known that Kenmotsu manifolds can be characterized through their Levi-Civita connection. He proved that such a manifold M^{2n+1} is locally a warped product (-ε,+ε)×$_f$N^{2n} being a Kaehlerian manifold and $f(t) = ce^{t}$ where c is a positive constant (see [8]).

Weakly symmetric and weakly Ricci-symmetric Riemannian manifolds are generalized locally symmetric manifolds and pseudo symmetric manifold, respectively. These are manifolds in which the covariants derivative ∇R of the curvature tensor R is a linear expression in R. The appearing coefficients of this expression are called associated 1-forms. They satisfy in the specified types of manifolds gradually weaker conditions.

Firstly, the notions of weakly symmetric and weakly Ricci-symmetric manifolds were introduced by L. Tamássy and T.Q. Binh in 1992 (see [14], [15]). In [14], the authors studied on weakly symmetric and weakly projective symmetric Riemannian manifolds. In 1993, the authors considered weakly symmetric and weakly Ricci symmetric Einstein and Sasakian manifolds (see [15]). In 2000, U. C. De, T.Q. Binh and A.A. Shaikh gave necessary conditions for the compatibility of several K -contact structure with weak symmetry and weakly Ricci-symmetry (see [5]). In 2002, C. Özgür, investigated weakly symmetric and weakly Ricci-symmetric Riemann-para Sasakian manifolds (see [11]).

The notion of special weakly Ricci symmetric manifolds was introduced andstudied by H. Singh, and Q. Khan in 2001(see [13]). The authors considered special weakly symmetric manifolds. Next, Q. Khan studied some geometric properties of conharmonic Sasakian manifolds in 2004 and he also obtained some results on special weakly Ricci-symmetric Sasakian manifolds (see [10]). This paper is devoted to obtain some results on α-Kenmotsu manifolds by choosing a real value-function α instead of any real number α (constant function) with the help of some certain curvature tensor fields. For this reason, we have an α-Kenmotsu structure if there exists a normal almost contact metric structure (φ,ξ,η,g) such that dη=0 and dΦ=2α(η∧Φ) for any vector fields X,Y on M^n, where α is a smooth function defined by dα∧η=0 on M^n.

In this paper, the flatness conditions of α-Kenmotsu manifolds are investigated where α is a smooth function defined by dα∧η=0 on M^n. In particular, we consider φ-conformally flat, φ-conharmonically flat and φ-projectively flat α-cosymplectic manifolds. We prove main results on these manifolds by using the class (3). Moreover, special weakly Ricci-symmetric α-cosymplectic manifolds are examined. Finally, we give an example on α-Kenmotsu manifolds.

PRELIMINARIES

Let (M^n,g) be an n-dimensional Riemannian manifold. We denote by ∇ the covariant differentiation with respect to the Riemannian metric g. Then we have

R(X,Y)Z = ∇$_X$∇$_Y$Z- ∇$_Y$∇$_X$Z- ∇$_{[X,Y]}$Z.

The Riemannian curvature tensor is defined by

R(X,Y,Z,W) = g (R(X,Y)Z,W).

The Ricci tensor of M^n is defined as

S(X,Y)=trace{Z→R(X,Z)Y}

Locally, S is given by

S(X,Y)=∑R(X,E$_i$,Y,E$_i$), for i=1,2,…,n,

where $\{E_1, E_2, ..., E_n\}$ is a local orthonormal frames field and X,Y,Z,W are vector fields on M^n.

The Ricci operator Q is a tensor field of type (1,1) on M^n defined by

$g(QX,Y)=S(X,Y)$,

for all vector fields on M^n.

Let (M^n,g), n=dimM , n > 3, be a connected Riemannian manifold of class C^∞ and ∇ be

its Riemannian connection. The Weyl conformal curvature tensor C, the conharmonic curvature tensor K and the projective curvature tensor P of (M^n,g) are defined by

(2.1) $C(X,Y)Z = R(X,Y)Z-(1/(n-2))[S(Y,Z)X-S(X,Z)Y+g(Y,Z)QX-g(X,Z)QY$

$+(r/((n-1)(n-2)))[g(Y,Z)X-g(X,Z)Y]$,

(2.2) $K(X,Y)Z = R(X,Y)Z-(1/(n-2))[S(Y,Z)X-S(X,Z)Y+g(Y,Z)QX-g(X,Z)QY]$,

(2.3) $P(X,Y)Z=R(X,Y)Z-(1/(n-1))[g(Y,Z)QX-g(X,Z)QY]$,

respectively, where Q is the Ricci operator, S is the Ricci tensor, r=trace(S) is the scalar curvature and $X,Y,Z \in \chi(M^n)$, $\chi(M^n)$ being the Lie algebra of vector fields of M^n (see [6,17]).

Let C be the Weyl conformal curvature tensor of M^n. Since at each point $p \in M^n$ the tangent space $T_P(M^n)$ can be decomposed into the direct sum $T_P(M^n)=\varphi(T_P(M^n) \oplus L(\xi_p)$, where $L(\xi_p)$ is a 1-dimensional linear subspace of $T_P(M^n)$ generated by ξ_p, we have a map:

$C: T_P(M^n) \times T_P(M^n) \times T_P(M^n) \rightarrow \varphi(T_P(M^n) \oplus L(\xi_p)$.

 It may be natural to consider the following particular cases:

(1) $C: T_P(M^n) \times T_P(M^n) \times T_P(M^n) \rightarrow L(\xi_p)$, i.e., the projection of the image of C in $\varphi(T_P(M^n)$ is zero,

(2) $C: T_P(M^n) \times T_P(M^n) \times T_P(M^n) \rightarrow \varphi(T_P(M^n)$, i.e., the projection of the image of C in $L(\xi_p)$ is zero,

(3) $C: \varphi(T_P(M^n) \times \varphi(T_P(M^n) \times \varphi(T_P(M^n) \rightarrow L(\xi_p)$, i.e., when C is restricted to $T_P(M^n) \times \varphi(T_P(M^n) \times \varphi(T_P(M^n)$, the projection of the image of in $\varphi(T_P(M^n)$ is zero (see [4]). This condition is equivalent to

(2.4) $\varphi^2 C(\varphi X, \varphi Y)\varphi Z=0$.

A differentiable manifold (M^n,g), n>3, satisfying (2.4) is called φ-conformally flat.

A differentiable manifold (M^n,g), n>3, satisfying the condition

(2.5) $\varphi^2 K(\varphi X, \varphi Y)\varphi Z=0$,

is called φ-conharmonically flat.

A differentiable manifold (M^n,g), n>3, satisfying the condition

(2.6) $\varphi^2 P(\varphi X, \varphi Y)\varphi Z=0$,

is called φ-projectively flat.

The cases (1) and (2) were considered in [18] and [19], respectively. The case (3) was considered in [4] for the case M^n is a K-contact manifold. In [2], the authors considered

(k,μ)-contact manifolds satisfying (2.5). Furthermore, the authors studied (k,μ)-contact metric manifolds satisfying (2.4) in [1].

In [12], the author proves that an n-dimensional (n>3) conformally flat Lorentzian para-Sasakian manifold is an η-Einstein manifold, conharmonically flat Lorentzian para-Sasakian manifold is an Einstein manifold with zero scalar curvature. Also the author showed that a projectively flat Lorentzian para-Sasakian manifold is an Einstein manifold with scalar curvature r=n(n-1).

α-KENMOTSU MANIFOLDS

Let M^n be an n-dimensional differentiable manifold equipped with a triple (φ,ξ,η), where φ is a (1,1)-tensor field, ξ is a vector field, η is a 1-form on M^n such that

(3.1) $$\eta(\xi)=1, \qquad \varphi^2=-I+\eta\otimes\xi,$$

which implies

(3.2) $$\varphi\xi=0, \ \eta\circ\varphi=0, \ rank(\varphi)=n-1.$$

If M^n admits a Riemannian metric g, such that

(3.3) $$g(\varphi X,\varphi Y)=g(X,Y)-\eta(X)\eta(Y), \ \eta(X)=g(X,\xi)$$

then M^n is said to admit almost contact structure (φ,ξ,η,g)

On such a manifold, the fundamental Φ of M^n is defined by

$$\Phi(X,Y)=g(\varphi X,Y),$$

for $X,Y\in\Gamma(TM)$.

An almost contact metric manifold (M,φ,ξ,η,g) is said to be almost cosymplectic if $d\eta=0$

and $d\Phi=0$, where d is the exterior differential operator. The products of almost Kaehlerian manifolds and the real line R or the S^1 circle are the simplest examples of almost cosymplectic manifolds. An almost contact manifold (M,φ,ξ,η) is said to be normal if the Nijenhuis torsion

$$N_\varphi(X,Y)=[\varphi X,\varphi Y]-\varphi[\varphi X,Y]-\varphi[X,\varphi Y]+\varphi^2[X,Y]+2d\eta(X,Y)\xi,$$

vanishes for any vector fields X and Y.

An almost contact metric manifold M^n is said to be almost α-Kenmotsu if $d\eta=0$ and $d\Phi=2\alpha\eta\wedge\Phi$, α being a non-zero real constant. It is worthwhile to note that almost α-Kenmotsu structures are related to some special local conformal deformations of almost cosymplectic structures.

Moreover, an α-Kenmotsu manifold satisfies the following relations

(3.4) $$\nabla_X\xi=-\alpha\varphi^2 X,$$

(3.5) $$(\nabla_X\eta)(Y)=\alpha[g(X,Y)-\eta(X)\eta(Y)],$$

(3.6) $$(\nabla_X\varphi)Y=-\alpha[g(X,\varphi Y)\xi+\eta(Y)\varphi X],$$

for any vector fields X,Y on M^n, where ∇ denotes the Riemannian connection of g.

An α-Kenmotsu manifold M^n is said to be Einstein if its Ricci tensor S is of the form

$$S(X,Y)=\lambda g(X,Y),$$

where λ is constant and it is called η-Einstein if its Ricci tensor S is of the form

(3.7) $$S(X,Y)=\lambda_1 g(X,Y)+\lambda_2\eta(X)\eta(Y),$$

for any vector fields X and Y where λ_1 and λ_2 are functions on M^n (see [3,17]).

BASIC CURVATURE PROPERTIES

By using Riemannian curvature tensor properties, the following relations are obtained on

α-Kenmotsu manifolds:

(4.1) $$R(X,Y)\xi=[\alpha^2+\xi(\alpha)](\eta(X)Y-\eta(Y)X),$$

(4.2) $$R(X,\xi)\xi=[\alpha^2+\xi(\alpha)](\eta(X)\xi-X),$$

(4.3) $R(\xi,X)Y=[\alpha^2+\xi(\alpha)](\eta(Y)X-g(X,Y)\xi),$

(4.4) $g(R(\xi,X)Y,\xi)=[\alpha^2+\xi(\alpha)](-g(X,Y)+\eta(X)\eta(Y)),$

(4.5) $S(X,\xi)=-(n-1)[\alpha^2+\xi(\alpha)]\eta(X),$

(4.6) $S(\xi,\xi)=-(n-1)(\alpha^2+\xi(\alpha)),$

(4.7) $S(\varphi X,\varphi Y)=S(X,Y)+2(n-1)[\alpha^2+\xi(\alpha)]\eta(X)\eta(Y),$

for any vector fields X,Y on M^n where α is a smooth function such that $d\alpha\wedge\eta=0$. In these formulas, ∇ is the Levi-Civita connection, R the Riemannian curvature tensor and S is the Ricci tensor of M^n.

Remark 4.1 In [20], the above curvature properties are obtained for $\alpha\in\mathbb{R}$, $\alpha\neq0$.

CERTAIN TENSOR FIELDS ON α-KENMOTSU MANIFOLDS

In this section, we consider conharmonically flat and special weakly Ricci-symmetric manifold. Thus we give the following results:

Theorem 5.1 Let M^n be an α-Kenmotsu manifold. If the manifold M^n is conharmonically flat Einstein manifold and α is parallel along ξ, then M^n is a manifold of constant curvature such that α^2 is constant.

Proof We suppose that M^n be an α-Kenmotsu manifold satisfying following condition

(5.1) $K(X,Y)Z=0.$

Then it follows from (2.2)

(5.2) $R(X,Y)Z=(1/(n-2))[S(Y,Z)X-S(X,Z)Y+g(Y,Z)Q(X)-g(X,Z)Q(Y)].$

Let the manifold be Einstein. Then (5.2) reduces to

(5.3) $R(X,Y)Z=(2\lambda/n-2)[g(Y,Z)X-g(X,Z)Y],$

or

(5.4) $g(R(X,Y)Z,W)=(2\lambda/n-2)[g(Y,Z)g(X,W)-g(X,Z)g(Y,W)].$

Taking $X=W=\xi$ in (5.4), then we get

(5.5) $g(R(\xi,Y)Z,\xi)=(2\lambda/n-2)[g(Y,Z)-\eta(Y)\eta(Z)].$

Using (5.5) with the help of (4.4), we obtain

$((2\lambda/n-2)+(\alpha^2+\xi(\alpha)))[g(Y,Z)-\eta(Y)\eta(Z)]=0.$

We observe that if $g(Y,Z)-\eta(Y)\eta(Z)=0$, then $g(\varphi Y,\varphi Z)=0$ can be obtained by using (3.3) which is a contradiction on this structure. So the relation $((2\lambda/n-2)+(\alpha^2+\xi(\alpha)))$ vanishes. If α is parallel along ξ, then

we obtain $(2\lambda/n-2)+\alpha^2=0$. Thus we find $\lambda=((2-n)\alpha^2)/2)$ with $\alpha\neq0$. Hence, α-Kenmotsu case has a constant curvature such that α^2 is constant.

Consequently, the manifold M^n has a constant curvature satisfying α^2 is constant. As a special case, if we choose $\alpha=1$, we obtain that a conharmonically α-Kenmotsu manifold is locally isometric with a unit sphere which is proved in [10].

Definition 5.1 An n-dimensional Riemannian manifold (M^n,g) is called a special weakly Ricci-symmetric manifold if

(5.6) $\qquad (\nabla_S)(Y,Z)=2\zeta(X)S(Y,Z)+\zeta(Y)S(X,Z)+\zeta(Z)S(Y,X),$

where ζ is a 1-form and is defined by

(5.7) $\qquad\qquad\qquad \zeta(X)=g(X,\rho),$

where ρ is the associated vector field.

Then we can give the following results:

Theorem 5.2 If a special weakly Ricci-symmetric α-Kenmotsu manifold admits a cyclic Ricci tensor and α is parallel along ξ, then the 1-form ζ must be vanished.

Proof Let (5.6) and (5.7) be satisfied in an α-Kenmotsu manifold M^n. Taking cyclic sum of (5.6), we have

 (5.8) $\qquad (\nabla_S)(Y,Z)+(\nabla_S)(Z,X)+(\nabla_S)(X,Y)=4[\zeta(X)S(Y,Z)+\zeta(Y)S(X,Z)+\zeta(Z)S(Y,X)].$

Let M^n admit a cyclic Ricci tensor. Then (5.8) reduces to

(5.9) $\qquad \zeta(X)S(Y,Z)+\zeta(Y)S(X,Z)+\zeta(Z)S(Y,X)=0.$

Taking $Z=\xi$ in (5.9), we have

(5.10) $\qquad \zeta(X)S(Y,\xi)+\zeta(Y)S(X,\xi)+\eta(\rho)S(Y,X)=0,$

also taking $Y=\xi$ in (5.10) we get

(5.11) $\qquad \zeta(X)S(\xi,\xi)+\eta(\rho)S(X,\xi)+\eta(\rho)S(\xi,X)=0.$

Using $X=\xi$ in (5.11) and (4.6), we find

(5.12) $\qquad -3(\alpha^2+\xi(\alpha))(n-1)\eta(\rho)=0,$

and

(5.13) $\qquad\qquad\qquad \eta(\rho)=0,$

for the parallelity of α along ξ. Then making use of (5.13) in (5.11) gives $\zeta(X)=0$, $\forall X \in T(M)$.

At this point we recall that 1-form ζ must be vanished for α-Kenmotsu. This completes the proof.

Theorem 5.3   If the 1-form ζ does not vanish, then there exists no special weakly Ricci-symmetric α-Kenmotsu Einstein manifold.

Proof   Since α-Kenmotsu is an Einstein manifold, it holds $(\nabla_S)(Y,Z)=0$ and $S(Y,Z)=\lambda g(Y,Z)$. Hence (5.6) implies

$$(5.14) \qquad 2\zeta(X)S(Y,Z)+\zeta(Y)S(X,Z)+\zeta(Z)S(Y,X)=0.$$

Putting Z=ξ in (5.14), we get

$$(5.15) \qquad 2\zeta(X)S(Y,\xi)+\zeta(Y)S(X,\xi)+\eta(\rho)S(Y,X)=0.$$

Further, taking X=ξ in (5.15), we have

$$(5.16) \qquad 2\eta(\rho)S(Y,\xi)+\zeta(Y)S(\xi,\xi)+\eta(\rho)S(Y,\xi)=0.$$

Then putting Y=ξ in (5.16) and with the help of (3.1) and (3.6) we have

$$(5.17) \qquad -4(\alpha^2+\xi(\alpha))(n-1)\eta(\rho)=0.$$

In view of (5.17) in (5.16), we find $\zeta(Y)=0$, $\forall Y \in T(M)$ for $\alpha^2 \neq 0$. Thus it completes the proof.

α-KENMOTSU MANIFOLDS SATISFYING FLATNESS CONDITIONS

In this section, we consider φ-conformally flat, φ-conharmonically flat and φ-projectively flat α-Kenmotsu manifolds. According to these statements, the following results are held under certain flatness conditions:

Theorem 6.1   Let M^n be an n-dimensional, (n>3), φ-conformally flat α-Kenmotsu manifold and α is parallel along ξ. Then M^n is an η-Einstein manifold.

 Proof   Suppose that (M^n,g), n>3, be a φ-conformally flat α-Kenmotsu manifold. It is easy to see that $\varphi^2 C(\varphi X,\varphi Y)\varphi Z=0$ holds if and only if

$$g(C(\varphi X,\varphi Y)\varphi Z,\varphi W)=0,$$

for any $X,Y,Z,W \in \chi(M^n)$. So by the use of (2.1) φ-conformally flat means

$$(6.1) \qquad g(R(\varphi X,\varphi Y)\varphi Z,\varphi W) = (1/(n-2))[g(\varphi Y,\varphi Z)S(\varphi X,\varphi W)$$
$$-g(\varphi X,\varphi Z)S(\varphi Y,\varphi W)+g(\varphi X,\varphi W)S(\varphi Y,\varphi Z)-g(\varphi Y,\varphi W)S(\varphi X,\varphi Z)]$$
$$-(r/((n-1)(n-2)))[g(\varphi Y,\varphi Z)g(\varphi X,\varphi W)-g(\varphi X,\varphi Z)g(\varphi Y,\varphi W).$$

Let $\{E_1,...,E_{n-1},\xi\}$ be a local orthonormal basis of vector fields in M^n. Using that $\{\varphi E_1,...,\varphi E_{n-1},\xi\}$ is also a local orthonormal basis, if we put $X=W=E_i$ in (6.1) and sum up with respect to i, then

$$(6.2) \qquad \sum g(R(\varphi E_i,\varphi Y)\varphi Z,\varphi E_i) = (1/(n-2))\sum g(\varphi Y,\varphi Z)S(\varphi E_i,\varphi E_i)$$
$$-g(\varphi Ei,\varphi Z)S(\varphi Y,\varphi E_i)+g(\varphi E_i,\varphi E_i)S(\varphi Y,\varphi Z)-g(\varphi Y,\varphi E_i)S(\varphi E_i,\varphi Z)]$$
$$-(r/(n-1)(n-2))\sum[g(\varphi Y,\varphi Z)g(\varphi E_i,\varphi E_i)-g(\varphi E_i,\varphi Z)g(\varphi Y,\varphi E_i), \qquad \text{for } \forall i=1,2,\ldots,n-1.$$

It can be verify that

(6.3) $\sum g(R(\varphi E_i,\varphi Y)\varphi Z,\varphi E_i)=S(\varphi Y,\varphi Z)+[\alpha^2+\xi(\alpha)]g(\varphi Y,\varphi Z),$

(6.4) $\sum S(\varphi E_i,\varphi E_i)=r+[\alpha^2+\xi(\alpha)](n-1),$

(6.5) $\sum g(\varphi E_i,\varphi Z)S(\varphi Y,\varphi E_i)=S(\varphi Y,\varphi Z),$

(6.6) $\sum g(\varphi E_i,\varphi E_i)=n-1,$

and

(6.7) $\sum g(\varphi E_i,\varphi Z)g(\varphi Y,\varphi E_i)=g(\varphi Y,\varphi Z),$

for $i=1,2,\ldots,n-1$.

So by virtue of (6.4) and (6.5), (6.7) can be written as

(6.8) $S(\varphi Y,\varphi Z)=(r/n-1+(\alpha^2+\xi(\alpha)))g(\varphi Y,\varphi Z).$

Then making use of (3.6) and (4.7), (6.8) takes the form

(6.9) $S(Y,Z) = (r/n-1+(\alpha^2+\xi(\alpha)))g(Y,Z)-(r/n-1+(\alpha^2+\xi(\alpha))n)\eta(Y)\eta(Z).$

It means that M^n is an η-Einstein manifold by virtue of (3.7). This completes the proof.

Theorem 6.2   Let M^n be an n-dimensional, (n>3), α-Kenmotsu manifold and α is parallel along ξ. Then there exists no φ-projectively flat α-Kenmotsu manifolds with zero scalar curvature and the manifold has negative scalar curvature.

Proof   Assume that M^n be an n-dimensional, (n>3), φ-projectively flat α-Kenmotsu manifold. It is clear that $\varphi^2 P(\varphi X,\varphi Y)\varphi Z=0$ holds if and only if

$g(R(\varphi X,\varphi Y)\varphi Z,\varphi W)=0,$

for any $X,Y,Z,W \in \chi(M^n)$. From (2.3) and (4.7), φ-projectively flat means

(6.10) $g(R(\varphi X,\varphi Y)\varphi Z,\varphi W) = (1/n-2)[g(\varphi Y,\varphi Z)S(\varphi X,\varphi W)-g(\varphi X,\varphi Z)S(\varphi Y,\varphi W)].$

Choosing $\{E_1,\ldots,E_{n-1},\xi\}$ as a local orthonormal basis of vector fields in M^n and using the fact that $\{\varphi E_1,\ldots,\varphi E_{n-1},\xi\}$ is also a local orthonormal basis, putting $X=W=E_i$ in (6.10) and summing up with respect to i, then we have

(6.11) $\sum g(R(\varphi E_i,\varphi Y)\varphi Z,\varphi E_i) = (1/n-2)\sum[g(\varphi Y,\varphi Z)S(\varphi E_i,\varphi E_i)-g(\varphi E_i,\varphi Z)S(\varphi Y,\varphi E_i)],$

for i=1,2,…,n-1.

Then applying (6.4) and (6.5) into (6.11) gives

(6.12) $S(\varphi Y,\varphi Z)=(r/n)g(\varphi Y,\varphi Z).$

By virtue of (3.6) and (4.7) in (6.12), we find

(6.13) $S(Y,Z)=(r/n)g(Y,Z)-(r/n+(\alpha^2+\xi(\alpha))(n-1))\eta(Y)\eta(Z),$

and contracting (6.13) with respect to Y and Z, we have

(6.14) $r+(\alpha^2+\xi(\alpha))n(n-1)=0,$

which contradicts our hypothesis for n=0 and n=1. Thus there exists no φ-projectively flat α-Kenmotsu manifolds with r=0. On the other hand, the manifold is of negative scalar curvature with r=-α²n(n-1). Therefore, the proof is completed.

Theorem 6.3 Let M^n be an n-dimensional, (n>3), φ-conharmonically flat α-Kenmotsu manifold. Then M^n is an η-Einstein manifold with zero scalar curvature.

Proof Suppose that (M^n,g), (n>3), be a φ-conformally flat α-Kenmotsu manifold. It obvious that $\varphi^2 K(\varphi X,\varphi Y)\varphi Z=0$ holds if and only if

$g(K(\varphi X,\varphi Y)\varphi Z,\varphi W)=0,$

for any $X,Y,Z,W\in\chi(M^n)$. Using (2.2), φ-conformally flat gives

(6.15) $g(R(\varphi X,\varphi Y)\varphi Z,\varphi W) = (1/n-2)[g(\varphi Y,\varphi Z)S(\varphi X,\varphi W)$

$-g(\varphi X,\varphi Z)S(\varphi Y,\varphi W)+g(\varphi X,\varphi W)S(\varphi Y,\varphi Z)-g(\varphi Y,\varphi W)S(\varphi X,\varphi Z)].$

In analogy with the proof of (6.2), we can suppose that $\{E_1,...,E_{n-1},\xi\}$ is a local orthonormal basis of vector fields in M^n. By using the fact that $\{\varphi E_1,...,\varphi E_{n-1},\xi\}$ is also a local orthonormal basis, if we put $X=W=E_i$ in (6.15) and sum up with respect to i, then

(6.16) $\sum g(R(\varphi E_i,\varphi Y)\varphi Z,\varphi E_i) = (1/n-2)\sum[g(\varphi Y,\varphi Z)S(\varphi E_i,\varphi E_i)$

$-g(\varphi E_i,\varphi Z)S(\varphi Y,\varphi E_i)+g(\varphi E_i,\varphi E_i)S(\varphi Y,\varphi Z)-g(\varphi Y,\varphi E_i)S(\varphi E_i,\varphi Z)],$

for i=1,2,…,n-1.

Making use of (6.4) and (6.5), (6.16) turns into

(6.17) $S(\varphi Y,\varphi Z)=(r+(\alpha^2+\xi(\alpha)))g(\varphi Y,\varphi Z).$

Then applying (3.6) and (4.7) into (6.17) we have

(6.18) $S(Y,Z)=(r+(\alpha^2+\xi(\alpha)))g(Y,Z)-(r+(\alpha^2+\xi(\alpha))n)\eta(Y)\eta(Z).$

Assume that α is parallel along ξ, then the manifold is η-Einstein manifold. By contracting (6.18) with respect to Y and Z, we obtain (2-n)r=0 which implies the scalar curvature r=0.

EXAMPLE IN THREE DIMENSIONAL CASE

Let us denote the standart coordinates of R^3(x,y,z) and consider 3-dimensional manifold $M \subset R^3$ defined by

$M=\{(x,y,z) \in R^3 : z \neq 0\}$.

The vector fields are

$e_1 = e^{z^3}(\partial/(\partial x))$, $e_2 = e^{z^3}(\partial/(\partial y))$ $e_3 = (\partial/(\partial z))$.

It is clear that $\{e_1, e_2, e_3\}$ are linearly independent at each point of M. Let g be the Riemannian metric defined by

$g(e_1,e_1)=g(e_2,e_2)=g(e_3,e_3)=1$, $g(e_1,e_2)=g(e_1,e_3)=g(e_2,e_3)=0$,

and given by the tensor product

$g=(1/e^{z^3}) (dx \otimes dx + dy \otimes dy) + dz \otimes dz$.

Let η be the 1-form defined by $\eta(X)=g(X,e_3)$ for any vector field X on M and φ be the (1,1) tensor field defined by $\varphi(e_1)=e_2$, $\varphi(e_2)= -e_1$, $\varphi(e_3)=0$. Let h be the (1,1) tensor field defined by $h(e_1)=-\lambda e_1$, $h(e_2)=\lambda e_2$ and $h(e_3)=0$. Then using linearity of g and φ, we have

$\varphi^2 X=-X+\eta(X)e_3$, $\eta(e_3)=1$, $g(\varphi X, \varphi Y)=g(X,Y)-\eta(X)\eta(Y)$,

for any vector fields on M.

Let ∇ be the Levi-Civita connection with respect to the metric g. Then we get

$[e_1,e_3] = -3z^2 e_1$, $[e_2,e_3] = -3z^2 e_2$, $[e_1,e_2]=0$.

It follows that the structure of (φ,ξ,η,g) can easily be obtained. So it is sufficient to check that the only non-zero components of the second fundamental form Φ are

$\Phi((\partial/(\partial x)),(\partial/(\partial y)))=-\Phi((\partial/(\partial y)),(\partial/(\partial x)))=-(1/e^{z^3})$,

and hence

(7.1) $\Phi = -(1/e^{z^3}) (dx \wedge dy)$,

where $\Phi(e_1,e_2) = -1$ and otherwise $\Phi(e_i,e_j) = 0$ for i≤j. Thus the exterior derivation of Φ is given by

(7.2) $\qquad d\Phi = 6z^2(1/e^{2z^3}), (dx \wedge dy \wedge dz).$

Since $\eta = dz$, with the help of (7.1) and (7.2), we have

(7.3) $\qquad d\Phi = -6z^2 (\eta \wedge \Phi),$

where α defined $\alpha(z) = -3z^2$. Moreover, it can be noted that Nijenhuis torsion tensor of φ vanishes. Hence, the manifold is an α-Kenmotsu.

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