ISSN 2347-1921
Volume 12 Number 10
Journal of Advances in Mathematics

Some extensions Hardy integral inequalities and their analogues on finite interval

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ABSTRACT

The aim of this paper is to give some extensions Hardy integral inequalities for sum and product of several functions and their analogues inequalities on finite interval. Some direct consequences are established. Also a partial answer of an open problem posed by Sroysang is obtained.

Indexing terms/Keywords

Hardy's integral inequality; Levinson's integral inequality.

Academic Discipline And Sub-Disciplines

Mathematical Analysis, Mathematical Inequalities, Applied Mathematics.

SUBJECT CLASSIFICATION

2010Mathematics Subject Classification: 26D15.

1. INTRODUCTION

In 1920, Hardy (see [1]) presented the following inequality

$$\int_0^{+\infty} \left(\frac{F(x)}{x}\right)^p dx \leq \left(\frac{p}{p-1}\right)^p \int_0^{+\infty} f^p(x)dx, \tag{1.1}$$

where $f \geq 0,\ p > 1$ and

$$F(x) = \int_0^x f(t)dt.$$

The constant $\left(\frac{p}{p-1}\right)^p$ is the best possible. This inequality is important in mathematical analysis and its applications.

In 1964, Levinson (see [2], [3]) presented the following analogue of Hardy's integral inequality on finite interval $[a, b]$

$$\int_a^b \left(\frac{F(x)}{x}\right)^p dx \leq \left(\frac{p}{p-1}\right)^p \int_a^b f^p(x)dx, \tag{1.2}$$

where $0 < a < b < +\infty, f \geq 0,\ p > 1$ and

$$F(x) = \int_0^x f(t)dt.$$

In 2006, Bougoffa (see [4]) prove the following theorem about Hardy's integral inequality for several functions.

Theorem 1.1. Let $f_1, f_2, \cdots, f_i$ be nonnegative integrable functions. Define $F(x) = \int_a^x f_k(t)dt$, where $k = 1,2,\cdots,i$.

Then for $p > 1$, we have

$$\int_0^{+\infty} \left(\frac{F_1(x)F_2(x)\cdots F_i(x)}{x^i}\right)^{\frac{p}{i}} dx \leq \left(\frac{p}{p-1}\right)^p \int_0^{+\infty} (f_1(x) + f_2(x) + \cdots + f_i(x))^p dx. \tag{1.3}$$

The main purpose of this paper is to give several extensions Hardy integral inequalities for sum and product of several functions and their analogues inequalities on finite interval $[a, b]$ using Levinson's integral inequality (1.2). Some direct consequences are established. Special case obtained give a partial answer of an open problem posed by Sroysang in 2014(see [3]).

2. MAIN RESULTS

Throughout this section, functions are assumed to be integrable. Let

$$F(x)=\int_0^x\left(\sum_{i=1}^n f_i(t)\right)dt,\quad G(x)=\int_0^x\left(\sum_{i=1}^m g_i(t)\right)dt.\ (2.1)$$

Now, we are in position to prove the following theorem for sum of several functions.

Theorem 2.1. Let $0\le\sum_{i=1}^n f_i\le\sum_{i=1}^m g_i$ and $\sum_{i=1}^m g_i$ nonidentically zero, for all $x\in[0,+\infty[$. Define $F(x),G(x)$ by (2.1). Then

$$\int_0^{+\infty}\frac{F^p(x)}{G^q(x)}dx\le\left(\frac{p-q}{p-q-1}\right)^{p-q}\int_0^{+\infty}\varphi^{p-q}(x)dx,\ (2.2)$$

where $\varphi(x)=x(\sum_{i=1}^m g_i(x))+G(x),\ p-q>1,\ p>0.$

Proof. Let $0\le\sum_{i=1}^n f_i\le\sum_{i=1}^m g_i$ and $\sum_{i=1}^m g_i$ nonidentically zero, for all $x\in[0,+\infty[$. Then, for all $p>0,\ 0\le F^p(x)\le G^p(x)$. By using inequality (1.1) we get

$$\int_0^{+\infty}\frac{F^p(x)}{G^q(x)}dx\le\int_0^{+\infty}\frac{G^p(x)}{G^q(x)}dx=\int_0^{+\infty}\left(\frac{G^*(x)}{x}\right)^{p-q}dx\le\left(\frac{p-q}{p-q-1}\right)^{p-q}\int_0^{+\infty}\varphi^{p-q}(x)dx,$$

where $G^*(x)=xG(x)$ and $\varphi(x)=x(\sum_{i=1}^m g_i(x))+G(x),\ p-q>1,\ p>0.$ ■

Analogue of Theorem 2.1 on finite interval $[a,b], 0<a<b<+\infty$ is as follows.

Theorem 2.2. Let $0\le\sum_{i=1}^n f_i\le\sum_{i=1}^m g_i$ and $\sum_{i=1}^m g_i$ nonidentically zero, for all $x\in[a,b], 0<a<b<+\infty$. Define $F(x),G(x)$ by (2.1). Then

$$\int_a^b\frac{F^p(x)}{G^q(x)}dx\le\left(\frac{p-q}{p-q-1}\right)^{p-q}\int_a^b\varphi^{p-q}(x)dx,\ (2.3)$$

where $\varphi(x)=x(\sum_{i=1}^m g_i(x))+G(x),\ p-q>1,\ p>0.$

Proof. Let $0\le\sum_{i=1}^n f_i\le\sum_{i=1}^m g_i$ and $\sum_{i=1}^m g_i$ nonidentically zero, for all $x\in[a,b], 0<a<b<+\infty$. Then, for all $p>0,\ 0\le F^p(x)\le G^p(x)$. By using inequality (1.2) we get

$$\int_a^b\frac{F^p(x)}{G^q(x)}dx\le\int_a^b\frac{G^p(x)}{G^q(x)}dx=\int_a^b\left(\frac{G^*(x)}{x}\right)^{p-q}dx\le\left(\frac{p-q}{p-q-1}\right)^{p-q}\int_a^b\varphi^{p-q}(x)dx,$$

where $G^*(x)=xG(x)$ and $\varphi(x)=x(\sum_{i=1}^m g_i(x))+G(x),\ p-q>1,\ p>0.$ ■

Let

$$F_k(x)=\int_0^x f_k(t)dt,\quad G_k(x)=\int_0^x g_k(t)dt \text{ and } k=1,2,\cdots,n. \quad (2.4)$$

Now, we are in position to prove the following theorem for product of several functions.

Theorem 2.3. Let $0\le f_k\le g_k$ for all $k=1,2,\cdots,n, g_k$ nonidentically zero for all for all $x\in[0,+\infty[$. Define $F_k(x),G_k(x)$ by (2.4). Then

$$\int_0^{+\infty}\frac{(\prod_{k=1}^n F_k(x))^p}{(\prod_{k=1}^n G_k(x))^q}dx\le\left(\frac{p-q}{n(p-q-1)}\right)^{p-q}\int_0^{+\infty}\left(\sum_{k=1}^n\varphi_k(x)\right)^{p-q}dx,\ (2.5)$$

where $\varphi_k(x)=nxg_k(x)G_k^{n-1}(x)+G_k^n(x)$ for all $k=1,2,\cdots,n,\ p-q>1,\ p>0.$

Proof. Let $0\le f_k\le g_k$ for all $k=1,2,\cdots,n,\ g_k$ nonidentically zero for all for all $x\in[0,+\infty[$. Then, for all $p>0,\ 0\le(\prod_{k=1}^n F_k(x))^p\le(\prod_{k=1}^n G_k(x))^p$. By using Theorem 1.1 we get

$$\int_0^{+\infty}\frac{(\prod_{k=1}^n F_k(x))^p}{(\prod_{k=1}^n G_k(x))^q}dx\le\int_0^{+\infty}\frac{(\prod_{k=1}^n G_k(x))^p}{(\prod_{k=1}^n G_k(x))^q}dx=\int_0^{+\infty}\left(\prod_{k=1}^n\Psi_k(x)\right)^{\frac{p-q}{n}}dx$$

$$=\int_0^{+\infty}\left(\frac{\prod_{k=1}^n\Phi_k(x)}{x^n}\right)^{\frac{p-q}{n}}dx\le\left(\frac{p-q}{n(p-q-1)}\right)^{p-q}\int_0^{+\infty}\left(\sum_{k=1}^n\varphi_k(x)\right)^{p-q}dx,$$

where $\Phi_k(x) = x\Psi_k(x) = xG_k^n(x) = \int_0^x \varphi_k(t)dt$, for all $k = 1,2,\cdots,n$.

After differentiation we get $\varphi_k(x) = nxg_k(x)G_k^{n-1}(x) + G_k^n(x)$ for all $k = 1,2,\cdots,n,\ p-q > 1,\ p > 0$. ■

Analogue of Theorem 2.3 on finite interval $[a,b], 0 < a < b < +\infty$ is as follows.

Theorem 2.4. Let $0 \le f_k \le g_k$ for all $k = 1,2,\cdots,n$, g_k nonidentically zero for all for all $x \in [a,b], 0 < a < b < +\infty$. Define $F_k(x), G_k(x)$ by (2.4). Then

$$\int_a^b \frac{(\prod_{k=1}^n F_k(x))^p}{(\prod_{k=1}^n G_k(x))^q} dx \le \left(\frac{p-q}{n(p-q-1)}\right)^{p-q} \int_a^b \left(\sum_{k=1}^n \varphi_k(x)\right)^{p-q} dx, (2.6)$$

where $\varphi_k(x) = nxg_k(x)G_k^{n-1}(x) + G_k^n(x)$ for all $k = 1,2,\cdots,n,\ p-q > 1,\ p > 0$.

Proof. Let $0 \le f_k \le g_k$ for all $k = 1,2,\cdots,n$, g_k nonidentically zero for all for all $x \in [a,b], 0 < a < b < +\infty$. Then, for all $p > 0,\ 0 \le (\prod_{k=1}^n F_k(x))^p \le (\prod_{k=1}^n G_k(x))^p$. By using Theorem 1.1 and inequality (1.2) we get

$$\int_a^b \frac{(\prod_{k=1}^n F_k(x))^p}{(\prod_{k=1}^n G_k(x))^q} dx \le \int_a^b \frac{(\prod_{k=1}^n G_k(x))^p}{(\prod_{k=1}^n G_k(x))^q} dx = \int_a^b \left(\prod_{k=1}^n \Psi_k(x)\right)^{\frac{p-q}{n}} dx$$

$$= \int_a^b \left(\frac{\prod_{k=1}^n \Phi_k(x)}{x^n}\right)^{\frac{p-q}{n}} dx \le \left(\frac{p-q}{n(p-q-1)}\right)^{p-q} \int_a^b \left(\sum_{k=1}^n \varphi_k(x)\right)^{p-q} dx,$$

where $\Phi_k(x) = x\Psi_k(x) = xG_k^n(x) = \int_0^x \varphi_k(t)dt$, for all $k = 1,2,\cdots,n$.

After differentiation we get $\varphi_k(x) = nxg_k(x)G_k^{n-1}(x) + G_k^n(x)$ for all $k = 1,2,\cdots,n,\ p-q > 1,\ p > 0$. ■

Remark 2.5. Special case: Theorem 2.3 for $k = 1$ has the following form

$$\int_0^{+\infty} \frac{F_1{}^p(x)}{G_1{}^q(x)} dx \le \left(\frac{p-q}{p-q-1}\right)^{p-q} \int_0^{+\infty} \varphi_1{}^{p-q}(x)dx, (2.7)$$

where $\varphi_1(x) = xg_1(x) + G_1(x),\ p-q > 1,\ p > 0$.

This is a partial answer of an open problem posed by Sroysang in 2014 (see [3]).

Analogue inequality on finite interval $[a,b], 0 < a < b < +\infty$, is as follows.

$$\int_a^b \frac{F_1{}^p(x)}{G_1{}^q(x)} dx \le \left(\frac{p-q}{p-q-1}\right)^{p-q} \int_a^b \varphi_1{}^{p-q}(x)dx, (2.8)$$

where $\varphi_1(x) = xg_1(x) + G_1(x),\ p-q > 1,\ p > 0$. Use Theorem 2.4 for $k = 1$.

Next, we give some direct consequences of Theorems 2.1, 2.2, 2.3 and 2.4.

3. APPLICATIONS

Corollary 3.1. Let $0 \le \sum_{i=1}^n f_i \le \sum_{i=1}^m g_i$ and $\sum_{i=1}^m g_i$ nonidentically zero, for all $x \in [0,+\infty[$. Define $F(x), G(x)$ by (2.1). Then

$$\int_0^{+\infty} \big(F(x)G(x)\big)^p dx \le \left(\frac{2p}{2p-1}\right)^{2p} \int_0^{+\infty} \varphi^{2p}(x)dx, (3.1)$$

where $\varphi(x) = x(\sum_{i=1}^m g_i(x)) + G(x)$ and $p > \frac{1}{2}$.

Proof. Let $q = -p$ and use Theorem 2.1. ■

Analogue inequality of Corollary 3.1 on finite interval $[a,b]$, is as follows.

Corollary 3.2. Let $0 \le \sum_{i=1}^n f_i \le \sum_{i=1}^m g_i$ and $\sum_{i=1}^m g_i$ nonidentically zero, for all $x \in [a,b], 0 < a < b < +\infty$. Define $F(x), G(x)$ by (2.1). Then

$$\int_a^b \big(F(x)G(x)\big)^p dx \le \left(\frac{2p}{2p-1}\right)^{2p} \int_a^b \varphi^{2p}(x)dx, (3.2)$$

where $\varphi(x) = x(\sum_{i=1}^{m} g_i(x)) + G(x)$ and $p > \frac{1}{2}$.

Proof. Let $q = -p$ and use Theorem 2.2. ■

Corollary 3.3. Let $0 \le f_k \le g_k$ for all $k = 1,2,\cdots,n$, g_k nonidentically zero for all for all $x \in [0,+\infty[$. Define $F_k(x), G_k(x)$ by (2.4). Then

$$\int_0^{+\infty} \left(\prod_{k=1}^{n} \mathrm{F}_k(x) G_k(x)\right)^p dx \le \left(\frac{2p}{n(2p-1)}\right)^{2p} \int_0^{+\infty} \left(\sum_{k=1}^{n} \varphi_k(x)\right)^{2p} dx, \quad (3.3)$$

where $\varphi_k(x) = nxg_k(x)G_k^{n-1}(x) + G_k^n(x)$ for all $k = 1,2,\cdots,n$ and $p > \frac{1}{2}$.

Proof. Let $q = -p$ and use Theorem 2.3. ■

Corollary 3.4. Let $f_k > 0$ for all $k = 1,2,\cdots,n$, and for all $x \in [0,+\infty[$. Define $F_k(x)$ by (2.4). Then

$$\int_0^{+\infty} \left(\prod_{k=1}^{n} F_k(x)\right)^{2p} dx \le \left(\frac{2p}{n(2p-1)}\right)^{2p} \int_0^{+\infty} \left(\sum_{k=1}^{n} \varphi_k(x)\right)^{2p} dx, \quad (3.4)$$

where $\varphi_k(x) = nxf_k(x)F_k^{n-1}(x) + F_k^n(x)$ for all $k = 1,2,\cdots,n$ and $p > \frac{1}{2}$.

Proof. Let $f_k = g_k$ for all $k = 1,2,\cdots,n$ and use Corollary 3.3. ■

Corollary 3.5. Let $0 \le f_k = f \le g = g_k$ for all $k = 1,2,\cdots,n$, g nonidentically zero for all for all $x \in [0,+\infty[$. Define

$$F(x) = \int_0^x f(t)dt, \qquad G(x) = \int_0^x g(t)dt.$$

Then

$$\int_0^{+\infty} \left(\frac{F^p(x)}{G^q(x)}\right)^n dx \le \left(\frac{p-q}{p-q-1}\right)^{p-q} \int_0^{+\infty} \varphi^{p-q}(x)dx, \quad (3.5)$$

where $\varphi(x) = nxg(x)G^{n-1}(x) + G^n(x)$, $p - q > 1$, $p > 0$.

Proof. Let $0 \le f_k = f \le g = g_k$ for all $k = 1,2,\cdots,n$, g nonidentically zero for all for all $x \in [0,+\infty[$. Then $F_k(x) = F(x)$ and $G_k(x) = G(x)$ for all $k = 1,2,\cdots,n$. Using Theorem 2.3, inequality (3.5) follows. ■

Remark 3.6. Inequality (3.5) is a generalization of inequality (2.7).

Analogues inequalities for Corollaries 3.3, 3.4 and 3.5 on finite interval are as follows.

Corollary 3.7. Let $0 \le f_k \le g_k$ for all $k = 1,2,\cdots,n$, g_k nonidentically zero for all for all $x \in [a,b], 0 < a < b < +\infty$. Define $F_k(x), G_k(x)$ by (2.4). Then

$$\int_a^{b} \left(\prod_{k=1}^{n} F_k(x) G_k(x)\right)^p dx \le \left(\frac{2p}{n(2p-1)}\right)^{2p} \int_a^{b} \left(\sum_{k=1}^{n} \varphi_k(x)\right)^{2p} dx, \quad (3.6)$$

where $\varphi_k(x) = nxg_k(x)G_k^{n-1}(x) + G_k^n(x)$ for all $k = 1,2,\cdots,n$ and $p > \frac{1}{2}$.

Proof. Let $q = -p$ and use Theorem 2.4. ■

Corollary 3.8. Let $f_k > 0$ for all $k = 1,2,\cdots,n$ and for all $x \in [a,b], 0 < a < b < +\infty$. Define $F_k(x)$ by (2.4). Then

$$\int_a^{b} \left(\prod_{k=1}^{n} F_k(x)\right)^{2p} dx \le \left(\frac{2p}{n(2p-1)}\right)^{2p} \int_a^{b} \left(\sum_{k=1}^{n} \varphi_k(x)\right)^{2p} dx, \quad (3.7)$$

where $\varphi_k(x) = nxf_k(x)F_k^{n-1}(x) + F_k^n(x)$ for all $k = 1,2,\cdots,n$ and $p > \frac{1}{2}$.

Proof. Let $f_k = g_k$ for all $k = 1,2,\cdots,n$ and use Corollary 3.7. ■

Corollary 3.9. Let $0 \le f_k = f \le g = g_k$ for all $k = 1,2,\cdots,n$, g nonidentically zero for all $x \in [a,b], 0 < a < b < +\infty$. Define

$$F(x) = \int_0^x f(t)dt, \qquad G(x) = \int_0^x g(t)dt.$$

Then

$$\int_a^b \left(\frac{F^p(x)}{G^q(x)}\right)^n dx \leq \left(\frac{p-q}{p-q-1}\right)^{p-q} \int_a^b \varphi^{p-q}(x)dx, (3.8)$$

where $\varphi(x) = nxg(x)G^{n-1}(x) + G^n(x),\ p-q > 1,\ p > 0.$

Proof. Let$0 \leq f_k = f \leq g = g_k$ for all $k = 1,2,\cdots,n,$ gnonidentically zero for all $x \in [a,b], 0 < a < b < +\infty$.Then $F_k(x) = F(x)$ and $G_k(x) = G(x)$for all $k = 1,2,\cdots,n$.Using Theorem 2.4, inequality (3.8)holds. ■

ACKNOWLEDGMENTS

We thank anonymous referee for his/her valuable suggestion regarding the manuscript.

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