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A probability space of continuous-time discrete value stochastic process with Markov property

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Abstract

Getting acquainted with the theory of stochastic processes we can read the following statement: "In the ordinary axiomatization of probability theory by means of measure theory, the problem is to construct a sigma-algebra of measurable subsets of the space of all functions, and then put a finite measure on it". The classical results for limited stochastic and intensity matrices goes back to Kolmogorov at least late 40-s. But for some infinity matrices the sum of probabilities of all trajectories is less than 1.

Some years ago I constructed physical models of simulation of any stochastic processes having a stochastic or an intensity matrices and I programmed it. But for computers I had to do some limitations - set of states at present time had to be limited, at next time - not necessarily. If during simulation a realisation accepted a state out of the set of limited states - the simulation was interrupted. I saw that I used non-quadratic, half-infinity stochastic and intensity matrices and that the set of trajectories was bigger than for quadratic ones. My programs worked good also for stochastic processes described in literature as without probability space. I asked myself: did the probability space for these experiments not exist or were only a set of events incompleted? This paper shows that the second hipothesis is true.

Keywords: Markov process; Intensity matrix; Caratheodory theorem in measure theory; Probability space for stochastic processes

Mathematics Subject Classification (2000) : 60J45

1 Introduction

For each continuous-time discrete value stochastic process we can define instantaneous probability rates. That idea is used by biologists [9], [6] and it corresponds to the intensity of probability used in the theory of stochastic processes. For a continuous-time discrete value stochastic Markov process it can be defined as a right differential coefficient of the conditional probability of change from one state to another.

Let $((X_t)_{t\in[0,T)}, P, \sigma(\Omega))$ be continuous-time discrete value stochastic process with Markov property, where: $X_t : [0,T) \to W$ are random variables, W is a set of events, $\sigma(\Omega)$ is σ-algebra on Ω, $P : \sigma(\Omega) \to [0.1]$ is probability, $T > 0$, and T can be equal to ∞, W is a finite or numerable set of states. The states in W can be numbered by integers, so $W = (E_n)_{n\in \mathrm{N}}$.

For all t and $\tau > 0$ we can calculate a conditional probability (state-transition probability) in period $[t, t+\tau)$:

$$P_{k,n}(t, t+\tau) = P\{X_{t+\tau} = E_n \mid X_t = E_k\} \tag{1}$$

where $E_k \in W$ and $E_n \in W$ are states.

Then we can define the instantaneous probability rate of the change of the state E_n to E_k at time t as equal to:

$$P'_{k,n}(t) = \lim_{\tau\to 0^+} \frac{P_{k,n}(t,t+\tau)}{\tau} = \Psi^t_{k,n}{}'(0^+) \text{ for } k \neq n \tag{2}$$

or

$$P'_{k,k}(t) = \lim_{\tau\to 0^+} \frac{P_{k,k}(t,t+\tau)-1}{\tau} = \Psi^t_{k,k}{}'(0^+) \text{ for } k = n \tag{3}$$

where $\Psi^t_{k,n}(\tau) = P_{k,n}(t, t+\tau)$. The function $P'_{k,n}$ is equal to $[0,T) \ni t \to \Psi^t_{k,n}{}'(0^+) \in \mathrm{N}$.

For all standard continuous-time discrete value stochastic processes those limits are presented. For homogenous processes functions $t \to P_{k,n}{}'(t)$ are constant and for many processes (described diurnal or seasonal fluctuations of probability) those functions are continuous. For the proofs of theorems in this article you will need only their integrability and the limitations during the limited time-periods.

Instantaneous probability rates can have values greater than 1 (but always non-negative for $k \neq n$ and non-positive for $k = n$) and they are measured by 1/year, 1/h, 1/s etc. According to the property:

$$\sum_k P_{k,n}(t,t+\tau) = 1 \text{ for all } t,\tau > 0 \text{ and } n \tag{4}$$

that is equivalent to

$$\sum_k \frac{P_{k,n}(t,t+\tau)}{\tau} + \frac{P_{k,n}(t,t+\tau)-1}{\tau} = 0 \text{ for all } t,\tau > 0 \text{ and } n \tag{5}$$

instantaneous probability rates fit the condition:

$$\sum_k P_{k,n}'(t) = 0 \text{ for all } t \text{ and } n \tag{6}$$

For every time t we can consider a matrix (finite or infinite) of instantaneous probability rates: $[P_{k,n}'(t)]_{k,n\in\mathrm{N}}$. It is a well-known intensity matrix [11], [1] or state-transition matrix [7], [12] used by mathematicians to create Kolmogorov equations. In this article such a matrix will be treated differently. This matrix is a based object creating probability space for Markov stochastic processes.

The aim of this paper is to prove that for all matrices of integrability functions $[f_{k,n}(t)]_{k,n\in I}$ limited on each time period $[0,T]$ $(T<\infty)$, non-negative for $k \neq n$ and non-positive for $k = n$, which satisfy the condition $\sum_k f_{k,n}'(t) = 0$ for all t and n we can construct a probability space for a continuous-time discrete value stochastic Markov process whose instantaneous probability rates $P_{k,n}'(t)$ are equal to the $f_{k,n}(t)$.

2 Half-infinite stochastic and intensity matrices

A modelling of Markov stochastic processes is an important branching in theoretical biology [5], [3]. But in biology stochastic and intensities matrices are defined by functions $P_{k,n}'(t)$ given by mathematical formulas of t, k and n, so the matrices have infinite rows and columns. In the other hand the simulation of this processes are provided by finite, assumed maximal time or by shorter time, if k shows an impossible state. The maximal possible state are always assumed - sometimes it is a subjectively assumed number in computer programme, sometimes it is maximal integer understanding by computers.

Stochastic and intensities matrices of these processes are half-infinite. They have a form: $[P_{k,n}']_{k\in[0,K],n\in[0,\infty]}$. All finite matrices $[P_{k,n}']_{k,n\in[0,K]}$ can be extended to half-infinite by assumption $P_{k,n}'(t) = 0$ if $n > K$. The set of realisations of these processes includes all time-depended functions $[0,t) \to W$, where if $t < t_{max}$ then the last state of the realisation is an impossible state.

Although half-infinite stochastic and intensity matrices give the impression of unnecessary mathematical entities, this paper shows that all continuous-time discrete value stochastic processes with Markov property defined by half-infinite intensity matrices have a correct defined probability spaces as Kolmogorov conception. These probability spaces allow to define a probability space for all (finite or infinite) intensity matrices and for infinite time period.

3 General theorem

For all matrices of integrability functions $[f_{k,n}(t)]_{k\in[0,K],n\in[0,\infty]}$, where K can be finite or infinite, which satisfy the following condition:

$$\forall_{k\le K}\forall_{n\neq k}\forall_{t\in[0,T)} f_{k,n}(t) \ge 0 \tag{7}$$

$$\forall_{k\le K}\forall_{t\in[0,T)} f_{k,k}(t) \le 0 \tag{8}$$

$$\forall_{T<\infty}\exists_{M>0}\forall_{k<K}\forall_{t\in[0,T]} f_{k,k}(t) > -M \tag{9}$$

$$\forall_{t\in[0,T)}\forall_{k\le K}\sum_n f_{k,n}(t) = 0 \tag{10}$$

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a correct probability space $(\Omega, \sigma(\Omega), P)$ for a continuous-time discrete value stochastic Markov process whose instantaneous probability rates $P_{k,n}'(t)$ are equal to the $f_{k,n}(t)$ can be constructed.

The states in W noted as E_k can be assumed. In set of these states the natural or unnatural order can be defined $(E_0 < E_1 < \cdots < E_K < \cdots)$ to form a matrix with columns and rows named by E_0, E_1, $\cdots$. The state E_K is a maximal possible state and for all $k > K$ the states E_k are impossible. The half-infinite matrix $[f_{k,n}(t)]_{k \in [0,K], n \in [0,\infty]}$ can be labelled by $(E_0 < E_1 < \cdots < E_K) \times (E_0 < E_1 < \cdots < E_K < \cdots)$.

4 The construction of probability space for half infinite matrices and $T < \infty$

4.1 Set of events Ω

We can consider time-depended, step functions (constant on for a finite or countable number of intervals): $\varphi : [0,T) \to W$ and their graphs on the coordinate system (Fig.1). We will consider functions constant on left closed intervals $[t_1, t_2)$. Such a function will be called the realization of the forming of a stochastic process. All those realizations form a set Ω.

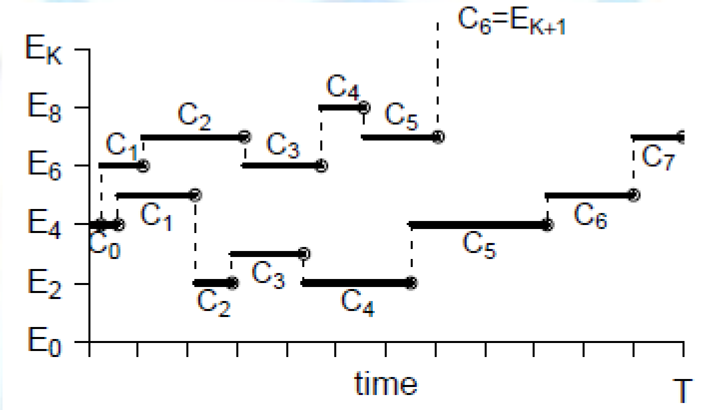

Figure 1: Two exemplary realizations.

States defined by names of rows or columns of matrix $[f_{k,n}(t)]_{k,n}$ are labelled by E_n. But for each realization we have a sequence of states successively appearing in this function. This sequence will be noted as $(C_i)_{i=0,1,2,\cdots}$.

4.2 Basis B of $\sigma(\Omega)$

Let $(\Delta_i)_{i=1,2,\cdots}$ be a finite or infinite sequence of time left-closed intervals such that $\Delta_i \cap \Delta_j = \varnothing$ for $i \neq j$ and $\bigcup \Delta_i = [0,S]$ for $S \leq T$. Such a sequence of intervals will be called the covering of interval $[0,S)$. A case that this sequence has one or more focusing points is not excluded. Let C_0 be the initial state at $t_0 = 0$ and $(C_i)_{i=1,2,\cdots}$ be any sequence of states from W. A case that $C_i = C_{i+1}$ is not excluded. We assume that C_0 is fixed, the same for all realizations.

Definition 1 The base set formed by the covering $(\Delta_i)_{i=1,2,\cdots,z}$ of $[0,S)$ and the sequence of states $(C_i)_{i=1,2,\cdots,z}$ is a set of all realizations $\varphi \in \Omega$, which satisfy the following conditions:

1. for each Δ_i there is $s \in \Delta_i$ that for $t \in \Delta_i$ $\varphi(t) = \begin{cases} C_i & \text{if } t < s \\ C_{i+1} & \text{if } t \geq s \end{cases}$

2. all states $(C_i)_{i=1,2,\cdots,z-1}$ are possible,

3. if $S < T$, the last state C_S is impossible,

4. if $S = T$ the last state C_T is possible.

Two realizations are contained in the same base set if the sequences of states of both realizations are the same and only times of change from one state to another are a little different (Fig.2).

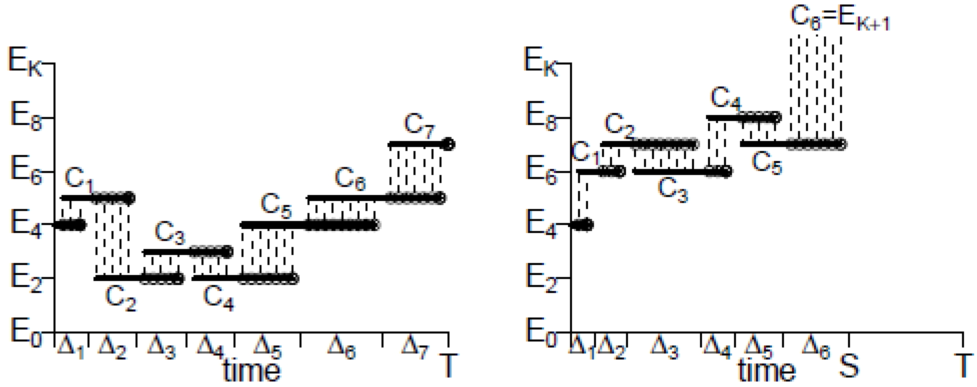

Figure 2: Realizations included in the same base set formed by sequence of intervals $(\Delta_i)_{i=1,2,...}$ and sequence of states $(C_i)_{i=1,2,...}$.

For the covering $(\Delta_i)_{i=1,2,..}$ without focusing points we will assume that Δ_i is followed by Δ_{i+1} for all i. For all coverings we can find such a sequence of index that finite number of chosen neighbouring intervals are noted Δ_i and Δ_{i+1}. Therefore, we will use a notation Δ_i, Δ_{i+1}, ..., Δ_{i+L} for neighbouring intervals.

Sometimes the same base set of realizations can be formed by different coverings and sequences of states. For instance, the covering:

$$([t_{2i}, t_{2i+1}), [t_{2i+1}, t_{2i+2}))_{i=0,1,2..}$$

with the sequence of states $(C_i, C_i)_{i=1,2,..}$ (realizations don't change states on intervals $[t_{2i+1}, t_{2i+2})$ and another covering

$$([t_{2i}, t_{2i+1}), [t_{2i+1}, 0.5(t_{2i+1} + t_{2i+2})), [0.5(t_{2i+1} + t_{2i+2}), t_{2i+2}))_{i=0,1,2,..}$$

with the sequence of states $(C_i, C_i, C_i)_{i=1,2,..}$ formed the same base set. But for all base sets there exists only one covering $(\Delta_i)_{i=1,2,..}$ and sequence of states $(C_i)_{i=1,2,..,z}$ such that $C_i = C_{i+1}$ imply $C_{i+1} \neq C_{i+2}$. It will be called the optimal covering.

Definition 2 Basis B is a set of all base sets formed by all possible coverings $(\Delta_i)_{i=1,2,..}$ of all possible intervals $[0, S) \subseteq [0, T)$ and all sequences of states $(C_i)_{i=1,2,..}$.

B is a subset of 2^{Ω} (all subsets of Ω). Basis B is required to define σ-algebra on Ω as minimal σ-algebra consisting B, but in this paper the construction of σ-algebra for probability space is a little different. It is similar to construction of Lebesgue measure domain [8], [10], [2], [4].

For any A and B from B such that $A \cap B \neq \emptyset$ the set $A \cap B \in \mathrm{B}$ because if A is formed by optimal covering $(\Delta_i^A)_{i=1,2,..}$ and the sequence of states $(C_i^A)_{i=1,2,..}$ and B is formed by the optimal covering $(\Delta_i^B)_{i=1,2,..}$ and the sequence of states $(C_i^B)_{i=1,2,..}$ ($A \cap B \neq \emptyset$ only if $C_i^A = C_i^B = C_i$ for all i), then $A \cap B$ is formed by the covering $(\Delta_i^B \cap \Delta_i^B)_{i=1,2,..}$ and the sequence of states $(C_i, C_i, C_{i+1})_{i=1,2,..}$.

For A and B from B formed by finite optimal coverings (noted as $(\Delta_i^A)_{i=1,2,..}$ and $(\Delta_i^B)_{i=1,2,..}$ respectively) and the same sequences of states $(C_i)_{i=1,2,..}$ there exists a finite sequence of disjoint sets $(D_k)_{k=1,2,..}$ from B such that

$A-B=\bigcup D_i$. Set D_k is formed by the covering $(\Delta_i^A \cap \Delta_{i-1}^B, \Delta_i^A \cap \Delta_i^B, \Delta_i^A \cap \Delta_{i+1}^B)_{i=1,2,..}$ and sequence of states $(C_{i'}, C_{i+1}', C_{i+2}')_{i=1,2,..}$ where $(C_{i'}, C_{i+1}', C_{i+2}') \in \{(C_i, C_{i+1}, C_{i+1}), (C_i, C_i, C_{i+1}), (C_i, C_i, C_i)\}$.

The properties of: $A \in \mathrm{B}$ and $B \in \mathrm{B}$ imply $A \cap B \in \mathrm{B}$, $A \in \mathrm{B}$ and $B \in \mathrm{B}$ imply that existed some $D_k \in \mathrm{B}$ that $A-B=\bigcup D_i$ mean that B is a semi-ring on Ω [8].

4.3 Base function P on B

For any interval $[t_1,t_2) \subseteq [0,S) \subseteq [0,T)$ and any pair of states E_k, E_n where $k \le K$, we can calculate:

$$P_{E_k,E_n}(t_1,t_2)=\begin{cases} exp(\int_{t_1}^{t_2} f_{n,n}(x)dx) \quad \text{if } k=n \\ \int_{t_1}^{t_2}[f_{k,n}(x)exp(\int_{t_1}^{x} f_{k,k}(s)ds+\int_{x}^{t_2} f_{n,n}(s)ds)]dx \\ \qquad \text{if } k \ne n \text{ and } n \le K \\ \int_{t_1}^{t_2}[f_{k,n}(x)exp(\int_{t_1}^{x} f_{k,k}(s)ds)]dx \\ \qquad \text{if } k \ne n \text{ and } n > K \end{cases} \tag{11}$$

If $t_1=t_2$ then the integrals are equal to 0. It means that if $\Delta=0$ then $P_{E_k,E_k}(\Delta)=1$ and if $\Delta=0$ and $n \ne k$, then $P_{E_k,E_n}(\Delta)=0$.

Theorem 1 For any E_k and E_n and $t \in [0,T)$ the function $[0,\infty) \ni \tau \to P_{E_k,E_n}(t,t+\tau)$ has the following properties:

1. its values are in interval [0,1]
2. it is not decreased for $k \ne n$ and not increased for $k=n$
3. if $\tau \to 0$, then this function tends to 1 for $k=n$ and it tends to 0 for $k \ne n$
4. $\sum_n P_{E_k,E_n}(t,t+\tau) \le 1$ for all k, t and τ.

Proof. Properties 2 and 3 result from the assumptions that $f_{n,n}(t)$ are non-positive, and $f_{k,n}$ are non-negative for $k \ne n$, and from the properties of the exponent function and integrals. Property 1 results from point 4. Therefore, only point 4 must be proved.11

For each $k \ne n$ we have:

$$P_{E_k,E_n}(t,t+\tau) \le \int_t^{t+\tau}[f_{k,n}exp(\int_t^x f_{k,k}(s)ds)]dx \tag{12}$$

So:

$$\sum_{n \ne k} P_{E_k,E_n}(t,t+\tau) \le \int_t^{t+\tau}[\sum_{n \ne k} f_{k,n}exp(\int_t^x f_{k,k}(s)ds)]dx = \tag{13}$$

$$\int_t^{t+\tau}[-f_{k,k}exp(\int_t^x f_{k,k}(s)ds)]dx = -\int_t^{t+\tau}\frac{d}{dx}[exp(\int_t^x f_{k,k}(s)ds)]dx =$$

$$-exp(\int_{t}^{t+\tau} f_{k,k}(x)dx)+1=-P_{E_k,E_k}([t,t+\tau))+1$$

According to this inequality the theorem 1 point 4 is truthful.

Definition 3 The base function $P:\mathrm{B}\to[0,1]$ is a function which for all base sets A formed by the covering $(\Delta_i)_{i=1,2,..}$ and sequence of states $(C_i)_{i=1,2,..}$ is equal to:

$$P(A)=\prod P_{C_i,C_{i+1}}(\Delta_i) \tag{14}$$

Theorem 2 The definition of base function P is good.

Proof. We must prove that for the optimal covering $(\Delta_i)_{i=1,2,..}$ and the sequence of states $(C_i)_{i=0,1,..}$ and another covering $(\Delta'_i)_{i=1,2,..}$ and the sequence of states $(C'_i)_{i=1,2,..}$ forming the same base set, we have:

$$\prod P_{C_i,C_{i+1}}(\Delta_i)=\prod P_{C'_i,C'_{i+1}}(\Delta'_i) \tag{15}$$

The same base set can be formed by $(\Delta_i)_{i=1,2,..}$ with $(C_i)_{i=1,2,..}$ and $(\Delta'_i)_{i=1,2,..}$ with $(C'_i)_{i=1,2,..}$ if for some m all $\Delta_i=\Delta'_i$ for all $i<m$ and $C_i=C'_i$ for all $i<m$ and $\Delta_m=\Delta'_m+\Delta'_{m+1}$ and $C_m=C_{m+1}=C'_m=C'_{m+1}=C'_{m+2}=E_k$. Then:

$$\begin{aligned}&P_{C'_m,C'_{m+1}}(\Delta'_i)P_{C'_{m+1},C'_{m+2}}(\Delta'_i)=P_{E_k,E_k}(\Delta'_i)P_{E_k,E_k}(\Delta'_i)=\\&exp(\int_{\Delta'_i} f_{k,k}(x)dx)exp(\int_{\Delta'_{i+1}} f_{k,k}(x)dx)=\\&exp(\int_{\Delta'_i\cup\Delta'_{i+1}} f_{k,k}(x)dx)=exp(\int_{\Delta_i} f_{k,k}(x)dx)=P_{C_m,C_{m+1}}(\Delta_m)\end{aligned} \tag{16}$$

The value of P for the base set is the same as the value for its optimal covering. So the P is a well-defined function.

Theorem 3 Base function P has the following properties:

1. $P(A)<1$.
2. If $A\subseteq B$, then $P(A)\le P(B)$.
3. If $\bigcup_{i=1}^{\infty} A_i\in\mathrm{B}$ and $A_i\cap A_j=\varnothing$ for all $i\ne j$ then $P(\bigcup_{i=1}^{\infty} A_i)=\sum_{i=1}^{\infty} P(A_i)$.

Proof. The property 1 is obvious because all $P_{E_k,E_n}(\Delta)$ are non-negatiwe and less than 1.

Let B be formed by the optimal covering $(\Delta_i^B)_{i=1,2,..}$, and the sequences of states $C^B_{i=1,2,..}$. Let $A\in\mathrm{B}$ and $A\subseteq B$. Then there exists a covering $(\Delta^A_{i_1},\Delta^A_{i_2},\Delta^A_{i_3})_{i=1,2,..}$ and the sequence of states $(C^A_{i_1},C^A_{i_2},C^A_{i_3})_{i=1,2,..}$ forming set A such that:

1. $\Delta_i^B=\Delta^A_{i_1}\cup\Delta^A_{i_2}\cup\Delta^A_{i_3}$ for all i,
2. $C^A_{i_1}=C^B_{i-1}$ and $C^A_{i_2}=C^A_{i_3}=C^B_i$ then for all $i\ge1$.

Sometimes $\Delta_i^A=\varnothing$ or $\Delta^A_{i+2}=\varnothing$.

If $C^B_{i-1}=E_k$ and $C^B_i=E_n$ and $\Delta_i^B=[t_1,t_2)$ and $\Delta^A_{i_1}=[t_1,y_1)$ and $\Delta^A_{i_3}=[y_2,t_2)$ then (if $k\ne n$):

$$P_{C_{i-1}^B, C_i^B}(\Delta_i^B) = P_{E_k, E_n}(\Delta_i^B) = \tag{17}$$

$$\int_{t_1}^{t_2}[f_{k,n}(x)exp(\int_{t_1}^{x} f_{k,k}(s)ds + \int_{x}^{t_2} f_{n,n}(s)ds)]dx =$$

$$\int_{t_1}^{y_1}[f_{k,n}(x)exp(\int_{t_1}^{x} f_{k,k}(s)ds + \int_{x}^{y_1} f_{n,n}(s) + \int_{y_1}^{t_2} f_{n,n}(s)ds)]dx +$$

$$\int_{y_1}^{y_2}[f_{k,n}(x)exp(\int_{t_1}^{y_1} f_{k,k}(s)ds + \int_{y_1}^{x} f_{k,k}(s)ds + \int_{x}^{y_2} f_{n,n}(s) + \int_{y_2}^{t_2} f_{n,n}(s)ds)]dx +$$

$$\int_{y_2}^{t_2}[f_{k,n}(x)exp(\int_{t_1}^{y_2} f_{k,k}(s)ds + \int_{y_2}^{x} f_{k,k}(s)ds + \int_{x}^{t_2} f_{n,n}(s)ds)]dx \geq$$

$$exp(\int_{t_1}^{y_1} f_{k,k}(x)dx)\cdot \int_{y_1}^{y_2}[f_{k,n}(x)exp(\int_{y_1}^{x} f_{k,k}(s)ds + \int_{x}^{y_2} f_{n,n}(s)ds)]dx \cdot exp(\int_{y_2}^{t_2} f_{n,n}(x)dx) =$$

$$= P_{C_i^A, C_{i_1}^A}(\Delta_{i_1}^A) P_{C_{i_1}^A, C_{i_2}^A}(\Delta_{i_2}^A) P_{C_{i_2}^A, C_{i_3}^A}(\Delta_{i_3}^A)$$

If $k = n$ then $P_{C_{i-1}^B, C_i^B}(\Delta_i^B) = P_{C_i^A, C_{i_1}^A}(\Delta_{i_1}^A) P_{C_{i_1}^A, C_{i_2}^A}(\Delta_{i_2}^A) P_{C_{i_2}^A, C_{i_3}^A}(\Delta_{i_3}^A)$. These formulas are true for all i, so $P(A) \leq P(B)$

Let base sets A and B are disjoint and $A \cup B \in \mathrm{B}$. We can make an assumption that the sets A and B are formed by optimal coverings $(\Delta_i^A)_{i=1,2,\cdots}$ and $(\Delta_i^B)_{i=1,2,\cdots}$, and the sequences of states $(C_i^A)_{i=1,2,\cdots}$ and $(C_i^B)_{i=1,2,\cdots}$ respectively. Then, the sets A and B are disjoint and $A \cup B$ is a base set from B if and only if:

1. $\Delta_i^A = \Delta_i^B$ for all i,
2. $C_i^A = C_i^B$ for almost all i (except any j and $j+1$)
3. Number j exists, such that Δ_j and Δ_{j+1} are adjacent, and

   $a)$ $C_{j-1}^A = C_{j-1}^B = C_j^B = E_k$ and $C_j^A = C_{j+1}^A = C_{j+1}^B = E_n$

   or

   $b)$ $C_{j-1}^A = C_j^A = C_j^B = E_k$ and $C_{j+1}^A = C_j^B = C_{j+1}^B = E_n$

for any $n \neq k$.

Cases a) and b) are symmetrical, so only a) will be proved.

Let $\Delta_i = [t_1, t_2)$ and $\Delta_{i+1} = [t_2, t_3)$. Then:

$$P_{E_k, E_n}(\Delta_i \cup \Delta_{i+1}) = \int_{t_1}^{t_3}[f_{k,n}(x)exp(\int_{t_1}^{x} f_{k,k}(s)ds + \int_{x}^{t_3} f_{n,n}(s)ds)]dx \tag{18}$$

$$\int_{t_1}^{t_2}[f_{k,n}(x)exp(\int_{t_1}^{x}f_{k,k}(s)ds+\int_{x}^{t_2}f_{n,n}(s)ds+\int_{t_2}^{t_3}f_{n,n}(s)ds)]dx+$$

$$\int_{t_1}^{t_2}[f_{k,n}(x)exp(\int_{t_1}^{t_2}f_{k,k}(s)ds+\int_{t_2}^{x}f_{k,k}(s)ds+\int_{x}^{t_3}f_{n,n}(s)ds)]dx=$$

$$\int_{t_1}^{t_2}[f_{k,n}(x)exp(\int_{t_1}^{x}f_{k,k}(s)ds+\int_{x}^{t_2}f_{n,n}(s)ds)]dx\cdot exp(\int_{t_2}^{t_3}f_{n,n}(s)ds)+$$

$$exp(\int_{t_1}^{t_2}f_{k,k}(s)ds)\cdot\int_{t_1}^{t_2}[f_{k,n}(x)exp(\int_{t_2}^{x}f_{k,k}(s)ds+\int_{x}^{t_3}f_{n,n}(s)ds)]dx=$$

$$P_{E_k,E_n}(\Delta_j)P_{E_n,E_n}(\Delta_{j+1})+P_{E_k,E_k}(\Delta_j)P_{E_k,E_n}(\Delta_{j+1})$$

so:

$$P(A\cup B)=\prod_{i\neq j,i\neq j+1}P_{C_i,C_i+1}(\Delta_i)\cdot P_{E_k,E_n}(\Delta_j\cup\Delta_{j+1})= \qquad (19)$$

$$\prod_{i\neq j,i\neq j+1}P_{C_i,C_i+1}(\Delta_i)\cdot P_{E_k,E_n}(\Delta_j)P_{E_n,E_n}(\Delta_{j+1})+\prod_{i\neq j,i\neq j+1}P_{C_i,C_i+1}(\Delta_i)\cdot P_{E_k,E_k}(\Delta_j)P_{E_k,E_n}(\Delta_{j+1})=$$

$$P(A)+P(B)$$

The same calculations can be made for countable number of the sets A_i which a state E_k transits to E_n on neighbouring intervals $[t_j,s_{j,1})$, $[s_{j,1},s_{j,2})$,... and $s_{j,n}\to t_{j+1}$.

Let $\bigcup_{m=1}^{\infty}A_m=A\in\mathrm{B}$ for disjoint base sets A_i. Let A be formed by optimal covering $(\Delta_i)_{i=1,2,\dots}$ and sequence of states $(C_i)_{i=1,2,\dots}$. Then each A_m is formed by the covering $(\Delta^m_{i,1},\Delta^m_{i,2},\Delta^m_{i,3})_{i=1,2,\dots}$ and seqence of states $(C_i,C_{i+1},C_{i+1})_{i=1,2,\dots}$. For each interval Δ_i the sets A_i can be choosen, wchich intervals of transition C_{i-1} to C_i are adjacent. For two different intervals Δ_i:

1. $\Delta^m_{i,2}=\Delta_i$ for all m, or
2. $\Delta^m_{i,2}$ are disjoint and $\bigcup_m\Delta^m_{i,2}=\Delta_i$.

For all possible sequences of coverings $\Delta^{m_1}_{1,2},\Delta^{m_2}_{2,2},\Delta^{m_3}_{3,2},\cdots$ there exists A_m such that $\Delta^{m_i}_{i,2}=\Delta^m_{i,2}$. Otherwise there would be a realization which would belong to A and would not belong to $\bigcup_m A_m$. It means that:

$$P(A)=\prod_{i=1}^{\infty}\sum_m P_{k_i,k_i}(\Delta^m_{i,1})P_{k_i,n_i}(\Delta^m_{i,2})P_{n_i,n_i}(\Delta^m_{i,3})= \qquad (20)$$

$$\sum_m\prod_{i=1}^{\infty}P_{k_i,k_i}(\Delta^m_{i,1})P_{k_i,n_i}(\Delta^m_{i,2})P_{n_i,n_i}(\Delta^m_{i,3})=\sum_m P(A_m)$$

where $C_i=E_{k_i}$ and $C_{i+1}=E_{n_i}$.

Basis B and base function P satisfy all of the assumption of Caratheodory's theorem [8], [10], [2], [4]. It means that function $P^*: 2^{\Omega} \to [0,\infty)$ such that:

$$P^*(D) = inf\{\sum_{i=1}^{\infty} P(A_i);\ A_i \in \mathrm{B}, D \subseteq \bigcup_{i=1}^{\infty} A_i\} \tag{21}$$

is an outer measure. A set of subsets satisfaying Caratheodory condition:

$$\overline{\mathrm{B}} = \{D \in 2^{\Omega};\ \forall_{A \subseteq \Omega} P^*(A) = P^*(A \cap D) + P^*(A - D)\}$$

is a σ-algebra and restriction P^* to $\overline{\mathrm{B}}$ is a measure. Moreother for base set $A \in \mathrm{B}$: $P^*(A) = P(A)$.

The set $\overline{\mathrm{B}}$ is a σ-algebra for the created probability space. Measure $P^*: \overline{\mathrm{B}} \to [0,\infty)$ will be noted $\overline{P}$

4.4 Measure $\overline{P}$ is the probability

$\overline{P}$ satisfies all properties of measures but we don,t know if it is probability, i.e. $\overline{P}(\Omega) = 1$. Base set B doesn,t include the set of all realizations Ω. It isn't a sum of finite or countable numbers of sets from B. A simple proof that $\overline{P}(\Omega) = 1$ doesn't exist.

Theorem 4 Let Ω_{∞} be a set of all realizations in Ω which have an infinite number of changes of states from C_i to C_{i+1} $(C_i \neq C_{i+1})$ during time $[0,T)$ (they don't finish before time T). Then $\overline{P}(\Omega) = 0$.

Proof. A realization from Ω has an infinite number of changes of states if the sequence of moments of this change has a focusing point. For each such realization we can find a covering $([w_{i-1}, w_i))_{i=1,2,..}$ such that $w_i \in \mathrm{Q}$ (set of rational numbers) and in each interval this realization changes the states. Let $\delta_i = w_i - w_{i-1}$. For a state from Ω_{∞} the product $\prod_{i=1}^{\infty} \delta_i = 0$ and for any finite number M also $\prod_{i=1}^{\infty} M\delta_i = 0$.

Let A be a set from B formed by covering $([w_{i-1}, w_i))_{i=1,2,..}$ and sequence of states $(C_i)_{i=1,2,..}$. Let M be a maximum value of all functions $-f_{k,k}(t)$ on interval $[0,T]$. Such M exists due to assumption (9). M is greater than the maximum value of each function $f_{k,n}$ on the interval $[0,T)$ because for $k \neq n$ $f_{k,n}(t) > 0$ and $\sum_{n \neq k} f_{k,n}(t) = -f_{k,k}(t) < -M$.

According to theorem 1 (point 4):

$$\sum_{n \neq k} P_{E_k,E_n}([w_{i-1}, w_i)) \le 1 - P_{E_k,E_n}([w_{i-1}, w_i)) = \tag{22}$$

$$1 - exp(-\int_{w_{i-1}}^{w_i} f_{k,k}(t)dt) \le 1 - exp(-M\delta_i) \le M\delta_i$$

So:

$$P(A) \le \prod_{i=1}^{\infty} M\delta_i = 0 \tag{23}$$

The countable sum of all sets from B formed by all coverings $([w_{i-1}, w_i))_{i=1,2,..}$ is also equal 0.

$\overline{P}$ is a measure so $\overline{P}(\Omega - \Omega_{\infty}) = \overline{P}(\Omega) - \overline{P}(\Omega_{\infty}) = \overline{P}(\Omega)$. The set $\Omega - \Omega_{\infty}$ will be noted Ω_0.

Theorem 5 $\overline{P}(\Omega_0) \le 1$

Proof. Let m be a natural number and $\delta_m = \frac{T}{2^m}$. Let Ψ_m be a sum of all base sets formed by covering $[0,\delta_m),[\delta_m,2\delta_m),...,[(2^m-1)\delta_m,T)$ (for realizations finishing at time S: $[0,\delta_m),[\delta_m,2\delta_m),...,[j\delta_m,S)$ where j is a maximal integer which $j\delta_m < S$) and different sequences of states. Each realization which has a finite number of changes of states belongs to any set from any Ψ_m.

Because $\Psi_1 \subseteq \Psi_2 \subseteq \cdots$ and $\bigcup_{m=1}^{\infty} \Psi_m = \Omega_0$ So:

$$\overline{P}(\Omega_0) = \lim_{m\to\infty} \overline{P}(\Psi_m) \tag{24}$$

All realizations formed by covering $[0,\delta_m),[\delta_m,2\delta_m),...,[(2^m-1)\delta_m,S)$ can be noted as a chain of states. For instance $C_0 \to C_1 \to \cdots \to C_S$ is a set of all realization which have a state C_i at time $i\delta_m$ and the state C_i is changed to C_{i+1} in time $[i\delta_m,(i+1)\delta_m)$. We will note as $C_0 \to C_1 \to \cdots \to C_i \Rightarrow$ a set of all realization which have the same sequence of states to time $i\delta_m$ and after this time they have all possible chains of states.

$$\overline{P}(C_0 \to C_1 \to \cdots \to C_{2^m-1} \Rightarrow) = \tag{25}$$

$$\overline{P}(\bigcup_{n=0}^{\infty} C_0 \to C_1 \to \cdots \to C_{2^m-1} \to E_n) =$$

$$\prod_{i=0}^{2^m-2} P_{C_i,Ci+1}(i\delta_m,(i+1)\delta_m)(\sum_{n=0}^{\infty} P_{C_{2^m-1},E_n}((2^m-1)i\delta_m,2^m\delta_m)) \le$$

$$\prod_{i=0}^{2^m-2} P_{C_i,Ci+1}(i\delta_m,(i+1)\delta_m)$$

according to theorem 1, point 4. Using mathematical induction we can prove that for any j:

$$\overline{P}(C_0 \to C_1 \to \cdots \to C_j \Rightarrow) \le \prod_{i=0}^{j-1} P_{C_i,Ci+1}(i\delta_m,(i+1)\delta_m) \tag{26}$$

In the end:

$$\overline{P}(C_0 \Rightarrow) \le \sum_{n=0}^{\infty} P_{C_0,E_n}(0,\delta_m) \le 1 \tag{27}$$

But $\{C_0 \Rightarrow 0\} = \Psi_m$ so for any m $\overline{P}(\Psi_m) \le 1$. It means that $\lim_{m\to\infty} \overline{P}(\Psi_m) \le 1$. Due to (24) $\overline{P}(\Omega_0) \le 1$.

Theorem 6 $\overline{P}(\Omega_0) \ge 1$

Proof. As in proof of theorem 5 we mark that $\delta_m = \frac{T}{2^m}$ for any number m, Ψ_m is a sum of all base sets formed by covering $[0,\delta_m),[\delta_m,2\delta_m),...,[(2^m-1)\delta_m,T)$. M be a finite number that for any n: $f_{n,n}(t) \ge -M$. Such number exists due to assumption (9). The proof of theorem 4 shows that M greater than all $f_{k,n}(t)$ for $t \in [0,T]$ and $k \ne n$. So:

$$P_{E_k,E_n}(i\delta_m,(i+1)\delta_m) = \tag{28}$$

$$\int_{i\delta_m}^{(i+1)\delta_m} f_{k,n}(x)exp(\int_{i\delta_m}^{x} f_{k,k}(s)ds + \int_{x}^{(i+1)\delta_m} f_{k,k}(s)ds)dx \geq$$

$$\int_{i\delta_m}^{(i+1)\delta_m} f_{k,n}(x)exp(\int_{i\delta_m}^{x} f_{k,k}(s)ds)exp(-M[(i+1)\delta_m - x])dx \geq$$

$$exp(-M\delta_m)\int_{i\delta_m}^{(i+1)\delta_m} f_{k,n}(x)exp(\int_{i\delta_m}^{x} f_{k,k}(s)ds)dx$$

because $exp(-M(i+1)\delta_m - Mx) \geq exp(-M\delta_m)$ for all $x \in [i\delta_m,(i+1)\delta_m)$.

Then:

$$\sum_{n\neq k} P_{E_k,E_n}(i\delta_m,(i+1)\delta_m) \geq \tag{29}$$

$$exp(-M\delta_m)\int_{i\delta_m}^{(i+1)\delta_m} \sum_{k\neq n} f_{k,n}(x)exp(\int_{i\delta_m}^{x} f_{k,k}(s)ds)dx =$$

$$-exp(-M\delta_m)\int_{i\delta_m}^{(i+1)\delta_m} f_{k,k}(x)exp(\int_{i\delta_m}^{x} f_{k,k}(s)ds)dx =$$

$$-exp(-M\delta_m)\int_{i\delta_m}^{(i+1)\delta_m} \frac{d}{dx}exp(\int_{i\delta_m}^{x} f_{k,k}(s)ds)dx =$$

$$exp(-M\delta_m)[1-exp(\int_{i\delta_m}^{(i+1)\delta_m} f_{k,k}(s)ds)]$$

and:

$$\sum_{n} P_{E_k,E_n}(i\delta_m,(i+1)\delta_m) \geq \tag{30}$$

$$exp(-M\delta_m)[1-exp(\int_{i\delta_m}^{(i+1)\delta_m} f_{k,k}(s)ds)] + exp(\int_{i\delta_m}^{(i+1)\delta_m} f_{k,k}(s)ds) =$$

$$exp(-M\delta_m) + [1-exp(-M\delta_m)]exp(\int_{i\delta_m}^{(i+1)\delta_m} f_{k,k}(s)ds) \geq$$

$$exp(-M\delta_m) + [1-exp(-M\delta_m)]exp(-M\delta_m) = 2exp(-M\delta_m) - exp(-2M\delta_m)$$

Above inequolities are true if all realizations, also finishing before time T due to last state E_L for $L > K$, are taken into account.

Function $2z - z^2$ increase in interval $[-\infty,1]$ and always $1 \geq exp(-x) \geq 1-x$. So:

$$2exp(-M\delta_m) - exp(-2M\delta_m) \geq 2(1-M\delta_m) - (1-M\delta_m)^2 = 1-(M\delta_m)^2 \tag{31}$$

The finished inequality has the following form:

$$\sum_n P_{E_k,E_n}(i\delta_m,(i+1)\delta_m) \geq 1-(M\delta_m)^2 \tag{32}$$

Using the same notation for $C_0 \to C_1 \to \cdots \to C_i \Rightarrow$ as in proof of theorem 5, we can note:

$$\overline{P}(C_0 \to C_1 \to \cdots \to C_{2^m-1} \Rightarrow) = \overline{P}(\bigcup_n C_0 \to C_1 \to \cdots \to C_{2^m-1} \to E_n) = \tag{33}$$

$$\prod_{i=0}^{2^m-2} P_{C_i,C_{i+1}}(i\delta_m,(i+1)\delta_m)(\sum_n P_{C_{2^m-1},E_n}((2^m-1)\delta_m, 2^m\delta_m)) \geq$$

$$(1-(M\delta_m)^2)\prod_{i=0}^{2^m-2} P_{C_i,C_{i+1}}(i\delta_m,(i+1)\delta_m)$$

Using mathematical induction we can prove that for any j

$$\overline{P}(C_0 \to C_1 \to \cdots \to C_j \Rightarrow) \geq (1-(M\delta_m)^2)^{2^m-j}\prod_{i=0}^{j-1} P_{C_i,C_{i+1}}(i\delta_m,(i+1)\delta_m) \tag{34}$$

In the end:

$$\overline{P}(\Psi_m) = P(C_0 \Rightarrow) \geq (1-(M\delta_m)^2)^{2^m} \tag{35}$$

Because of: $\delta_m = \dfrac{T}{2^m}$:

$$\overline{P}(\Psi_m) \geq (1-\frac{(MT)^2}{2^{2m}})^{2^m} \tag{36}$$

This inequality is true for all k, so:

$$\overline{P}(\Omega_0) = \lim_{m\to\infty}\overline{P}(\Psi_m) \geq \lim_{m\to\infty}(1-\frac{(MT)^2}{2^{2m}})^{2^m} = exp(-\lim_{m\to\infty}\frac{(MT)^2}{2^m}) = 1 \tag{37}$$

Theorems 4, 5 and 6 have shown that measure $\overline{P}$ is the probability for the half-infinite matrix $[f_{k,n}(t)]_{k,n}$ and finite time T. The finiteness of K is significant in both proofs 4 and 6 (the existence of $M < \infty$ that is grather than all $f_{k,n}(t)$).

5 The construction of probability space for the infinite matrix or/and $T = \infty$

A set of events Ω includes all the functions $\varphi:[0,T) \to W$ which are constant on intervals $[t_i, t_{i+1})$. The T can be equal to infinity, but for each $T \leq \infty$ the Ω consists also all step functions $\varphi:[0,T) \to W$ such that $\lim_{t\to T}\varphi(t) = E_\infty$ or such limits exist in the subset of values of φ. Such function will be called realizations and will be noted $C_0 \to C_{t_1} \to C_{t_2} \to \ldots$, where $C_0, C_{t_1}, C_{t_2}\ldots$ are succesive states in realizations, $t_1, t_2, \ldots$ are times of transition the state C_0 to C_1, C_1 to C_2, $\cdots$.

For any $K < \infty$ and $T < \infty$ let Ω_T^K be a set of realizations formed by the half-finite submatrix $[f_{k,n}(t)]_{k=0,1,\cdots,K,n=0,1,\ldots\infty}$ of the given infinite matrix and finite time T. Between Ω and Ω_T^K there is a natural surjection that every step function φ assigns the same function truncated to the interval [0, T), or/and states from the set $\{E_0, E_1, \cdots, E_K\}$.

Let $g:\Omega \to \Omega_T^K$ be a function such that:

1. $g(C_0 \to C_{t_1} \to C_{t_2} \to ...) = C_0 \to C_{t_1} \to C_{t_2} \to ... \to C_S$ where $S \leq T$ and $t_{z+1} > T$ if all $C_0, C_{t_1}, C_{t_2}, ..., C_S$ belongs to set $\{E_0, E_1, \cdots, E_K\}$

2. $g(C_0 \to C_{t_1} \to C_{t_2} \to ...) = C_0 \to C_{t_1} \to C_{t_2} \to ... \to C_S$ if all $C_0, C_{t_1}, C_{t_2}, ..., C_{S-1}$ belongs to set $\{E_0, E_1, \cdots, E_K\}$ and $C_S \in \{E_{K+1}, E_{K+2}, ...\}$ and $S < T$.

Probabilitic structure on Ω_T^K showed in subsections 4.1-4.4 will be noted

$(\Omega_T^K, \sigma(\Omega_T^K), \overline{P}_T^K)$. We can transfer it to 2^Ω using function g.

Definition 4 Probabilistic space of K and T limits on set of events Ω is a space $(\Omega, \overleftarrow{\sigma(\Omega_T^K)}, \overleftarrow{\overline{P}}_T^K)$ where:

1. $\overleftarrow{\sigma(\Omega_T^K)} = \{g^{-1}(A); A \in \sigma(\Omega_T^K)\}$

2. $\overleftarrow{\overline{P}}_T^K(g^{-1}(A)) = \overline{P}_T^K(A)$

The base set in probabilistic space of K and T limits is formed by coverings of $[0, Min\{S,T\})$ and the sequence of states $C_0, C_{t_1}, C_{t_2} ... C_S$ and after the time $Min\{S,T\}$ the realizations in the base set can be run in all possible ways (Fig.3). Probability of this set is equal to:

$$\overleftarrow{\overline{P}}_T^K(g^{-1}(A)) = \prod_{t_i < Min\{S,T\}} P_{E_k, E_n}(t_{i-1}, t_i)$$

and states occurring after the time $Min\{S,T\}$ are omitted.

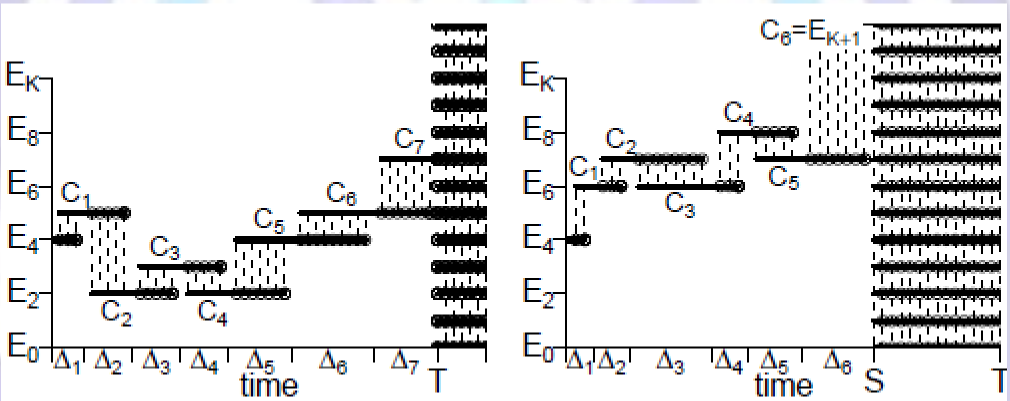

Figure 3: Realizations included in the same base set of K and T limits formed by sequence of intervals $(\Delta_i)_{i=1,2,...}$ and sequence of states $(C_i)_{i=1,2,...}$.

For different K and T the sequence of probability spaces $(\Omega, \overleftarrow{\sigma(\Omega_T^K)}, \overleftarrow{\overline{P}}_T^K)$ satisfy the following conditions:

1. if $K < N$ or/and $T < S$ then $\overleftarrow{\sigma(\Omega_T^K)} \subseteq \overleftarrow{\sigma(\Omega_S^N)}$,

2. if $K < N$ or/and $T < S$ then $\overleftarrow{\overline{P}}_T^K(A) = \overleftarrow{\overline{P}}_S^N(A)$ for all $A \in \overleftarrow{\sigma(\Omega_T^K)}$.

Definition 5 A σ-algebra on Ω is a set:

$$\sigma(\Omega) = \bigcup_{T>0} \bigcup_{K>0} \overleftarrow{\sigma(\Omega_T^K)}$$

Probability on the $\sigma(\Omega)$ is a function:

$$\overline{P}(A)=\overline{P}_T^K(A) \text{ if } A\in\sigma(\Omega_T^K)$$

where $(\Omega,\overleftarrow{\sigma(\Omega_T^K)},\overleftarrow{\overline{P}_T^K})$ is a probability space of K and T limits.

$(\Omega,\sigma(\Omega),\overline{P})$ is a probability space created by infinite matrices of functions $f_{k,n}(t)$ and $t\in[0,\infty)$. This probability space has following property: if $A\subseteq\Omega$ such that $A=\{\varphi;\varphi(t)=E_k\}$ for any k and t. Then time t can be treated as $t=0$ in probability space on set $\Omega_{(t,\infty)}=\{\varphi|_{[t,\infty)};\varphi(t)=E_k\}$, where $\varphi|_{[t,\infty)}$ is a restriction φ to $[t,\infty)$. The construction of the probability space on this set is the same. It will be noted: $(\Omega_{[t,\infty)},\sigma(\Omega_{[t,\infty)}),\overline{P}_{[t,\infty)})$. This probability space limited by time T only will be noted: $(\Omega_{[t,T)},\sigma(\Omega_{[t,T)}),\overline{P}_{[t,T)})$.

For all sets $A=\{\varphi;\varphi(t_1)=E_{k_1},\varphi(t_2)=E_{k_2},\cdots\varphi(S)=E_{k_z}\}$ is true:

$$\overline{P}(A)=\overline{P}_{[0,t_1]}(A|_{[0,t_1)})\overline{P}_{[t_1,t_2]}(A|_{[t_1,t_2)})\cdots\overline{P}_{[S-1,S]}(A|_{[S-1,S]}) \tag{38}$$

where $A|_{[t_i,t_j)}$ is a set of restricting realizations.

6 Stochastic process defined by the probability space

$(\Omega,\sigma(\Omega),\overline{P})$ is a probability space of events, which changed in time. But if we put: $X_t:\Omega\ni\varphi\to\varphi(t)\in W$ then X_t is a random variable and $(X_t)_{t\in[0,T)}$ is a stochastic process with probability space $(\Omega,\sigma(\Omega),\overline{P})$. This is a stochastic process defined by matrix $[f_{k,n}(t)]_{k,n}$.

Stochastic process defined by matrix $[f_{k,n}(t)]_{k,n}$ has Markov property. This is a simple conclusion of equation (38).

At the end, the last theorem will be proved.

Theorem 7 For any k, n and t the instantaneous probability rate of stochastic process defined by matrix $[f_{k,n}(t)]_{k,n}$ are equal to $P_{k,n}'(t)=f_{k,n}(t)$

Proof. $\overline{P}\{X_{t+\tau}=E_n\,|\,X_t=E_k\}=\overline{P}\{X_t=E_k \text{ and } X_{t+\tau}=E_n\}/\overline{P}\{X_t=E_k\}=$

$$\overline{P}_{[0,t)}\{X_t=E_k\}\overline{P}_{[t,t+\tau)}\{X_t=E_k \text{ and } X_{t+\tau}=E_n\}/\overline{P}_{[0,t)}\{X_t=E_k\}=$$

$$\overline{P}_{[t,t+\tau)}\{X_t=E_k \text{ and } X_{t+\tau}=E_n\}$$

Let A_1 be a set of realizations form $\Omega_{[t,t+\tau)}$ such that a state E_k transits to E_n during time $[t,t+\tau)$. Let A_2 be a set of realizations such thet E_k transits to any state and this state transits to E_n during time $[t,t+\tau)$. Generally A_i is a set of realizations statrting from E_k and finishing to E_n after i transitions.

$$\{X_t=E_k \text{ and } X_{t+\tau}=E_n\}=\bigcup_i A_i \tag{39}$$

The sets A_i are disjoint, so:

$$\overline{P}\{X_t=E_k and X_{t+\tau}=E_n\}=\sum_i\overline{P}(A_i) \tag{40}$$

It show that:

$$P_{k,n}(t,t+\tau)=\sum_i \overline{P}(A_i) \tag{41}$$

But for $i \geq 2$ it can be proved that $\lim_{\tau\to 0}\overline{P}(A_i)=0$ (this is a derivation of product of equations converging to 0). So for $k \neq n$ the function $\Psi^t_{k,n}(\tau)=P_{k,n}(t,t+\tau)$ has the derivation in 0^+ equal to:

$$\Psi^t_{k,n}{}'(0^+)=[\int_t^{t+\tau} f_{k,n}(x)exp(\int_t^x f_{k,k}(s)ds+\int_x^{t+\tau} f_{n,n}(s)ds)dx]'(0^+)= \tag{42}$$

$$=f_{k,n}(t)exp(\int_t^t f_{k,k}(s)ds+\int_t^t f_{n,n}(s)ds)=f_{k,n}(t)$$

Similarly:

$$\Psi^t_{k,k}{}'(0^+)=[exp(\int_t^{t+\tau} f_{k,k}(x)dx)]'(0^+)= \tag{43}$$

$$=[\int_t^{t+\tau} f_{k,k}(x)dx]'(0^+)exp(\int_t^t f_{k,k}(x)dx)=f_{k,k}(t)$$

7 Conclusion

The true value of the work is to describe the detailed construction of a probability space in which the events are realizations of the stochastic process. Equations (11) and (14) allow calculating the probability of occurrence of a given type of realization of a stochastic process during the simulation. It is a skill comparable to the possibility of calculating the volume of a cube, what allows calculation of the volume of other geometric solids.

Probability space exists for all discrete stochastic processes with Markov property. It exists also for processes $(X_t)_{t\in[0,T)}$ which are calculating for processes $(Y_t)_{t\in[0,T)}$ with Markov property by formula $X_t=F(Y_t)$ where F is a some function. For instance, the events of $(Y_t)_{t\in[0,T)}$ maybe a form $E=(n_1,n_2,\cdots,n_k)$ and function F may have a formula $F(E)=\sum_{i=1}^k n_i$. Process $(X_t)_{t\in[0,T)}$ may not have Markov property but it has the same probability space than $(Y_t)_{t\in[0,T)}$. The class of these processes is very broad.

Continuous time-discrete value stochastic processes are often defined with the use of the Kolmogorov equations. The relation between the system of Kolmogorov equations and the matrix of instantaneous probability rates $f_{k,n}(t)$ is very easy: if

$$\Phi_n:[0.T)\ni t\to\overline{P}\{X_t=E_n\}\in[0,1],$$

then:

$$\frac{d\Phi_n}{dt}=\sum_{k=0}^{\infty} f_{k,n}(t)\Phi_k(t).$$

After reading this article it is easy to understand why sometimes $\sum_n \Phi_n<1$. Function $\Phi(t)$ can be defined only for realization surviving to time t. The Kolmogorov equations are defined only on the part of events belonging to Ω. For all t the set of realizations which interrupt or approach the infinity before time t is infinite. For some matrices $[f_{k,n}(t)]_{k,n}$ the probability of this set is greater than 0 and equal to $1-\sum_n \Phi_n(t)$.

The existence of Kolmogorov equations isn't sufficient for the existence of probability of continuous time Markov stochastic processed of discrete values. Just consider the matrix of the functions $[f_{k,n}(t)]_{k,n}$ non-differentiable at any point.

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