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ON CJ-TOPLOGICAL SPACES

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Abstract. In this paper we introduced new types of spaces as CJ-space and strong CJ-space, also we studied the relationship between them and the relation of them with J-space and strong J-space researched by E.Michael.

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Introduction and preliminaries

In 2000 E.Michael [1] introduced the concepts of J-space and strong J-space. A space X is J-space if, whenever {A, B} is a closed cover of X with A⋂B compact, then A or B is compact. A space X is strong J-space if every compact K⊂X is contained in a compact L⊂X with X\L is connected. Every strong J-space is J-space, but the converse is not true in general [1]. There is a common generalization of countably-compact space and J-space, this generalization helped us to define new concepts which are CJ-spaces and strong CJ-spaces.

CJ- Spaces and Strong CJ-space

Definition 1.1: A topological space(X,τ)is a CJ- space if, whenever {A, B} is a closed cover of X with A⋂ B countably compact, then A or B is countably compact.

Definition 1.2: A topological space (X,τ)is a strong CJ-space if every countably compact K⊂ X is contained in a countably compact L⊂X with X \ L is connected.

Remark 1.3: Every finite space is a CJ-space.

Proposition 1.4: Every countably compact space is a strong CJ-space.

Proof: Let X be a countably compact space and let K⊂X be a countably compact, then X is a countably compact with K⊂X and X \ X= ∅ is connected.

Remark 1.5: The converse of Proposition (1.4) is not true in general.For example;Let us take $\mathbb{R}^+$ as a subspace of ℝ with the usual topology which is strong CJ-space, but not countably compact.

Proposition1.6: Every strong CJ-space is a CJ-space.

Proof: Let X be a strong CJ- space and let {A,B} be a closed cover of X with A⋂B countably compact, so there exists a countably compact L⊂X such that A⋂B⊂L and X\ L is connected. Therefore {A⋂ X\L, B⋂X\ L} is a disjoint closed cover of X\ L, but X\L is connected, so X\ L must be in A⋂X\L or in B⋂X\L, it follows that X\L⊂ A or X\L⊂B. By complementation we have $A^c \subset L$ or $B^c \subset L$, and since A⋂B⊂ L, so A⊂ L or B⊂ L. Thus A or B is countably compact. Hence X is CJ-space.

Remark 1.7: The converse of proposition (1.6) is not true in general.For example;Consider the Odd-Even topology defined on the set of natural numbers ℕ. This topology is generated by the partition P={{2k-1,2k}; k∊ℕ}. The only countably compact subsets of ℕ are the finite subsets, so if we take a closed cover {A,B} of ℕ with A⋂B countably compact, that is mean A⋂B is finite set and since the intersection of any two infinite sets in this space must be an infinite set, so A or B must be finite, that is mean A or B is countably compact. Hence ℕ is CJ-space.

Butℕ is not strong CJ-space since every countably compact subset of ℕ is finite and hence its complement is infinite and every infinite subset of ℕ is disconnected.

Proposition1.8: Every countably compact space is CJ-space.

Proof: Follows from Propositions (1.4) and (1.6).

Remark 1.9: The converse of Proposition (1.8) is not true in general.For example;Let us take the set of real numbers with the particular point topology τ, such that $\tau = \{U \subseteq \mathbb{R} | 0 \in U \text{ or } U = \emptyset\}$.Then every closed cover of ℝ must contain ℝ. Let us take {ℝ, A} as a closed cover of ℝ with ℝ ∩ A countably compact, but ℝ ∩ A= A, so A is countably compact. Hence (ℝ, τ) is CJ- space.

But this space is not countably compact, since the open cover $C = \{(-n, n); n \in \mathbb{N}\}$ of ℝ has no finite open subcover.

Proposition1.10: Every topological linear space X≠ℝ is a strong CJ-space.

Proof: Let X be any topological linear space and let K⊂X be acountably compact and let L={αx: α∊[0, 1] and x∊K}, then K⊂L and L is a countably compact subset of X with X\ L is connected.

Theorem 1.11: In any metric space (X, d), the following concepts are equivalent:

i. X is CJ-space.
ii. X is strong CJ-space.
iii. X is J-space.
iv. X is strong J-space.

Proof: Follows from the fact, which token from [2], that if X is any metric space then the following statements are equivalent:

i. X is compact.
ii. X is countably compact.

Lemma1.12: If B is a closed non-countably compact subset of any topological space X and C⊂ B is countably compact, then there is a closed non-countably compact D⊂ B with D∩C = Ø.

Proof :Let $\mathfrak{A}$ be a countably open cover of B with no finite subcover, and let C⊂ B be a countably compact, then $\mathfrak{A}$ is a countable open cover of C. Pick a finite $\mathfrak{J} \subset \mathfrak{A}$ covering C. Then $D = B \setminus \cup \mathfrak{J}$is a closed non–countably compact subset of B with D ∩ C = Ø.

Theorem1.13: Let X be any topological space, then the following conditions are equivalent:

i. X is a CJ-space.
ii. For any A⊂X with countably compact boundary, cl(A) or cl(X\A) is countably compact.
iii. If A and B are disjoint closed subsets of X with ∂A or ∂B countably compact, then A or B is countably compact.
iv. If K⊂ X is countably compact, and if $\mathcal{W}$ is a disjoint open cover of X\ K, then X\W is countably compact for someW ∊ $\mathcal{W}$.
v. Same as (iv), but with card $\mathcal{W} = 2$.

Proof: (i)⟹ (ii)**:** Let A⊂X such that ∂A is countably compact , but ∂A=cl(A)⋂cl(X\A) ,now we have a closed cover {cl(A),cl(X\A)} of X with cl(A)⋂cl(X\A) is countably compact and X is CJ-space , so cl(A) or cl(X\A) is countably compact.

(ii)⟹ (iii): Let A and B be disjoint closed subsets of X with ∂A is countably compact. By (ii) we can get cl(A) or cl(X\A) is countably compact. But cl(A) =A, so A or cl(X\A) iscountably compact and since B⊂ cl(X\A), so A or B is countably compact (since B is closed and cl(X\A) is countably compact).

(iii) ⟹ (i): Let {A, B} be a closed cover of X with A⋂B iscountably compact. Suppose that B is non-countably compact and since A⋂B⊂B is countably compact, so by Lemma (1.12) there is a closed non-countably compact D⊂ B such that D⋂(A⋂B)=Ø, it follows that D⋂A=Ø. Thus A and D are disjoint closed subsets of X and ∂A is a closed subset of A⋂B and thus countablycompact. By (iii) A or D is countably compact, but D is non-countably compact. Hence A must be countably compact.

(iv) ⟹ (v): Clear.

(v) ⟹ (iv): Let K⊂X be a countably compact and let $\mathcal{W}$ be a disjoint open cover of X \ K. To show that X\W is countably compact for some W ∊ $\mathcal{W}$ we shall follow three demarches.

First, we prove that if U is open subset of X containing K, then $\mathcal{W}' = \{W \in \mathcal{W} : W \nsubseteq U\}$ is finite. Suppose that it is not finite, then $\mathcal{W} = W_1 \cup W_2$ with $W_1 \cap W_2 = \emptyset$ and $W_1 \cap \mathcal{W}'$ and $W_2 \cap \mathcal{W}'$ both finite.

Let $V_1 = \cup W_1$ and $V_2 = \cup W_2$, then $\{V_1, V_2\}$ is a disjoint open cover of X\K, so by (iv) $X \backslash V_1$ or $X \backslash V_2$ is countably compact, but $V_1 \subseteq X \backslash V_2$ and $V_2 \subseteq X \backslash V_1$ since V_1 and V_2 are disjoint. It follows that $cl(V_1) \subseteq cl\,(X \backslash V_2) = X \backslash V_2$ and $cl(V_2) \subseteq cl\,(X \backslash V_1) = X \backslash V_1$, so we get $cl(V_1)$ or $cl(V_2)$ is countably compact (since closed subset of countably compact is countably compact).

Suppose that $cl(V_1)$ is countably compact, then $C = cl(V_1) \backslash U$ is countably compact. Now let $\mathcal{W}'_1 = W_1 \cap \mathcal{W}'$, then $\mathcal{W}_1'$ covers C and each $W \in \mathcal{W}'$ intersects C, so C is not countably compact since $\mathcal{W}_1'$ is infinite and disjoint, which is a contradiction. Hence $\mathcal{W}'$ is finite.

Second, we show that if cl(W) is countably compact for all $W \in \mathcal{W}$, then X is countably compact. Let $\mathcal{V}$ be a countably open cover of X, then $\mathcal{V}$ is a countably open cover of K, which is countably compact, so $\mathcal{V}$ has a finite subcover $\mathcal{F}$ covers K. Let $U = \bigcup \mathcal{F}$, by step one we get a finite family $\mathcal{W}' = \{W \in \mathcal{W} : W \nsubseteq U\}$, so $\bigcup \{cl(W) : W \in \mathcal{W}'\}$ is countably compact and since $\mathcal{V}$ is an open cover of it therefore it is covered by some finite $\mathcal{E} \subset \mathcal{V}$. But $\bigcup \mathcal{E} \subset \mathcal{V}$ is finite and covers X, so X is countably compact.

Finally, let us show that X\W is countably compact for some $W \in \boldsymbol{\mathcal{W}}$. If cl(W) is countably compact for all $W \in \boldsymbol{\mathcal{W}}$, then X is countably compact by step (ii) and since X\W is a closed subset of X, so X\W is countably compact. Suppose that there exists $W_0 \in \mathcal{W}$ such that $cl(W_0)$ is not countably compact.

Let $W^* = \cup \{W \in \mathcal{W} : W \neq W_0\}$, then, $\{W_0, W^*\}$ is a disjoint open cover of $X \backslash K$, so $X \backslash W_0$ or $X \backslash W^*$ is countably compact, by (v). If $X \backslash W^*$ is countably compact, and since $cl(W_0)$ is a closed subset of $X \backslash W^*$, so $cl(W_0)$ is countably compact which is a contradiction, so $X \backslash W^*$ is not countably compact, it follows that $X \backslash W_0$ is countably compact .

(v)$\Longrightarrow$ (i): Let {A,B} be a closed cover of X with A⋂B countably compact, then {X\A, X\B} is a disjoint open cover of X\A⋂B, then by (v) X\(X\A) or X\(X\B) is countably compact, that is A or B is countably compact. Hence X is CJ-space.

(i)$\Longrightarrow$ (v): Let K be a countably compact subset of X and let $\{W_1, W_2\}$ be a disjoint open cover of X\K, then $\{X \backslash W_1, X \backslash W_2\}$ is a closed cover of X with $X \backslash W_1 \cap X \backslash W_2 = X \backslash (W_1 \cup W_2)$ countably compact, since $X \backslash K \subset W_1 \cup W_2$, and so $X \backslash (W_1 \cup W_2) \subset K$ which is countably compact and by the closed subset of countably compact space is countably compact. But X is CJ-space, so $X \backslash W_1$ or $X \backslash W_2$ is countably compact.

Theorem1.14: A locally connected space X is CJ-space if and only if it is a strong CJ-space.

Proof: By Proposition (1.6), every strong CJ-space is a CJ-space. Suppose, then, X is CJ-space. To show that X is strong CJ-space, let K⊂X be countably compact. Since X is locally connected, there is a disjoint open cover $\mathcal{W}$ of X\K with each $W \in \mathcal{W}$ connected. By Theorem (1.13), there is $W_0 \in \mathcal{W}$ such that $X \backslash W_0$ is countably compact. Letting L= $X \backslash W_0$, then L is countably compact containing K and X\L is connected.

Remark1.15: A closed subset of CJ-space need not be CJ-space. For example; let us take the same topological space $\mathbb{R}$ in the example of Remark (1.9), this space is CJ-space, but the closed subspace $\mathbb{N}$ with the induced topology, which is the discrete topology, is not CJ-space since $\{O^+, E^+\}$ is a closed cover of $\mathbb{N}$ with $O^+ \cap E^+ = \emptyset$ is countably compact, but neither O^+ nor E^+ is countablycompact.

This remark shows that the property of being CJ-space is not hereditary property.

Proposition1.16: A clopen subset of a CJ- space is CJ-space.

Proof: Let X be any CJ-space and let W be a clopen subspace of it, we have to show that W is CJ-space, let {A,B} be a closed cover of W with A⋂B countably compact, then {A⋃X\W,B} is a closed cover of X with (A⋃X\W)⋂B = A⋂B countably compact, so A⋃X\W or B is countably compact. If B is countably compact, then we have got the proof. If B is not countably compact, then A⋃X\W is countably compact, but A is a closed subset of A⋃X\W, so A is countably compact and hence W is CJ-space.

2. Functional Characterizations of CJ-Spaces

Remark2.1: The continuous image of CJ-space is not CJ-space in general.For example;Let $f:(\mathbb{N},\tau)\to(\mathbb{N},\tau')$such that $f(2k)=f(2k-1)=k$; $k\in\mathbb{N}$, where τ is the odd-even topology, (see example of Remark (1.7)), and τ' is the discrete topology.

Clear that f is continuous and onto map and$(\mathbb{N},\tau)$ is CJ-space, but $(\mathbb{N},\tau')$is not CJ-space.

Definition 2.2 [3]: A map $f:(X,\tau)\to(Y,\tau')$ is said to be countably compact map if inverse image of each countably compact set is countably compact.

Definition 2.3[3]: A map $f:(X,\tau)\to(Y,\tau')$ is said to be countably compact preserving if image of each countably compact set is countably compact.

Proposition 2.4: If $f:(X,\tau)\to(Y,\tau')$ is continuous countably compact map from a CJ-space X onto a topological space Y, then Y is CJ-space.

Proof: Let {A,B} be a closed cover of Y with A⋂B countably compact,then $\{f^{-1}(A),f^{-1}(B)\}$ is a closed cover of X with $f^{-1}(A)\cap f^{-1}(B)=f^{-1}(A\cap B)$ which is countably compact, so$f^{-1}(A)$ or $f^{-1}(B)$ is countably compact. It follows that A or B is countably compact. Hence Y is CJ-space.

Proposition 2.5: Let f: X→Y be an injective, closed, countably compact and countably compact preserving map from a topological space X into a CJ-space Y, then X is also CJ-space.

Proof: Let {A,B} be a closed cover of X with A⋂B countably compact, then {f(A),f(B)} is a closed cover of Y with f(A)⋂f(B) = f(A⋂B) countably compact, but Y is CJ-space, so f(A) or f(B) is countably compact. Then $f^{-1}f(A)$ or $f^{-1}f(B)$ is countably compact, it follows that A or B is countably compact. Hence X is CJ-space.

Corollary 2.6: The property of being "CJ-space" is a topological property.

Proof: Follows from the fact that every homeomorphism map is a countably compact and countably compact preserving map.

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