



Two Absolute Index-Summability Methods

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Abstract

In this paper we have established a relation between the Summability methods $X - \left| \overline{N}, p_n \right|_k$ and $Y - \left| A, \delta, \gamma \right|_k, k \geq 1, \delta \geq 0$.

Keywords: $X - \left| \overline{N}, p_n \right|_k, X - \left| A \right|_k, Y - \left| A, \delta \right|_k, Y - \left| A, \delta, \gamma \right|_k, k \geq 1, \delta \geq 0$ - summabilities.

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1. Introduction :

Let $\sum a_n$ be an infinite series and $\{s_n\}$ the sequence of partial sums. Let $\{p_n\}$ be a sequence of non-negative numbers with $P_n = \sum_{v=0}^n p_v$ for all $n \in N$. The sequence -to-sequence transformation

$$(1.1) \quad t_n = \frac{1}{P_n} \sum_{v=0}^n p_v s_v, P_n \neq 0$$

defines $\left| \overline{N}, p_n \right|_k$ -mean of the sequence $\{s_n\}$ generated by the sequence of coefficients $\{p_n\}$. The series $\sum a_n$ is said to be summable $\left| \overline{N}, p_n \right|_k, k \geq 1, [4]$ if

$$(1.2) \quad \sum_{n=1}^{\infty} \left(\frac{P_n}{p_n} \right)^{k-1} |t_n - t_{n-1}|^k < \infty.$$

The sequence -to-sequence transformation

$$(1.3) \quad \tau_n = \frac{1}{P_n} \sum_{v=0}^n p_{n-v} s_v, P_n \neq 0$$

defines $\left| N, p_n \right|_k$ -mean of the sequence $\{s_n\}$. The series $\sum a_n$ is said to be summable $\left| N, p_n \right|_k, k \geq 1$, if

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$$(1.4) \quad \sum_{n=1}^{\infty} \left(\frac{P_n}{p_n} \right)^{k-1} |\tau_n - \tau_{n-1}|^k < \infty.$$

The series $\sum a_n$ is said to be summable $X - \left| \overline{N}, p_n \right|_k, k \geq 1$, if



$$(1.5) \quad \sum_{n=1}^{\infty} X_n^{k-1} |t_n - t_{n-1}|^k < \infty$$

where $\{X_n\}$ is a sequence of positive real constants. Similarly, $\sum a_n$ is said to be summable $X - |N, p_n|_k, k \geq 1$, if

$$(1.6) \quad \sum_{n=1}^{\infty} X_n^{k-1} |\tau_n - \tau_{n-1}|^k < \infty.$$

Let $A = (a_{nk})$ be a $\infty \times \infty$ matrix. The series $\sum a_n$ is said to be summable $X - |A|_k, k \geq 1$, if

$$(1.7) \quad \sum_{n=1}^{\infty} X_n^{k-1} |T_n - T_{n-1}|^k < \infty,$$

$\sum a_n$ is said to be summable $X - |A, \delta|_k, k \geq 1, \delta \geq 0$, if

$$(1.8) \quad \sum_{n=1}^{\infty} X_n^{\delta k + k - 1} |T_n - T_{n-1}|^k < \infty$$

and for any real number γ , $\sum a_n$ is said to be summable $X - |A, \delta, \gamma|_k, k \geq 1, \delta \geq 0$, if

$$(1.9) \quad \sum_{n=1}^{\infty} X_n^{\gamma(\delta k + k - 1)} |T_n - T_{n-1}|^k < \infty,$$

where the sequence –to–sequence transformation $\{T_n\}$ is given by

$$(1.10) \quad T_n = \sum_{k=1}^{\infty} a_{nk} S_k.$$

2. Known Theorems:

Dealing with the index summability method Bor has established the following theorems:

Theorem-A[1]:

Let $\{p_n\}$ be a sequence of positive real constants such that as $n \rightarrow \infty$

$$i) np_n = O(P_n)$$

$$ii) P_n = O(np_n).$$

If $\sum a_n$ is summable $|C, 1|_k$ then it is summable $|\bar{N}, p_n|_k, k \geq 1$.

Theorem-B [2]:

Let $\{p_n\}$ be a sequence of positive real constants such that as $n \rightarrow \infty$

$$(i) np_n = O(P_n)$$

$$(ii) P_n = O(np_n).$$

If $\sum a_n$ is summable $|\bar{N}, p_n|_k$ then it is summable $|C, 1|_k, k \geq 1$.

Subsequently Bor and Thorpe established the following result.

Theorem-C [3]:

Let $\{p_n\}$ and $\{q_n\}$ be the sequence of positive real constants such that

$$(i) p_n Q_n = O(P_n q_n)$$

$$(ii) P_n q_n = O(p_n Q_n).$$

then the series $\sum a_n$ is summable $|\bar{N}, q_n|_k$ whenever it is summable $|\bar{N}, p_n|_k, k \geq 1$.

Further, Tripathi established

Theorem-D [5]:

Suppose $\{p_n\}, \{q_n\}, \{X_n\}$ and $\{Y_n\}$ are sequences of positive real constants such that

$$(i) q_n p_n = O(p_n q_n)$$

$$(ii) Q_n = O(q_n X_n)$$

$$(iii) Y_n p_n = O(p_n).$$

If $\sum a_n$ is summable $X - |\bar{N}, p_n|_k$, then it is summable $Y - |\bar{N}, q_n|_k, k \geq 1$.

Extending the above result, Misra, Misra and Routa established the following theorem replacing

$Y - |\bar{N}, q_n|_k, k \geq 1$ by $Y - |N, q_n|_k, k \geq 1$, in Theorem-D.

Theorem-E [6]:

Suppose $\{p_n\}$, $\{q_n\}$, $\{X_n\}$ and $\{Y_n\}$ are sequences of positive real constants such that

$$(i) \quad q_n p_n = O(p_n q_n) \quad (ii) \quad Q_n = O(q_n X_n) \quad (iii) \quad Y_n p_n = O(p_n)$$

$$(iv) \quad \sum_{n=v+1}^{n+1} \frac{1}{Q_n} = O\left(\frac{1}{Q_v}\right) \quad (v) \quad \sum_{v=1}^{n-1} a_v^{k-1} = O(1), k \neq 1.$$

If $\sum a_n$ is summable $X - \left| \overline{N}, p_n \right|_k$, then it is summable $Y - \left| N, q_n \right|_k, k \geq 1$.

Subsequently Misra et al established the following theorem:

Theorem-F [7]:

Suppose $\{p_n\}$, $\{q_n\}$, $\{X_n\}$ and $\{Y_n\}$ are sequences of positive real constants such that

$$(i) \quad a_{nn} P_n = O(p_n),$$

$$(ii) \quad \frac{1}{a_{nn}} = O(X_n),$$

$$(iii) \quad Y_n p_n = O(P_n),$$

$$(iv) \quad \sum_{n=r}^{m+1} \left(\frac{P_n}{p_n} \right)^{\delta k + k - 1} A_{nr} = O(a_{rr}), \text{ where } A_{nk} = \sum_{v=k}^n a_{nv}$$

$$\text{and } (v) \quad \sum_{r=1}^n A_{nr} = O(1).$$

Then $\sum a_n$ is $Y - \left| A, \delta \right|_k, k \geq 1, \delta \geq 0$ summable whenever $\sum a_n$ is summable $X - \left| \overline{N}, p_n \right|_k, k \geq 1$.

In what follows, in the present paper we generalize the above theorem the matrix Summability.

3. Main Theorem:

Suppose $\{p_n\}$, $\{q_n\}$, $\{X_n\}$ and $\{Y_n\}$ are sequences of positive real constants such that

$$(3.1) \quad a_{nn} P_n = O(p_n),$$

$$(3.2) \quad \frac{1}{a_{nn}} = O(X_n),$$

$$(3.3) \quad Y_n p_n = O(P_n),$$

$$(3.4) \quad \sum_{n=r}^{m+1} \left(\frac{P_n}{p_n} \right)^{\gamma(\delta k + k - 1)} A_{nr} = O(a_{rr}), \text{ where } A_{nk} = \sum_{v=k}^n a_{nv}$$

and

$$(3.5) \quad \sum_{r=1}^n A_{nr} = O(1).$$

Then $\sum a_n$ is $Y - \left| A, \delta \right|_k, k \geq 1, \delta \geq 0$ summable whenever $\sum a_n$ is summable $X - \left| \overline{N}, p_n \right|_k, k \geq 1$.

4. Proof of the Theorem:

If $\{t_n\}$ is the nth $\left| \overline{N}, p_n \right|$ -mean of $\sum a_n$, then

$$\begin{aligned} t_n &= \frac{1}{P_n} \sum_{v=0}^n p_v s_v \\ &= \frac{1}{P_n} \{p_0 a_0 + p_1(a_0 + a_1) + \dots + p_n(a_0 + a_1 + \dots + a_n)\} \\ &= \frac{1}{P_n} \{P_n a_0 + (P_n - p_0)a_1 + \dots + (P_n - p_{n-1})a_n\} \\ &= \frac{1}{P_n} \sum_{v=0}^n (P_n - P_{v-1})a_v \end{aligned}$$

Then

$$\begin{aligned}
 \Delta t_n &= t_n - t_{n-1} \\
 &= \frac{1}{P_n} \sum_{v=0}^n (P_n - P_{v-1}) a_v - \frac{1}{P_{n-1}} \sum_{v=0}^{n-1} (P_{n-1} - P_{v-1}) a_v \\
 &= \frac{1}{P_n} \sum_{v=1}^n (P_n - P_{v-1}) a_v - \frac{1}{P_{n-1}} \sum_{v=0}^{n-1} (P_{n-1} - P_{v-1}) a_v \\
 &= \left(\frac{1}{P_{n-1}} - \frac{1}{P_n} \right) \sum_{v=1}^n P_{v-1} a_v \\
 &= \frac{P_n}{P_n P_{n-1}} \sum_{v=1}^n P_{v-1} a_v
 \end{aligned}$$

Hence,

$$\frac{P_n P_{n-1}}{P_n} \Delta t_n = \sum_{v=1}^n P_{v-1} a_v$$

and

$$\frac{P_{n-1} P_{n-1}}{P_{n-1}} \Delta t_{n-1} = \sum_{v=1}^{n-1} P_{v-1} a_v$$

Thus,

$$a_n = \frac{P_n}{P_n} \Delta t_n - \frac{P_{n-2}}{P_{n-1}} \Delta t_{n-1}$$

Further, if $\{T_n\}$ is the n th A -mean of $\sum a_n$, where $A = (a_{nk})_{\infty \times \infty}$ triangular matrix,

$$\begin{aligned}
 T_n &= \sum_{k=0}^n a_{nk} s_k \\
 &= \sum_{k=0}^n A_{nk} a_k, \quad A_{nk} = \sum_{v=k}^n a_{nv}
 \end{aligned}$$

Then,

$$\begin{aligned}
 T_n - T_{n-1} &= \sum_{k=0}^n A_{nk} a_k - \sum_{k=0}^{n-1} A_{n-1,k} a_k \\
 &= \sum_{k=1}^n (A_{nk} - A_{n-1,k}) a_k \\
 &= \sum_{k=1}^n (A_{nk} - A_{n-1,k}) \left(\frac{P_k}{P_k} \Delta t_k - \frac{P_{k-2}}{P_{k-1}} \Delta t_{k-1} \right) \\
 &= \sum_{k=1}^n A_{nk} \frac{P_k}{P_k} \Delta t_k - \sum_{k=1}^n A_{nk} \frac{P_{k-2}}{P_{k-1}} \Delta t_{k-1} - \sum_{k=1}^{n-1} A_{n-1,k} \frac{P_k}{P_k} \Delta t_k + \sum_{k=1}^{n-1} A_{n-1,k} \frac{P_{k-2}}{P_{k-1}} \Delta t_{k-1} \\
 &= S_1 + S_2 + S_3 + S_4 \text{ (say)}.
 \end{aligned}$$

Now,

$$\sum_{n=1}^{m+1} Y_n^{\delta k+k-1} |T_n - T_{n-1}|^k \leq \sum_{n=1}^{m+1} Y_n^{\gamma(\delta k+k-1)} |S_1 + S_2 + S_3 + S_4|^k = \sum_{i=1}^4 \sum_{n=1}^{m+1} Y_n^{\gamma(\delta k+k-1)} |S_i|^k$$

(By Minokowski's inequality)

Our Theorem will be established if we show that $\sum_{n=1}^{m+1} Y_n^{\delta k+k-1} |S_i|^k < \infty$, $\forall i = 1, 2, 3, 4$.

$$\sum_{n=1}^{m+1} Y_n^{\gamma(\delta k+k-1)} |S_1|^k = \sum_{n=1}^{m+1} Y_n^{\gamma(\delta k+k-1)} \left| \sum_{r=1}^n A_{nr} \frac{P_r}{P_r} \Delta t_r \right|^k$$

$$\begin{aligned}
 &\leq \sum_{n=1}^{m+1} Y_n^{\gamma(\delta k+k-1)} \sum_{r=1}^n \left(\frac{P_r}{p_r}\right)^k |\Delta t_r|^k A_{nr} \left(\sum_{r=1}^n A_{nr}\right)^{k-1} \\
 &\quad \text{(Using Holder's inequality)} \\
 &= O(1) \sum_{r=1}^{m+1} \left(\frac{P_r}{p_r}\right)^k |\Delta t_r|^k \sum_{n=r}^{m+1} Y_n^{\gamma(\delta k+k-1)} A_{nr}, \text{ by (3.5)} \\
 &= O(1) \sum_{r=1}^{m+1} \left(\frac{P_r}{p_r}\right)^k |\Delta t_r|^k \sum_{n=r}^{m+1} \left(\frac{P_n}{p_n}\right)^{\gamma(\delta k+k-1)} A_{nr}, \text{ by (3.3)} \\
 &= O(1) \sum_{r=1}^{m+1} \left(\frac{1}{a_{rr}}\right)^k |\Delta t_r|^k a_{rr}, \text{ (using 3.4)} \\
 &= O(1) \sum_{r=1}^{m+1} X_r^{k-1} |\Delta t_r|^k, \text{ (using 3.2)} \\
 &= O(1).
 \end{aligned}$$

Next,

$$\begin{aligned}
 \sum_{n=1}^{m+1} Y_n^{\gamma(\delta k+k-1)} |S_2|^k &= \sum_{n=1}^{m+1} Y_n^{\gamma(\delta k+k-1)} \left| \sum_{r=1}^n A_{nr} \frac{P_{r-2}}{p_{r-1}} \Delta t_{r-1} \right|^k \\
 &\leq \sum_{n=1}^{m+1} Y_n^{\gamma(\delta k+k-1)} \sum_{r=1}^n \left(\frac{P_{r-1}}{p_{r-1}}\right)^k |\Delta t_{r-1}|^k A_{nr} \left(\sum_{r=1}^n A_{nr}\right)^{k-1} \\
 &= O(1) \sum_{r=1}^{m+1} \left(\frac{P_{r-1}}{p_{r-1}}\right)^k |\Delta t_{r-1}|^k \sum_{n=r}^{m+1} Y_n^{\gamma(\delta k+k-1)} A_{nr}, \text{ by (3.5)} \\
 &= O(1) \sum_{r=1}^{m+1} \left(\frac{P_{r-1}}{p_{r-1}}\right)^k |\Delta t_{r-1}|^k \sum_{n=r}^{m+1} \left(\frac{P_n}{p_n}\right)^{\gamma(\delta k+k-1)} A_{nr}, \text{ by (3.3)} \\
 &= O(1) \sum_{r=2}^{m+1} \left(\frac{P_r}{p_r}\right)^k |\Delta t_r|^k a_{rr}, \text{ (using 3.4)} \\
 &= O(1) \sum_{r=2}^{m+1} \left(\frac{1}{a_{rr}}\right)^k |\Delta t_r|^k a_{rr} \\
 &= O(1) \sum_{r=2}^{m+1} X_r^{k-1} |\Delta t_r|^k, \text{ (using 3.2)} \\
 &= O(1).
 \end{aligned}$$

Also,

$$\begin{aligned}
 \sum_{n=1}^{m+1} Y_n^{\gamma(\delta k+k-1)} |S_3|^k &= \sum_{n=1}^{m+1} Y_n^{\gamma(\delta k+k-1)} \left| \sum_{r=1}^{n-1} A_{n-1,r} \frac{P_r}{p_r} \Delta t_r \right|^k \\
 &\leq \sum_{n=1}^{m+1} Y_n^{\gamma(\delta k+k-1)} \sum_{r=1}^{n-1} \left(\frac{P_r}{p_r}\right)^k |\Delta t_r|^k A_{n-1,r} \left(\sum_{r=1}^n A_{n-1,r}\right)^{k-1} \\
 &= O(1) \sum_{r=1}^{m+1} \left(\frac{P_r}{p_r}\right)^k |\Delta t_r|^k \sum_{n=r+1}^{m+1} \left(\frac{P_n}{p_n}\right)^{\gamma(\delta k+k-1)} A_{n-1,r}, \text{ using (3.5)} \\
 &= O(1) \sum_{r=1}^{m+1} \left(\frac{1}{a_{rr}}\right)^k |\Delta t_r|^k a_{rr}, \text{ (using 3.4)}
 \end{aligned}$$



$$= O(1) \sum_{r=1}^{m+1} X_r^{k-1} |\Delta t_r|^k, \text{ (using 3.2)}$$

$$= O(1).$$

Finally,

$$\begin{aligned} \sum_{n=1}^{m+1} Y_n^{\gamma(\delta k+k-1)} |S_4|^k &= \sum_{n=1}^{m+1} Y_n^{\gamma(\delta k+k-1)} \left| \sum_{r=1}^{n-1} A_{n-1,r} \frac{P_{r-2}}{P_{r-2}} \Delta t_{r-1} \right|^k \\ &\leq \sum_{n=1}^{m+1} Y_n^{\gamma(\delta k+k-1)} \sum_{r=1}^{n-1} \left(\frac{P_{r-2}}{P_{r-2}} \right)^k |\Delta t_{r-1}|^k A_{n-1,r} \left(\sum_{r=1}^n A_{n-1,r} \right)^{k-1} \\ &= O(1) \sum_{n=1}^{m+1} Y_n^{\gamma(\delta k+k-1)} \sum_{r=1}^{n-1} \left(\frac{P_{r-1}}{P_{r-1}} \right)^k |\Delta t_{r-1}|^k A_{n-1,r} \\ &= O(1) \sum_{r=1}^m \left(\frac{P_{r-1}}{P_{r-1}} \right)^k |\Delta t_{r-1}|^k \sum_{n=r+1}^{m+1} A_{n-1,r} Y_n^{\gamma(\delta k+k-1)} \\ &= O(1) \sum_{r=1}^m \left(\frac{1}{a_{r-1,r-1}} \right)^k |\Delta t_{r-1}|^k a_{r-1,r-1}, \text{ (using 3.4)} \\ &= O(1) \sum_{r=1}^m X_{r-1}^{k-1} |\Delta t_{r-1}|^k, \text{ (using 3.2)} \\ &= O(1). \end{aligned}$$

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