ISSN 2347-1921
Volume 12 Number 10
Journal of Advances in Mathematics

On the classification of $(1+n)_{n\geq 2}-$dimensional non-linear Klein-Gordon equation via Lie and Noether approach

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ABSTRACT

A complete group classification for the Klein-Gordon equation is presented. Symmetry generators, up to equivalence transformations, are calculated for each $f(u)$ when the principal Lie algebra extends. Further, considered equation is investigated by using Noether approach for the general case $n \geq 2$. Conserved quantities are computed for each calculated Noether operator. At the end, a brief conclusion is presented.

Keywords

Klein-Gordon equation; Group classification; Noether approach; Conserved vectors.

SUBJECT CLASSIFICATION

35R01, 76M60

INTRODUCTION

The (1+n)-dimensional Klein-Gordon equation

$$u_{tt} = \Delta_2 u + f(u), \qquad f_{uu} \neq 0, \tag{1}$$

where $u = u(t, x_1, ..., x_n)$ *with* $\Delta_2 u = \sum_{i=1}^{n} u_{x_i x_i}$.

In the past, the authors of [3, 10] have studied Eq. (1) for different values of n, for exact solutions, compatibility of the conditions for the reduction and reduced equations by consideration of an ansatz which reduces the dimension of the corresponding PDE (see [11]). In [9], the author discussed the symmetry properties and found particular solutions for some cases of Eq. (2). Tajiri [20] proposed some similarity and soliton solutions for the three-dimensional Klein-Gordon equation by means of similarity variables. Fushchych et al. [12] investigated the reductions and solutions by using the broken symmetry for Eq.(1) with n=3. In [8], Fedorchuk considered the reductions of Eq.(1) for n=4 by using decomposable subgroups of the generalized Poincare group P(1,4). Fushchych [10] invoked an ansatz of the form $u = f(x)\phi(\omega) + g(x)$ to analyze exact solutions of Eq. (1). Description of such an ansatz for the Eq. (1) can be a difficult problem. That problem can be simplified by using symmetry methods.

Lie symmetry analysis is a systematic way to construct an ansatz which further reduces the dimension of the differential equation. The symmetry method also plays a central role in the algebraic analysis of the differential equation. There are nonlinear equations with arbitrary coefficients which possess nontrivial Lie point symmetries. Such nonlinear differential equations can be classified, with respect to unknown functions, according to the nontrivial Lie point symmetries they admit. This classification is known as group classification. The problem of group classification is one of the central aspects of modern symmetry analysis of differential equations. It was performed in the classical works of Lie.

For the nonlinear wave equation: $u_{tt} = (f(u)u_{x_1})_{x_1}$, group properties are deduced by Ames [1]. Pucci [18] discussed the group classification of $u_{tt} + u_{x_1 x_1} = f(u, u_{x_1})$. A list of symmetries of the equation $u_{tt} = u_{x_1 x_1} + f(t, x_1, u, u_{x_1})$ is presented in [16]. Furthermore, the group classification of

$u_{tt} = u_{x_1x_1} + u_{x_2x_2} + u_{x_3x_3} + f(u)$ was studied by Rudra [19]. The authors in [2] performed the group classification of the (1+1)-dimensional Klein-Gordon equation by using [7].

One of the classical aspects of the Lie theory is the computation of conservation laws. The existence of a large number of conserved quantities of a PDE or system of PDEs is a strong indication of its integrability. An efficient method to compute conservation laws is given by Noether [6,17]. The theorem states that there is a conservation law for the Noether symmetry of the differential equation. Conservation laws for the nonlinear (1+1)-dimensional wave equation viz $u_{tt} - (k(u)u_x)_x - (k(u))_x = 0$ are discussed in [15]. Bokhari et al. constructed the conservation laws [5] for the nonlinear (1+n)-dimensional wave equation $u_{tt} - (f(u)u_{x_i})_{x_i} = 0$ via partial Noether approach. Conserved quantities for the (1+1)-dimensional nonlinear Klein-Gordon equation are reported in [14].

Fundamental operators

Consider the 2nd order PDE of the type

$$E(t, x_i, u, u_t, u_{x_i}, u_{tt}, u_{x_ix_i}) = 0 \qquad (2)$$

where u is dependent variable, $t, x_i \; (i = 1, 2, \ldots, n)$ are independent variables.

(I) The Euler operator is

$$\frac{\delta}{\delta u} = \frac{\partial}{\partial u} - D_t \frac{\partial}{\partial u_t} - \sum_{i=1}^{n} D_i \frac{\partial}{\partial u_i} + \ldots , \qquad (3)$$

where

$$D_t = \frac{\partial}{\partial t} + u_t \frac{\partial}{\partial u} + u_{tj} \frac{\partial}{\partial u_j} + \ldots, \qquad (4)$$

$$D_i = \frac{\partial}{\partial x_i} + u_i \frac{\partial}{\partial u} + u_{ij} \frac{\partial}{\partial u_j} + \ldots, \quad i = 1, \ldots, n \qquad (5)$$

are known as the total derivative operators.

The generalized or Lie Backlund operator is defined by:

$$Y = \tau \frac{\partial}{\partial t} + \xi^i \frac{\partial}{\partial x_i} + \phi \frac{\partial}{\partial u} + \phi^t \frac{\partial}{\partial u_t} + \phi^{x_i} \frac{\partial}{\partial u_{x_i}} + \ldots . \qquad (6)$$

(II) Suppose $L = L(t, x_i, u, u_t, u_{x_i}) \in A$ (space of differential functions) is a differentiable function such that L is said to be a standard Lagrangian if

$$\frac{\delta L}{\delta u} = 0. \qquad (7)$$

(III) The generalized operator (6) satisfying

$$Y(L) + L(D_t\tau + \sum_{i=1}^{n} D_i \xi^i) = D_t B^0 + \sum_{i=1}^{n} D_i B^i \qquad (8)$$

is known as the Noether operator associated with a Lagrangian L.

In Eq. (8), B^i for $i = 0, 1, 2, \ldots, n$ are known as the gauge terms.

(IV) The equation

$$D_t T^0 + \sum_{i=1}^{n} D_i T^i = 0$$

evaluated on the solution space given by (2) is known as the conservation law for Eq. (2) and vector $T = (T^0, T^1, \cdots, T^n)$ is said to be a conserved vector.

(V) The conserved vectors of the system (2) associated with a Noether operator X can be determined from the formula

$$T^i = B^i - N^i(L). \tag{9}$$

In Eq. (9),

$$N^0 = \tau + W\frac{\delta}{\delta u_t}, \qquad N^i = \xi^i + W\frac{\delta}{\delta u_i},$$

where W is known as the Lie characteristic function and can be found from

$$W = \phi - \tau u_t - \sum_{j=1}^{n} \xi^j u_j. \tag{10}$$

The outline of this paper is as follows. In Section 2, the group classification of the (1+n) -dimensional Klein-Gordon equation is given. Section 3 is for the Noether symmetry operators and conserved vectors of Eq. (1). Finally, conclusions are summarized at the end.

Lie point symmetries

In this section, we discuss the group classification for the (1+n)-dimensional Klein-Gordon equation, i.e. Eq. (1) for arbitrary n. We apply the 2nd prolongation vector i.e

$$Y^{[2]}\left(u_{tt} - \sum_{i=1}^{n} u_{x_i x_i} - f(u)\right)\Big|_{u_{tt}=\sum_{i=1}^{n} u_{x_i x_i} + f(u)} = 0. \tag{11}$$

Eq. (11) yields the following determining equations:

$$(i)\xi^i_u = 0, \quad (ii)\tau_u = 0, \quad (iii)\phi_{uu} = 0, \tag{12}$$

$$(i)\xi^i_{x_j} + \xi^j_{x_i} = 0, \ (ii)\tau_t - \xi^i_{x_i} = 0, \ (iii)\xi^i_t - \tau_{x_i} = 0, \tag{13}$$

$$-\xi^i_{tt} + \sum_{j=1}^{n} \xi^i_{x_j x_j} - 2\phi_{x_i u} = 0, \ -\tau_{tt} + \sum_{i=1}^{n} \tau_{x_i x_i} + 2\phi_{tu} = 0, \tag{14}$$

$$\phi_{tt} - \sum_{i=1}^{n} \phi_{x_i x_i} - 2f\tau_t + f\phi_u - f_u\phi = 0. \tag{15}$$

Eq. (13) forms a set of equations for an infinitesimal conformal transformation on R^{n+1} with Lorentz metric and thus the unknowns appearing in these equations are quadratic polynomials of $t, x_1, \cdots, x_n$ (see [21]) and Eq. (14) implies

$$\phi_{x_i u} = constant, \ \phi_{tu} = constant, \quad i = 1, 2, \cdots, n. \tag{16}$$

Differentiating Eq. (15) with respect to u and using the results given in Eq. (16), yields

$$\left(\frac{f_u}{f_{uu}}\right)_{uu} = 0. \tag{17}$$

The solutions of Eq. (17) yield the following functions

$$(i)\, f(u) = ae^{bu} + c, \qquad (ii)\, f(u) = c\ln(au+b), \qquad (iii)\, f(u) = (au+b)^m + c,\ m \neq 0,1.$$

Lie algebra

In this section, we discuss the different forms of $f(u)$, up to equivalence transformations, which lead to an extension of the principal Lie algebra of Eq. (1) for $n = 2,3\dots$.

For $n = 1$, the results are presented in [2].

When f(u) is arbitrary

The minimal algebra for the arbitrary case is:

$$Y_0 = \frac{\partial}{\partial t},\quad Y_i = \frac{\partial}{\partial x_i},\quad Y_{n+i} = x_i\frac{\partial}{\partial t} + t\frac{\partial}{\partial x_i},\quad Y_{2n+i} = x_i\frac{\partial}{\partial x_j} - x_j\frac{\partial}{\partial x_i},\quad j > i \qquad (18)$$

and appeared in all the rest of the considered cases, thus we shall only present the additional algebra(s). The principal Lie algebra for this case is of dimension n(3+n)/2+1.

$f(u) = ae^{bu} + c$

For $c = 0$ the principal algebra extends and additional generators will be:

$$Y_{3n+p} = t\frac{\partial}{\partial t} + \sum_{i=1}^{n} x_i\frac{\partial}{\partial x_i} - 2\frac{\partial}{\partial u}, \qquad (19)$$

where

$$P = \begin{cases} 0, & \text{if } n \text{ is even} \\ 1, & \text{if } n \text{ is odd.} \end{cases} \qquad (20)$$

The Lie algebra is of dimension **n(n+3)/2+2**.

$f(u) = c\ln(au+b)$

There is no extension in the principal algebra.

$f(u) = (au+b)^m + c \quad m \neq 0,1$

In this case, for $b = 0 = c$ leads to an extension of the principal algebra and additional generators will be:

$$Y_{3n+p} = t\frac{\partial}{\partial t} + \sum_{i=1}^{n} x_i\frac{\partial}{\partial x_i} - \frac{2u}{m-1}\frac{\partial}{\partial u}, \qquad (21)$$

where p is defined in (20) and the Lie algebra is of dimension **n(n+3)/2+2**.

Noether symmetries

In this section, we will use Noether approach for finding the conserved vectors of Eq. (1) for arbitrary n, taking $n \geq 2$.

Case 1: n=1

Noether operators and conserved vectors of Eq. (1) for $n=1$ are reported in [14].

Case 2: n=2

The standard Lagrangian for Eq. (1) will be

$$L=\frac{u_t^2}{2}-\sum_{i=1}^{n}\frac{u_{x_i}^2}{2}+F(u), \quad F'(u)=f(u). \tag{22}$$

The Noether determining equation (8) with the help of Eq. (22) after some lengthy manipulation gives the following set of determining

equations: $(i)\tau_u=0, \quad (ii)\xi_u^i=0, \quad (iii)2\phi_u=\tau_t-\sum_{i=1}^{n}\xi_{x_i}^i, \ (i)2\phi_u=\xi_{x_i}^i-\tau_t-\sum_{j=1,\, j\neq i}^{n}\xi_{x_j}^j,$

(23)

$$(i)\tau_{x_i}-\xi_t^i=0, \ (i)\xi_{x_j}^i+\xi_{x_i}^j=0, \quad (i)B_u^0=\phi_t, \quad (ii)B_u^i=-\phi_{x_i}, \tag{24}$$

$$\phi f(u)+F(u)[\tau_t+\sum_{i=1}^{n}\xi_{x_i}^i]=B_t^0+\sum_{i=1}^{n}B_{x_i}^i. \tag{25}$$

Hence doing the routine calculation, Eq. (25) yields:

$$\phi\big((2n+1)f_{uu}f_{uuuu}-2nf_{uuu}^2\big)=0. \tag{26}$$

Eq. (26) further divides two cases and discussed in the following sections.

$\phi=0$

For this case, the Noether operators will be:

$$Y_0, \ Y_i, \ Y_{n+i}, \ Y_{2n+i}.$$

This forms the minimal algebra and thus thus we shall only present the additional algebras in the next section.

$\phi\neq 0$

For this case, the additional Noether operators will

be: $Y_{3n+i}=x_i t\frac{\partial}{\partial t}+\frac{1}{2}[x_i^2+t^2-\sum_{j=1}^{n}x_j^2]\frac{\partial}{\partial x_i}+\sum_{j=1}^{n}x_i x_j\frac{\partial}{\partial x_j}-\frac{x_i u}{2}\frac{\partial}{\partial u}, \ j\neq i,$

$$Y_{4n+1}=t\sum_{i=1}^{n}x_i\frac{\partial}{\partial x_i}+\frac{1}{2}[t^2+\sum_{i=1}^{n}x_i^2]\frac{\partial}{\partial t}-\frac{ut}{2}\frac{\partial}{\partial u}, \quad Y_{4n+2}=t\frac{\partial}{\partial t}+\sum_{i=1}^{n}x_i\frac{\partial}{\partial x_i}-\frac{u}{2}\frac{\partial}{\partial u}.$$

Conserved quantities

(I) The vector T_0 corresponding to Y_0 has the following components:

$$T_0^0=\frac{1}{2}[u_t^2+\sum_{i=1}^{n}u_{x_i}^2]-F(u), \quad T_0^i=-u_{x_i}u_t.$$

(II) The components of the vector T_i are:

$$T_i^0=u_{x_i}u_t, \quad T_i^i=-\frac{1}{2}[u_t^2+u_{x_i}^2-\sum_{j=1}^{n}u_{x_j}^2]-F(u), \ T_i^j=-u_{x_i}u_{x_j}, \quad j\neq i.$$

(III) For Y_{n+i} the components of T_{n+i} are:

$$T^0_{n+i}=\frac{x_i}{2}\Big[u_t^2+\sum_{j=1}^{n}u_{x_j}^2-2F(u)\Big]+tu_{x_i}u_t,\quad T^i_{n+i}=\frac{-t}{2}\Big[u_t^2+u_{x_i}^2-\sum_{j=1,j\neq i}^{n}u_{x_j}^2+2F(u)\Big]-x_iu_{x_i}u_t,$$

$$T^j_{n+i}=-(x_iu_t+tu_{x_i})u_{x_j},\quad j\neq i.$$

(IV) For Y_{2n+i} with $j \succ i$ the conserved vector T_{2n+i} has the following components:

$$T^0_{2n+i}=(x_ju_{x_i}-x_iu_{x_j})u_t,\quad T^i_{2n+i}=\frac{-x_j}{2}\Big[u_t^2+u_{x_i}^2-\sum_{k=1,k\neq i}^{n}u_{x_k}^2+2F(u)\Big]+x_iu_{x_i}u_{x_j},$$

$$T^j_{2n+i}=\frac{x_i}{2}\Big[u_t^2+u_{x_j}^2-\sum_{k=1,k\neq j}^{n}u_{x_k}^2+2F(u)\Big]-x_ju_{x_i}u_{x_j},\quad T^k_{2n+i}=-(x_ju_{x_i}-x_iu_{x_j})u_{x_k},\ k\neq i,j.$$

(V) For Y_{3n+i} the components of T_{3n+i} are:

$$T^0_{3n+i}=\frac{x_it}{2}\Big[u_t^2+\sum_{k=1}^{n}u_{x_k}^2\Big]+\Big[\frac{x_iu}{2}+\frac{1}{2}(x_i^2+t^2-\sum_{j=1}^{n}x_j^2)u_{x_i}+\sum_{j=1}^{n}x_ix_ju_{x_j}\Big]u_t,$$

$$T^i_{3n+i}=\frac{u^2}{22}-\frac{1}{4}(x_i^2+t^2-\sum_{j=1}^{n}x_j^2)\Big[u_t^2+u_{x_i}^2-\sum_{j=1}^{n}u_{x_j}^2\Big]-\Big[\frac{u}{2}+tu_t+\sum_{j=1}^{n}x_ju_{x_j}\Big]x_iu_{x_i},$$

$$T^j_{3n+i}=-\Big[\frac{x_iu}{2}+x_itu_t+\frac{1}{2}(x_i^2+t^2-\sum_{j=1}^{n}x_j^2)u_{x_i}+\sum_{m=1}^{n}x_ix_mu_{x_m}\Big]u_{x_j}-$$

$$\frac{x_ix_j}{2}\Big[u_t^2+u_{x_j}^2-\sum_{m=1}^{n}u_{x_m}^2\Big],\ m\neq i,j.$$

(VI) For Y_{4n+1} the components of T_{4n+1} are:

$$T^0_{4n+1}=-\frac{u^2}{4}+\frac{1}{4}(t^2+\sum_{i=1}^{n}x_i^2)\Big[u_t^2+\sum_{i=1}^{n}u_{x_i}^2\Big]+\Big[\frac{u}{2}+\sum_{i=1}^{n}x_iu_{x_i}\Big]tu_t,$$

$$T^i_{4n-p+1}=-\frac{x_it}{2}\Big[u_t^2+u_{x_i}^2-\sum_{j=1}^{n}u_{x_j}^2\Big]-\Big[\frac{ut}{2}+\frac{1}{2}(t^2+\sum_{i=1}^{n}x_i^2)u_t+\sum_{j=1}^{n}x_jtu_{x_j}\Big]u_{x_i},\quad j\neq i.$$

(VII) For Y_{4n+2}, the vector T_{4n+2} has the following components:

$$T^0_{4n+2}=\frac{t}{2}\Big[u_t^2+\sum_{i=1}^{n}u_{x_i}^2\Big]+\Big[\frac{u}{2}+\sum_{i=1}^{n}x_iu_{x_i}\Big]u_t,$$

$$T^i_{4n-p+2}=-\frac{x_i}{2}\Big[u_t^2+u_{x_i}^2-\sum_{j=1}^{n}u_{x_j}^2\Big]-\Big[\frac{u}{2}+tu_t+\sum_{j=1}^{n}x_ju_{x_j}\Big]u_{x_i},$$

Where p is defined in (20).

Conclusion

A complete classification of the $(1+n)$-dimensional Klein-Gordon equation is reported. The procedure is carried out for arbitrary n. A class of functions is obtained which possess the non-trivial Lie point symmetries. It is observed that obtained class of function does not depend on the number of independent variables. Extensions of the principal algebras up to equivalence transformations are constructed for each $f(u)$. These symmetry algebras can be further used to construct ansatz or similarity variables.

It should be noted that in the entire obtained Lie point symmetries, the coefficients of ∂_t and ∂_{x_i} are independent of the dependent variable u and hence the obtained symmetries are fiber preserving or projective transformations. Such transformations allow one to calculate the expression for the group transformations on the actual function $u(t,x_1,\ldots,x_n)$ with less difficulty. Noether operators and conservation laws of the considered equation are computed by using Noether approach for $n \geq 2$.

ACKNOWLEDGMENTS

Authors acknowledge Qassim University, represented by the Deanship of Scientific Research, on the material support for this research under the number 2955 during the academic year 1436 AH/ 2015 AD.

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