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The global attractors and exponential attractors for a class of nonlinear damping Kirchhoff equation

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Abstract

This paper consider the long time behavior of a class of nonlinear damped Kirchhoff equation $u_{tt}+\alpha_1 u_t-\gamma\Delta u_t-\left(\alpha+\beta\|\nabla u\|^2\right)^{\rho}\Delta u=f(x)$. Study the attractor problem with initial boundary value conditions, then using priori estimate and the Galerkin method prove existence and uniqueness of solution, we obtain to the existence of the global attractors. The squeezing property of the nonlinear semi-group associated with this equation and the existence of exponential attractors are also proved.

Key words: Kirchhoff equation; Priori estimate; Galerkin method; Global attractors; Squeezing property; Exponential attractors

1 Introduction

In this paper,we study the existence of the global attractors and the existence of exponential attractors for a class of nonlinear damping Kirchhoff equation:

$$u_{tt}+\alpha_1 u_t-\gamma\Delta u_t-\left(\alpha+\beta\|\nabla u\|^2\right)^{\rho}\Delta u=f(x), \tag{1.1}$$

$$u(x,0)=u_0(x);u_t(x,0)=u_1(x) \ , \tag{1.2}$$

$$u(x,t)|_{\partial\Omega}=0,\Delta u(x,t)|_{\partial\Omega}=0, \tag{1.3}$$

Where Ω is a bounded domain of $R^n\,(n\geq 1)$ with a smooth boundary $\partial\Omega$, $\alpha_1,\gamma,\alpha,\beta$ are positive constants, and the assumptions on $\left(\alpha+\beta\|\nabla u\|^2\right)^{\rho}\Delta u$ will be specified later, $f(x)$ is an external force item.

Since the 1980s, on one hand, because of the practical problems and the push of the other disciplines, on the other hand because of the mathematics development deeply, the infinite dimensional dynamical system research to be one of the important research subject in power system. From the point of a partial differential equation theory research, the infinite dimensional dynamical system problem is mainly to full time solution of progressive qualitative research, its core and the key problem is to the solution of a prior estimate of time t consistency. An infinite dimensional dynamical system is an important concept of global attractor,global attractor is the largest of all the attractor and it is sole. For ordinary differential equations, H has a finite dimensional, existence of global attractor has been studied. For infinite dimensional dynamical system, the existence of the attractor is behind have been proved [1]. R.Teman, C.Foias and so on put forward the concept of exponential attractor in 1990s [2]. exponential attractor is one who has a finite dimensional fractal dimension of the tight is invariant set and the exponential attractor like in space orbit.Because of its solution is exponentially attractor orbit, so it has better stability than the global attractor,and the exponential attractor is not the only one [3].

In [4] ,Perikles G.Papadopoulos ,Nikos M.Stavrakakis study the global existence and blow-up results for an equation of Kirchhoff type on R^N

$$u_{tt}-\phi(x)\|\nabla u(t)\|^2\Delta u+\delta u_t=|u|^\alpha u, x\in R^N, t\geq 0 \quad , \tag{1.4}$$

with initial conditions $u(x,0)=u_0(x), u_t(x,0)=u_1(x)$.

In [5], Zhijian Yang, Pengyan Ding, Zhiming Liu are concerned with the existence of global attractor for the Kirchhoff type equation with strong nonlinear damping and supercritical nonlinearity

$$u_{tt}-\sigma\left(\|\nabla u\|^2\right)\Delta u_t-\phi\left(\|\nabla u\|^2\right)\Delta u+f(u)=h(x), in\Omega\times R^+ \quad , \tag{1.5}$$

with initial conditions $u|_{\partial\Omega}=0, u(x,0)=u_0(x), u_t(x,0)=u_1(x)$.

In [6], GUO Chun-xiao, MU Chun-lai consider the classical diffusion equation of exponential attractor

$$u_t-\Delta u_t-\Delta u=f(u)+g(x), in\Omega\times R^+ \quad , \tag{1.6}$$

In [7], Ke Li, Zhijian Yang are concered with the following strongly damped wave equation on a bounded domain $\Omega\subset R^3$ with smooth boundary $\partial\Omega$

$$u_{tt}-\Delta u_t-\Delta u+\varphi(u)=f \quad , \tag{1.7}$$

More research on the global attractor and exponential attractor see reference [2-17] and [21-23].

This paper is structured as follows. In section 2, some preliminaries are stated. In 3, global attractor is proved, In section 4, exponential attractor is proved.

2. Preliminaries

For brevity, we define the Sobolev space as follows

$$H=L^2(\Omega), V_1=H_0^1(\Omega)\times L^2(\Omega), V_2=\left(H^2(\Omega)\cap H_0^1(\Omega)\right)\times H_0^1(\Omega),$$

where $(\cdot,\cdot)$ and $\|\cdot\|$, are the inner product and norm of H . The inner product and the norm in V_1 space is defined as follows

$$\forall U_i=(u_i,v_i)\in V_i, i=1,2 \quad ,$$

we have

$$(U_1,U_2)=(\nabla u_1,\nabla u_2)+(v_1,v_2) \quad , \tag{2.1}$$

$$\|U\|_{V_1}^2 = (U,U)_{V_1} = \|\nabla u\|^2 + \|v\|^2 \ , \tag{2.2}$$

set $U=(u,v)\in V_1$, $v=u_t+\varepsilon u$, the equation (1.1) is equivalent to

$$U_t + H(U) = F(U) \ , \tag{2.3}$$

where

$$H(U)=\begin{pmatrix} \varepsilon u - v \\ (\alpha_1-\varepsilon)v + (\varepsilon^2-\varepsilon\alpha_1)u + \gamma\varepsilon\Delta u - \gamma\Delta v - \left(\alpha+\beta\|\nabla u\|^2\right)^{\rho}\Delta u \end{pmatrix}, F(U)=\begin{pmatrix} 0 \\ f(x) \end{pmatrix}.$$

3. Global attractors

Lemma 3.1 $(u_0,u_1)\in H_0^1(\Omega)\times L^2(\Omega)$, $f\in L^2(\Omega)$, $v=u_t+\varepsilon u$, then the solution (u,v) of equation (1.1) satisfies

$(u,v)\in H_0^1(\Omega)\times L^2(\Omega)$ and

$$\|(u,v)\|_{H_0^1\times L^2}^2 = \|v\|^2 + \|\nabla u\|^2 \le \frac{W(0)}{k}e^{-\delta_1 t} + \frac{C_1}{\delta_1 k}\left(1-e^{-\delta_1 t}\right), \tag{3.1}$$

here

$$W(0) = \|v_0\|^2 + 3\gamma\varepsilon\|\nabla u_0\|^2 + \alpha_1\varepsilon\|u_0\|^2 + \frac{\left(\alpha+\beta\|\nabla u_0\|^2\right)^{\rho+1}}{(\rho+1)\beta} \ , \tag{3.2}$$

thus there exist M_0 and $t_1=t_1(\Omega)>0$, such that

$$\|(u,v)\|_{H_0^1\times L^2}^2 = \|v(t)\|^2 + \|\nabla u(t)\|^2 \le M_0 \ (t>t_1) \ , \tag{3.3}$$

Proof : We use $v=u_t+\varepsilon u$ and equation (1.1) to inner product and obtain

$$\left(u_{tt} + \alpha_1 u_t - \gamma\Delta u_t - \left(\alpha+\beta\|\nabla u\|^2\right)^{\rho}\Delta u, v\right) = (f, v), \tag{3.4}$$

by using *Holder* inequality , *Young* inequality and *Poincare* inequality , we have

$$
\begin{aligned}
(u_{tt},v) &= (v_t-\varepsilon u_t,v)=\frac{1}{2}\frac{d}{dt}\|v\|^2-\varepsilon(u_t,v)\\
&=\frac{1}{2}\frac{d}{dt}\|v\|^2-\varepsilon(v-\varepsilon u,v)\\
&=\frac{1}{2}\frac{d}{dt}\|v\|^2-\varepsilon\|v\|^2+\varepsilon^2(u,v)\\
&\ge\frac{1}{2}\frac{d}{dt}\|v\|^2-\varepsilon\|v\|^2-\frac{\varepsilon^2}{2}\|u\|^2-\frac{\varepsilon^2}{2}\|v\|^2\\
&\ge\frac{1}{2}\frac{d}{dt}\|v\|^2-\varepsilon\|v\|^2-\frac{\varepsilon^2}{2\lambda_1}\|\nabla u\|^2-\frac{\varepsilon^2}{2}\|v\|^2,
\end{aligned}
\tag{3.5}
$$

and

$$
\begin{aligned}
(\alpha_1 u_t,v) &= (\alpha_1 u_t,u_t+\varepsilon u)=\alpha_1(v-\varepsilon u,v-\varepsilon u)+(\alpha_1 u_t,\varepsilon u)\\
&=\alpha_1\|v\|^2-2\alpha_1\varepsilon(u,v)+\alpha_1\varepsilon^2\|u\|^2+\alpha_1\varepsilon\cdot\frac{1}{2}\frac{d}{dt}\|u\|^2\\
&\ge\alpha_1\|v\|^2-\alpha_1\varepsilon\|u\|^2-\alpha_1\varepsilon\|v\|^2+\alpha_1\varepsilon^2\|u\|^2+\alpha_1\varepsilon\cdot\frac{1}{2}\frac{d}{dt}\|u\|^2\\
&=(\alpha_1-\alpha_1\varepsilon)\|v\|^2+(\alpha_1\varepsilon^2-\alpha_1\varepsilon)\|u\|^2+\alpha_1\varepsilon\cdot\frac{1}{2}\frac{d}{dt}\|u\|^2,
\end{aligned}
\tag{3.6}
$$

we use $-\gamma\Delta u_t$ and v to inner product and we have

$$
\begin{aligned}
(-\gamma\Delta u_t,v) &= \gamma\left(-\Delta(v-\varepsilon u),u_t+\varepsilon u\right)\\
&=\gamma\left(-\Delta(u_t+\varepsilon u),u_t+\varepsilon u\right)+\gamma\varepsilon\left(\Delta u,u_t+\varepsilon u\right)\\
&=\gamma\|\nabla u_t\|^2+\frac{\gamma\varepsilon}{2}\frac{d}{dt}\|\nabla u\|^2+\frac{\gamma\varepsilon}{2}\frac{d}{dt}\|\nabla u\|^2+\gamma\varepsilon^2\|\nabla u\|^2+\gamma\varepsilon\left(\Delta u,u_t+\varepsilon u\right)\\
&=\gamma\|\nabla u_t\|^2+\gamma\varepsilon\frac{d}{dt}\|\nabla u\|^2+\gamma\varepsilon^2\|\nabla u\|^2+\frac{\gamma\varepsilon}{2}\frac{d}{dt}\|\nabla u\|^2+\gamma\varepsilon^2\|\nabla u\|^2\\
&=\gamma\|\nabla u_t\|^2+\frac{3\gamma\varepsilon}{2}\frac{d}{dt}\|\nabla u\|^2+2\gamma\varepsilon^2\|\nabla u\|^2,
\end{aligned}
\tag{3.7}
$$

we use $-\left(\alpha+\beta\|\nabla u\|^2\right)^\rho\Delta u$ and v to inner product and we get

$$
\begin{aligned}
\left(-\left(\alpha+\beta\|\nabla u\|^2\right)^\rho\Delta u,v\right) &= \left(\alpha+\beta\|\nabla u\|^2\right)^\rho\cdot(-\Delta u,v)\\
&=\left(\alpha+\beta\|\nabla u\|^2\right)^\rho\cdot(-\Delta u,u_t+\varepsilon u)\\
&=\left(\alpha+\beta\|\nabla u\|^2\right)^\rho\left(\frac{1}{2}\frac{d}{dt}\|\nabla u\|^2+\varepsilon\|\nabla u\|^2\right)\\
&=\left(\alpha+\beta\|\nabla u\|^2\right)^\rho\cdot\frac{1}{2}\frac{d}{dt}\|\nabla u\|^2+\left(\alpha+\beta\|\nabla u\|^2\right)^\rho\cdot\varepsilon\|\nabla u\|^2\\
&=\frac{1}{2(\rho+1)\beta}\frac{d}{dt}\left[\left(\alpha+\beta\|\nabla u\|^2\right)^{\rho+1}\right]+\left(\alpha+\beta\|\nabla u\|^2\right)^\rho\cdot\varepsilon\|\nabla u\|^2,
\end{aligned}
\tag{3.8}
$$

take proper ε, α, β such that $\varepsilon\|\nabla u\|^2\ge\alpha+\beta\|\nabla u\|^2$ then (3.7) have

$$\begin{aligned}&\frac{1}{2(\rho+1)\beta}\frac{d}{dt}\left[\left(\alpha+\beta\|\nabla u\|^2\right)^{\rho+1}\right]+\left(\alpha+\beta\|\nabla u\|^2\right)^{\rho}\cdot\varepsilon\|\nabla u\|\\&\geq\frac{1}{2(\rho+1)\beta}\frac{d}{dt}\left[\left(\alpha+\beta\|\nabla u\|^2\right)^{\rho+1}\right]+\left(\alpha+\beta\|\nabla u\|^2\right)^{\rho+1},\end{aligned}\tag{3.9}$$

by using *Holder* inequality and *Young* inequality, we get

$$\left(f(x),v\right)\leq\|f\|\cdot\|v\|\leq\frac{\gamma_1}{2}\|v\|^2+\frac{1}{2\gamma_1}\|f\|^2,\tag{3.10}$$

then we have

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\|v\|^2-\varepsilon\|v\|^2-\frac{\varepsilon^2}{2\lambda_1}\|\nabla u\|^2-\frac{\varepsilon^2}{2}\|v\|^2+\left(\alpha_1-\alpha_1\varepsilon\right)\|v\|^2+\left(\alpha_1\varepsilon^2-\alpha_1\varepsilon\right)\|u\|^2\\&+\alpha_1\varepsilon\cdot\frac{1}{2}\frac{d}{dt}\|u\|^2+\gamma\|\nabla u_t\|^2+\frac{3\gamma\varepsilon}{2}\frac{d}{dt}\|\nabla u\|^2+2\gamma\varepsilon^2\|\nabla u\|^2\\&+\frac{1}{2(\rho+1)\beta}\frac{d}{dt}\left[\left(\alpha+\beta\|\nabla u\|^2\right)^{\rho+1}\right]+\left(\alpha+\beta\|\nabla u\|^2\right)^{\rho+1}\leq\frac{\gamma_1}{2}\|v\|^2+\frac{1}{2\gamma_1}\|f\|^2,\end{aligned}\tag{3.11}$$

that is

$$\begin{aligned}&\frac{d}{dt}\left[\|v\|^2+3\gamma\varepsilon\|\nabla u\|^2+\alpha_1\varepsilon\|u\|^2+\frac{\left(\alpha+\beta\|\nabla u\|^2\right)^{\rho+1}}{(\rho+1)\beta}\right]+\left(2\alpha_1-2\alpha_1\varepsilon-2\varepsilon-\varepsilon^2-\gamma_1\right)\\&\|v\|^2+\left(4\gamma\varepsilon^2-\frac{\varepsilon^2}{\lambda_1}\right)\|\nabla u\|^2+\left(2\alpha_1\varepsilon^2-2\alpha_1\varepsilon\right)\|u\|^2+2\left(\alpha+\beta\|\nabla u\|^2\right)^{\rho+1}\leq\frac{1}{\gamma_1}\|f\|^2;\end{aligned}\tag{3.12}$$

next, we take proper $\gamma,\varepsilon,\alpha_1,\alpha,\beta$ such that

$$\begin{cases}a_1=2\alpha_1-2\alpha_1\varepsilon-2\varepsilon-\varepsilon^2-\gamma_1\geq 0\\a_2=4\gamma\varepsilon^2-\dfrac{\varepsilon^2}{\lambda_1}\geq 0\\a_3=2\alpha_1\varepsilon^2-2\alpha_1\varepsilon\geq 0\end{cases}$$

taking $\delta_1=\min\left\{a_1,\frac{a_2}{3\gamma\varepsilon},\frac{a_3}{\alpha_1\varepsilon}\right\}$,

$$\frac{d}{dt}W(t)+\delta_1 W(t)\leq\frac{1}{\gamma_1}\|f\|^2:=C_1,\tag{3.13}$$

where

$$W(t)=\|v\|^2+3\gamma\varepsilon\|\nabla u\|^2+\alpha_1\varepsilon\|u\|^2+\frac{\left(\alpha+\beta\|\nabla u\|^2\right)^{\rho+1}}{(\rho+1)\beta},\tag{3.14}$$

by using *Gronwall* inequality, we obtain

$$W(t) \le W(0)e^{-\delta_1 t} + \frac{C_1}{\delta_1}\left(1-e^{-\delta_1 t}\right), \tag{3.15}$$

let $k=\min\{1,3\gamma\varepsilon\}$, so we have

$$\|(u,v)\|^2_{H_0^1\times L^2} = \|v\|^2 + \|\nabla u\|^2 \le \frac{W(0)}{k}e^{-\delta_1 t} + \frac{C_1}{\delta_1 k}\left(1-e^{-\delta_1 t}\right), \tag{3.16}$$

$$\overline{\lim_{t\to\infty}}\|(u,v)\|^2 \le \frac{C_1}{\delta_1 k}, \tag{3.17}$$

so, there exist M_0 and $t_1 = t_1(\Omega) > 0,$ such that

$$\|(u,v)\|^2_{H_0^1\times L^2} = \|v(t)\|^2 + \|\nabla u(t)\|^2 \le M_0\,(t>t_1), \tag{3.18}$$

Lemma 3.2 $(u_0,u_1)\in H^2(\Omega)\times H_0^1(\Omega)$, $f\in H_0^1(\Omega)$, $v=u_t+\varepsilon u$, then the solution (u,v) of equation (1.1) satisfies $(u,v)\in H^2(\Omega)\times H_0^1(\Omega)$, and

$$\|(u,v)\|^2_{H^2\times H^1} = \|\nabla v\|^2 + \|\Delta u\|^2 \le \frac{W(0)}{k}e^{-\delta_2 t} + \frac{C_5}{\delta_2 k}\left(1-e^{-\delta_2 t}\right), \tag{3.19}$$

where

$$V_{(0)} = \|\nabla v_0\|^2 + \gamma\varepsilon\|\Delta u_0\|^2, \tag{3.20}$$

thus there exist M_1 and $t_2 = t_2(\Omega) > 0$, such that

$$\|(u,v)\|^2_{H^2\times H^1} = \|\nabla v(t)\|^2 + \|\Delta u(t)\|^2 \le M_1\,(t>t_2), \tag{3.21}$$

Proof : We multiply $-\Delta v = -\Delta u_t - \varepsilon\Delta u$ with both sides of equation (1.1) and obtain

$$\left(u_{tt} + \alpha_1 u_t - \gamma\Delta u_t - \left(\alpha + \beta\|\nabla u\|^2\right)^{\rho}\Delta u, -\Delta v\right) = (f, -\Delta v), \tag{3.22}$$

by using *Holder* inequality, *Young* inequality, *Poincare* inequality, we have

$$
\begin{aligned}
&(u_{tt},-\Delta v)=(v_t-\varepsilon u_t,-\Delta v)=\frac{1}{2}\frac{d}{dt}\|\nabla v\|^2-\varepsilon(u_t,-\Delta v)\\
&=\frac{1}{2}\frac{d}{dt}\|\nabla v\|^2-\varepsilon(v-\varepsilon u,-\Delta v)\\
&=\frac{1}{2}\frac{d}{dt}\|\nabla v\|^2-\varepsilon\nabla\|v\|^2+\varepsilon^2(\nabla u,\nabla v)\\
&\geq\frac{1}{2}\frac{d}{dt}\|\nabla v\|^2-\varepsilon\|\nabla v\|^2-\frac{\varepsilon^2}{2}\|\nabla u\|^2-\frac{\varepsilon^2}{2}\|\nabla v\|^2\\
&\geq\frac{1}{2}\frac{d}{dt}\|\nabla v\|^2-\varepsilon\|\nabla v\|^2-\frac{\varepsilon^2}{2\lambda_1}\|\Delta u\|^2-\frac{\varepsilon^2}{2}\|\nabla v\|^2,
\end{aligned}
\tag{3.23}
$$

we use $\alpha_1 u_t$ and $-\Delta v$ to inner product and we get

$$
\begin{aligned}
&(\alpha_1 u_t,-\Delta v)=\alpha_1(v-\varepsilon u,-\Delta v)\\
&=\alpha_1\|\nabla v\|^2+\alpha_1\varepsilon(u,\Delta v)\\
&\geq\alpha_1\|\nabla v\|^2-\frac{\alpha_1\varepsilon}{2}\|\nabla u\|^2-\frac{\alpha_1\varepsilon}{2}\|\nabla v\|^2,
\end{aligned}
\tag{3.24}
$$

and

$$
(-\gamma\Delta u_t,-\Delta v)=(-\gamma\Delta u_t,-\Delta u_t-\varepsilon\Delta u)=\gamma\|\Delta u_t\|^2+\frac{\gamma\varepsilon}{2}\frac{d}{dt}\|\Delta u\|^2,
\tag{3.25}
$$

we use $-\left(\alpha+\beta\|\nabla u\|^2\right)^\rho\Delta u$ and $-\Delta v$ to inner product and we obtain

$$
\begin{aligned}
&\left(-\left(\alpha+\beta\|\nabla u\|^2\right)^\rho\Delta u,-\Delta v\right)=-\left(\alpha+\beta\|\nabla u\|^2\right)^\rho(\Delta u,-\Delta v)\\
&=\left(\alpha+\beta\|\nabla u\|^2\right)^\rho(\Delta u,\Delta v)\\
&=\left(\alpha+\beta\|\nabla u\|^2\right)^\rho(\Delta u,\Delta u_t+\varepsilon\Delta u)\\
&=\left(\alpha+\beta\|\nabla u\|^2\right)^\rho(\Delta u,\Delta u_t)+\left(\alpha+\beta\|\nabla u\|^2\right)^\rho(\Delta u,\varepsilon\Delta u)\\
&=\left(\alpha+\beta\|\nabla u\|^2\right)^\rho(\Delta u,\Delta u_t)+\left(\alpha+\beta\|\nabla u\|^2\right)^\rho\cdot\varepsilon\|\Delta u\|^2,
\end{aligned}
\tag{3.26}
$$

take a right α,β, such that $0<C_2\leq\left(\alpha+\beta\|\nabla u\|^2\right)^\rho\leq C_3$, then

$$
\begin{aligned}
&\left(\alpha+\beta\|\nabla u\|^2\right)^\rho(\Delta u,\Delta u_t)+\left(\alpha+\beta\|\nabla u\|^2\right)^\rho\cdot\varepsilon\|\Delta u\|^2\\
&\geq-\frac{C_3}{2}\|\Delta u\|^2-\frac{C_3}{2}\|\Delta u_t\|^2+C_2\varepsilon\|\Delta u\|^2,
\end{aligned}
\tag{3.27}
$$

by using *Holder* inequality and *Young* inequality, we get

$$
(f(x),-\Delta v)\leq\|\nabla f\|\cdot\|\nabla v\|\leq\frac{\lambda_1\varepsilon_1}{4}\|\nabla v\|^2+\frac{1}{\lambda_1\varepsilon_1}\|\nabla f\|^2,
\tag{3.28}
$$

then we have

$$\frac{1}{2}\frac{d}{dt}\|\nabla v\|^2-\varepsilon\|\nabla v\|^2-\frac{\varepsilon^2}{2\lambda_1}\|\Delta u\|^2-\frac{\varepsilon^2}{2}\|\nabla v\|+\alpha_1\|\nabla v\|^2-\frac{\alpha_1\varepsilon}{2}\|\nabla u\|^2$$
$$-\frac{\alpha_1\varepsilon}{2}\|\nabla v\|^2+\gamma\|\Delta u_t\|^2+\frac{\gamma\varepsilon}{2}\frac{d}{dt}\|\Delta u\|^2-\frac{C_3}{2}\|\Delta u\|^2-\frac{C_3}{2}\|\Delta u_t\|^2+C_2\varepsilon\|\Delta u\|^2 \tag{3.29}$$
$$\le\frac{\lambda_1\varepsilon_1}{4}\|\nabla v\|^2+\frac{1}{\lambda_1\varepsilon_1}\|\nabla f\|^2,$$

reduction to

$$\frac{d}{dt}\left[\|\nabla v\|^2+\gamma\varepsilon\|\Delta u\|^2\right]+\left(2\alpha_1-2\varepsilon-\varepsilon^2-\alpha_1\varepsilon-\frac{\lambda_1\varepsilon_1}{2}\right)\|\nabla v\|^2$$
$$+\left(2C_2\varepsilon-\frac{\varepsilon^2}{\lambda_1}-C_3\right)\|\Delta u\|^2\le\frac{2}{\lambda_1\varepsilon_1}\|\nabla f\|^2+C_4, \tag{3.30}$$

take proper α_1,ε, such that

$$\begin{cases} b_1=2\alpha_1-2\varepsilon-\varepsilon^2-\alpha_1\varepsilon-\frac{\lambda_1\varepsilon_1}{2}\ge 0 \\ b_2=2C_2\varepsilon-\frac{\varepsilon^2}{\lambda_1}-C_3\ge 0, \end{cases}$$

taking $\delta_2=\min\left\{b_1,\frac{b_2}{\gamma\varepsilon}\right\}$

$$\frac{d}{dt}V(t)+\delta_2 V(t)\le\frac{2}{\lambda_1\varepsilon_1}\|\nabla f\|^2+C_4=C_5, \tag{3.31}$$

here $V(t)=\|\nabla v\|^2+\gamma\varepsilon\|\Delta u\|^2$, by using $Gronwall$, we get

$$V(t)\le V(0)e^{-\delta_2 t}+\frac{C_5}{\delta_2}\left(1-e^{-\delta_2 t}\right), \tag{3.32}$$

let $k=\min\{1,\varepsilon_1\varepsilon\}$, we have

$$\|(u,v)\|^2_{H^2\times H^1}=\|\nabla v\|^2+\|\Delta u\|^2\le\frac{W(0)}{k}e^{-\delta_2 t}+\frac{C_5}{\delta_2 k}\left(1-e^{-\delta_2 t}\right), \tag{3.33}$$

$$\overline{\lim_{t\to\infty}}\|(u,v)\|^2_{H^2\times H^1}\le\frac{C_5}{\delta_2 k}, \tag{3.34}$$

so, there exists $M_1>0$ and $t_2=t_2(\Omega)>0$, such that

$$\|(u,v)\|^2_{H^2\times H^1}=\|\nabla v(t)\|^2+\|\Delta u(t)\|^2\le M_1\ (t>t_2), \tag{3.35}$$

Theorem 3.1 (see[12]) $(u_0,u_1)\in H^2(\Omega)\times H_0^1(\Omega)$, $f\in H_0^1(\Omega)$,equation (1.1) exists a unique smooth solution $(u,v)\in L^\infty([0,+\infty);H^2(\Omega)\times H_0^1(\Omega))$.

Proof: By the method of Galerkin and Lemma1 –Lemma2 , we can easily obtain the existence of solutions. Next, we prove the uniqueness of solutions in detail.

Assume u, v are two solutions of equation, let $w = u - v$, then, the two equations subtract and obtain

$$w_{tt} + \alpha_1 w_t - \gamma \Delta w_t - \left(\alpha + \beta\|\nabla u\|^2\right)^\rho \Delta u + \left(\alpha + \beta\|\nabla v\|^2\right)^\rho \Delta v = 0, \qquad (3.36)$$

by multiplying the equation by w_t we get

$$\left(w_{tt} + \alpha_1 w_t - \gamma \Delta w_t - \left(\alpha + \beta\|\nabla u\|^2\right)^\rho \Delta u + \left(\alpha + \beta\|\nabla v\|^2\right)^\rho \Delta v, w_t\right) = 0, \qquad (3.37)$$

by use w_{tt} and w_t to inner product and we have

$$(w_{tt}, w_t) = \frac{1}{2}\frac{d}{dt}\|w_t\|^2, \qquad (3.38)$$

In a similar way, we get the following

$$(\alpha_1 w_t, w_t) = \alpha_1\|w_t\|^2, \qquad (3.39)$$

$$(-\gamma \Delta w_t, w_t) = \gamma\|\nabla w_t\|^2 \geq \gamma\lambda_1\|w_t\|^2, \qquad (3.40)$$

$$\begin{aligned}
&\left(-\left(\alpha + \beta\|\nabla u\|^2\right)^\rho \Delta u + \left(\alpha + \beta\|\nabla v\|^2\right)^\rho \Delta v, w_t\right) \\
&= \left(-\left(\alpha + \beta\|\nabla u\|^2\right)^\rho \Delta u + \left(\alpha + \beta\|\nabla u\|^2\right)^\rho \Delta v - \left(\alpha + \beta\|\nabla u\|^2\right)^\rho \Delta v + \left(\alpha + \beta\|\nabla v\|^2\right)^\rho \Delta v, w_t\right) \\
&= -\left(\alpha + \beta\|\nabla u\|^2\right)^\rho (\Delta w, w_t) + \left(-\left(\alpha + \beta\|\nabla u\|^2\right)^\rho \Delta v + \left(\alpha + \beta\|\nabla v\|^2\right)^\rho \Delta v, w_t\right) \\
&= \left(\alpha + \beta\|\nabla u\|^2\right)^\rho \cdot \frac{1}{4}\frac{d}{dt}\|\nabla w\|^2 + \left(\alpha + \beta\|\nabla u\|^2\right)^\rho \left(\Delta w, \frac{w_t}{2}\right) \\
&+ \left(-\left(\alpha + \beta\|\nabla u\|^2\right)^\rho \Delta v + \left(\alpha + \beta\|\nabla v\|^2\right)^\rho \Delta v, w_t\right) \\
&\geq \left(\alpha + \beta\|\nabla u\|^2\right)^\rho \cdot \frac{1}{4}\frac{d}{dt}\|\nabla w\|^2 - \frac{C_3}{2}\|\nabla w\|^2 - \frac{C_3}{2}\|\nabla w_t\|^2 \\
&+ \left(-\left(\alpha + \beta\|\nabla u\|^2\right)^\rho \Delta v + \left(\alpha + \beta\|\nabla v\|^2\right)^\rho \Delta v, w_t\right) \\
&\geq \left(\alpha + \beta\|\nabla u\|^2\right)^\rho \cdot \frac{1}{4}\frac{d}{dt}\|\nabla w\|^2 - \frac{C_3}{2}\|\nabla w\|^2 - \frac{C_3\lambda_1}{2}\|w_t\|^2 \\
&+ \left(-\left(\alpha + \beta\|\nabla u\|^2\right)^\rho \Delta v + \left(\alpha + \beta\|\nabla v\|^2\right)^\rho \Delta v, w_t\right),
\end{aligned}$$

by lemma 2 $0 < C_2 \leq \left(\alpha + \beta\|\nabla u\|^2\right)^\rho \leq C_3$, $0 < C_2 \leq \left(\alpha + \beta\|\nabla v\|^2\right)^\rho \leq C_3$, then

$$\left(\alpha + \beta\|\nabla v\|^2\right)^\rho \Delta v - \left(\alpha + \beta\|\nabla u\|^2\right)^\rho \Delta v \leq (C_3 - C_2)\Delta v,$$

$$\begin{aligned}
&\left(\left(C_3-C_2\right)\Delta v,w_t\right)=\int_\Omega\left(C_3-C_2\right)\Delta v\cdot w_t dx=\left(C_3-C_2\right)\int_\Omega \Delta v\cdot w_t dx\\
&\le\left(C_3-C_2\right)\left(\int_\Omega \Delta v^2 dx\right)^{\frac{1}{2}}\left(\int_\Omega w_t^{\ 2}dx\right)^{\frac{1}{2}}\\
&\le\left(C_3-C_2\right)\cdot C\cdot\left(\int_\Omega 1^2 dx\right)^{\frac{1}{4}}\left(\int_\Omega w_t^{\ 4}dx\right)^{\frac{1}{4}}\\
&\le\left(C_3-C_2\right)\cdot|\Omega|^{\frac{1}{4}}\|w_t\|_4\\
&\le C_0\|w_t\|_4\le C_0\|\nabla w_t\|^{\frac{n}{4}}\|w_t\|^{1-\frac{n}{4}}\\
&\le C_0\frac{\|\nabla w_t\|^2}{2}+C_0\frac{\|w_t\|^2}{2},
\end{aligned}\tag{3.41}$$

then

$$\begin{aligned}
&\left(-\left(\alpha+\beta\|\nabla u\|^2\right)^\rho\Delta u+\left(\alpha+\beta\|\nabla v\|^2\right)^\rho\Delta v,w_t\right)\\
&\ge\left(\alpha+\beta\|\nabla u\|^2\right)^\rho\cdot\frac{1}{4}\frac{d}{dt}\|\nabla w\|^2-\frac{C_3}{2}\|\nabla w\|^2-\frac{C_3\lambda_1}{2}\|w_t\|^2\\
&-C_0\frac{\|\nabla w_t\|^2}{2}-C_0\frac{\|w_t\|^2}{2}\\
&\ge\left(\alpha+\beta\|\nabla u\|^2\right)^\rho\cdot\frac{1}{4}\frac{d}{dt}\|\nabla w\|^2-\frac{C_3}{2}\|\nabla w\|^2-\frac{C_3\lambda_1}{2}\|w_t\|^2\\
&-C_0\lambda_1\frac{\|w_t\|^2}{2}-C_0\frac{\|w_t\|^2}{2},
\end{aligned}\tag{3.42}$$

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\|w_t\|^2+\alpha_1\|w_t\|^2+\gamma\lambda_1\|w_t\|^2+\left(\alpha+\beta\|\nabla u\|^2\right)^\rho\cdot\frac{1}{4}\frac{d}{dt}\|\nabla w\|\\
&-\frac{C_3}{2}\|\nabla w\|^2-\frac{C_3\lambda_1}{2}\|w_t\|^2-C_0\lambda_1\frac{\|w_t\|^2}{2}-C_0\frac{\|w_t\|^2}{2}\le 0,
\end{aligned}\tag{3.43}$$

$$\begin{aligned}
&\frac{d}{dt}\left[\|w_t\|^2+\frac{\left(\alpha+\beta\|\nabla u\|^2\right)^\rho}{2}\frac{d}{dt}\|\nabla w\|^2\right]\le\left(C_3\lambda_1+C_0\lambda_1+C_0-2\alpha_1-2\gamma\lambda_1\right)\\
&\|w_t\|^2+C_3\|\nabla w\|^2,
\end{aligned}\tag{3.44}$$

taking $\delta_3=\max\left\{C_3\lambda_1+C_0\lambda_1+C_0-2\alpha_1-2\gamma\lambda_1,\frac{2C_3}{\left(\alpha+\beta\|\nabla u\|^2\right)^\rho}\right\}$, we have

$$\frac{d}{dt}\left[\|w_t\|^2+\left(\alpha+\beta\|\nabla u\|^2\right)^\rho\cdot\frac{1}{2}\|\nabla w\|^2\right]\le\delta_3\left(\|w_t\|^2+\frac{\left(\alpha+\beta\|\nabla u\|^2\right)^\rho}{2}\|\nabla w\|^2\right),\tag{3.45}$$

by using Gronwall inequality , we get

$$\|w_t\|^2+\frac{\left(\alpha+\beta\|\nabla u\|^2\right)^\rho}{2}\|\nabla w\|^2\le\left(\|w_t\|^2+\frac{\left(\alpha+\beta\|\nabla u\|^2\right)^\rho}{2}\|\nabla w\|^2\right)e^{\delta_3 t}=0, \qquad (3.46)$$

so we have $w(t)\equiv 0$, the uniqueness is proved to be.

Theorem 3.2 Let F be a *Banach* space, and $S(t)$ are the semi-group operator on F .

$$S(t):F\to F, S(t+x)=S(t)\cdot S(x)(\forall x,t\ge 0), S(0)=I,$$

where I is a unit operator. Set $S(t)$ satisfy the follow conditions:

(1) $S(t)$ is bounded, namely $\forall R>0$, $\|u\|_F\le R$, it exists a constant $C(R)$, so that $\|S(t)u\|_F\le C(R)(\forall t\in[0,+\infty))$,

(2) It exists a bounded absorbing set $B_0\in F$, namely, $\forall B\subset B_0$, it exists a constant t_0 , so that

$$S(t)B\subset B_0(\forall t>t_0),$$

(3) When $t>0$, $S(t)$ is a completely continuous operator A .

Therefore, the semi-group operator $S(t)$ exists a compact global attractor.

Theorem 3.3 Under the assume of Theorem 1, equations have a global attractor

$$A=\omega(B_0)=\bigcap_{x\ge 0}\overline{\bigcup_{t\ge x}S(t)B_0},$$

here $B_0=\left\{(u,v)\in H^2\times H^1:\|(u,v)\|^2_{H^2\times H^1}=\|u\|^2_{H^2}+\|v\|^2_{H^1}\le M_0+M_1\right\},$

B_0 is the bounded absorbing set of $H^2(\Omega)\times H_0^1(\Omega)$ and satisfies

(1) $S(t)A=A,$

(2) $\lim_{t\to\infty}dist(S(t)B_0)=0$, here $B\subset H^2(\Omega)\times H_0^1(\Omega)$ and it is a bounded set,

$$dist(Y,Z)=\sup_{y\in Y}\inf_{z\in Z}\|y-z\|_{H^2\times H^1},$$

Proof: Under the conditions of theorem1, it exists the solution semi-group $S(t)$, where $Y = H^2(\Omega)\times H_0^1(\Omega)$, $S(t): H^2 \times H^1 \to H^2 \times H^1$,

(1) From Lemma1 – Lemma2, we can get that $\forall B \subset H^2(\Omega)\times H_0^1(\Omega)$ is a bounded set that includes in the ball $\left\{\|(u,v)\|_{H^2\times H^1} \le R\right\}$,

$$\|S(t)(u_0,v_0)\|_{H^2\times H^1}^2 = \|u\|_{H^2}^2 + \|v\|_{H^1}^2 \le \|u_0\|_{H^2}^2 + \|v_0\|_{H^1}^2 + C_4 \le R^2 + C_4 \left(\forall t \ge 0, (u_0,v_0)\in B\right),$$

this shows that $S(t)(t\ge 0)$ is uniformly bounded in $H^2(\Omega)\times H_0^1(\Omega)$,

(2) Furthermore, for any $(u_0,v_0)\in H^2(\Omega)\times H_0^1(\Omega)$, when $t \ge \max\{t_1,t_2\}$, we get

$$\|S(t)(u_0,v_0)\|_{H^2\times H^1}^2 = \|u\|_{H^2}^2 + \|v\|_{H^1}^2 \le M_0 + M_1.$$

So, we get B_0 is the bounded absorbing set.

(3) Since $H^2(\Omega)\times H_0^1(\Omega) \to H_0^1(\Omega)\times L^2(\Omega)$ is tightly embedded, which means that the bounded set in $H^2(\Omega) H_0^1(\Omega)$ is the tight set in $H_0^1(\Omega)\times L^2(\Omega)$,s o the semi-group operator $S(t)$ is completely continuous.

So, the semi-group operator $S(t)$ exists a compact global attractor A.

4 Exponential attractors

We will use the following notations, let V_1, V_2 are two Hilbert spaces, we have $V_2 \to V_1$, with dense and continuous injection, and $V_2 \to V_1$ is compact. Let $S(t)$ is a map from V_i into $V_i, i = 1,2$.

Definition 4.1 The semi-group $S(t)$ possesses a (V_2,V_1) compact attractor A. If it exists a compact set $A \subset V_1$, A attracts all bounded subsets of V_2, and is invariant under the action of $S(t)$, that is $S(t)A = A, \forall t \ge 0$.

Definition 4.2 A compact set M is called a $(V_2,V_1)-$ exponential attractor for the system $(S(t),B)$, if $A \subseteq M \subseteq B$ and

(1) $S(t)M \subseteq M, \forall t \ge 0,$

(2) M has finite fractal dimension $d_f(\mathrm{M}) < +\infty,$

(3)there exist positive constants m_0, m_1 such that $dist\left(S(t)B, M\right) \le m_0 e^{-m_1 t}, t > 0$, where $dist_{V_1}(A,B) = \sup_{x\in A}\inf_{y\in B}|x-y|_{V_1}$,

$B \subset V_1$ is a positive invariant set of $S(t)$.

Definition 4.3 $S(t)$ is said to satisfy the discrete squeezing property on B, if there exists and orthogonal projection p_N of rank equal to N such that for every u and v in B, either

$$\left|S(t_*)u - S(t_*)v\right|_{V_1} \le \delta |u-v|_{V_1}, \delta \in \left(0, \frac{1}{8}\right),$$

or

$$\left|Q_N\left(S(t_*)u - S(t_*)v\right)\right|_{V_1} \le \left|P_N\left(S(t_*)u - S(t_*)v\right)\right|_{V_1},$$

where $Q_N = I - P_N$.

Theorem 4.1 Assuming that

(1) $S(t)$ possesses a $(V_2, V_1)-$ compact attractor A,

(2) it exists a positive invariant compact set $B \subset V_1$ of $S(t)$,

(3) $S(t)$ Is a Lipschitz continuous map with Lipschitz constant l on B, and satisfies the discrete squeezing property on B.

Then $S(t)$ has a $(V_2, V_1)-$ exponential attractor $A \subset M$ on B, and

$$M = \bigcup_{0\le t\le t_*} S(t)M_*, M_* = A \cup \left(\bigcup_{j=1}^{\infty}\bigcup_{k=1}^{\infty} S(t_*)^j (E)^k\right),$$

moreover, the fractal dimension of M satisfies $d_f(\mathrm{M}) \le N_0 \max\left\{1, \frac{\ln(16L_B+1)}{\ln 2}\right\}$,

where N_0, $E^{(k)}$ are defined as in.

Theorem 4.2 $(u_0, v_0) \in V_k, k = 1,2$, then the problem (1.1)-(1.3) admits a unique solution $(u,v) \in L^\infty(R^+, V_k)$. This solution possesses the following properties:

$$\|(u,v)\|_{V_1}^2 = \|\nabla u\|^2 + \|v\|^2 \le M_0, \|(u,v)\|_{V_1}^2 = \|\nabla v\|^2 + \|\Delta u\|^2 \le M_1, t \ge t_k, k = 1,2.$$

we denote the solution in Theorem 4.1, by $S(t)(u_0,v_0)=\left(u_{(t)},v_{(t)}\right)$, then $S(t)$ is a continuous semi-group in V_1, we have the ball:

$$B_1=\left\{(u,v)\in V_1:\|(u,v)\|_{V_1}^2\le M_0\right\},$$

respectively is a absorbing set of $S(t)$ in V_1 and V_2.

We notice, there exists $t_0(B_2)$ such that $B=\overline{\bigcup_{0\le t\le t_0(B_2)} S(t)B_2}$ is a positive invariant compact set of $S(t)$ in V_1, and absorbs all of the bounded subsets of V_2. According to and Theorem 4.1,we can have the semi-group $\{S(t)\}_{t\ge 0}$ possesses $(V_2,V_1)-$ compact attractor

$$A=\bigcap_{s\ge 0}\overline{\bigcup_{t\ge s} S(t)B_2},$$

where the bar means the closures in V_1 and A is bounded in V_2.

Lemma 4.1 For any $U=(u,v)\in V_1$, we have $(H(U),U)\ge\delta_4\|U\|_{V_1}^2+\delta_5\|\nabla v\|^2$.

Proof: By (2.1) and (2.2), we have

$$\begin{aligned}(H(U),U)_{V_1}&=\varepsilon\|\nabla u\|^2-(\nabla v,\nabla u)+(\alpha_1-\varepsilon)\|v\|^2+\left(\varepsilon^2-\varepsilon\alpha_1\right)(u,v)\\&+\gamma\varepsilon(\Delta u,v)+\gamma\|\nabla v\|^2+\left(\alpha+\beta\|\nabla u\|^2\right)^\rho(\nabla u,\nabla v),\end{aligned}\tag{4.1}$$

By using Holder inequality, Young inequality and Poincare inequality, we deal with the terms in (4.1) one by as follows

$$\begin{aligned}&\left(\varepsilon^2-\varepsilon\alpha_1\right)(u,v)\ge\left(\varepsilon^2-\varepsilon\alpha_1\right)\lambda_1^{-\frac{1}{2}}\|\nabla u\|\|v\|\ge-\varepsilon\alpha_1\lambda_1^{-\frac{1}{2}}\left(\frac{\lambda_1^{\frac{1}{2}}}{4\alpha_1}\|\nabla u\|^2+\alpha_1\lambda_1^{-\frac{1}{2}}\|v\|^2\right)\\&\ge-\frac{\varepsilon}{4}\|\nabla u\|^2-\frac{\varepsilon\alpha_1^2}{\lambda_1}\|v\|^2,\end{aligned}\tag{4.2}$$

$$\beta\varepsilon(\Delta u,v)=-\beta\varepsilon(\nabla u,\nabla v)\ge-\frac{\varepsilon}{4}\|\nabla u\|^2-\beta^2\varepsilon\|\nabla v\|^2,\tag{4.3}$$

$$\left(\left(\alpha+\beta\|\nabla u\|^2\right)^\rho-1\right)(\nabla u,\nabla v)\ge C(\nabla u,\nabla v)\ge-\frac{C}{2}\|\nabla u\|^2-\frac{C}{2}\|\nabla v\|^2\tag{4.4}$$

substituting (4.2), (4.3), (4.4) into (4.1) available

$$\left(H(U),\,U\right)_{V_1} \ge \varepsilon\|\nabla u\|^2 + (\alpha_1-\varepsilon)\|v\|^2 + \gamma\|\nabla v\|^2 - \frac{\varepsilon}{4}\|\nabla u\|^2 - \frac{\varepsilon\alpha_1{}^2}{\lambda_1}\| \,\| v$$

$$-\frac{\varepsilon}{4}\|\nabla u\|^2 - \gamma^2\varepsilon\|\nabla v\|^2 - \frac{C}{2}\|\nabla u\|^2 - \frac{C}{2}\|\nabla v\|^2 \qquad (4.5)$$

$$= \left(\frac{\varepsilon}{2}-\frac{C}{2}\right)\|\nabla u\|^2 + \left(\alpha_1 - \varepsilon - \frac{\varepsilon\alpha_1{}^2}{\lambda_1}\right)\|v\|^2 + \left(\gamma - \gamma^2\varepsilon - \frac{C}{2}\right)\|\nabla v\|^2,$$

when $\frac{\varepsilon}{2}-\frac{C}{2}\ge 0$, $\alpha_1-\varepsilon-\frac{\varepsilon\alpha_1{}^2}{\lambda_1}\ge 0$, $\gamma-\gamma^2\varepsilon-\frac{C}{2}\ge 0$, that is $C\le\varepsilon\le\frac{\lambda_1\alpha}{\lambda_1+\alpha^2}$,

let $\delta_4=\min\left\{\frac{\lambda_1\alpha_1}{\lambda_1+\alpha_1{}^2},\alpha_1-\varepsilon-\frac{\varepsilon\alpha_1{}^2}{\lambda_1}\right\},\delta_5=\gamma-\gamma^2\varepsilon-\frac{C}{2}$, then we obtain

$$\left(H(U),U\right)\ge\delta_4\|U\|_{V_1}^2+\delta_5\|\nabla v\|^2,$$

let $S(t)U_0=U(t)=\left(u(t),v(t)\right)^T$, where $v=u_t(t)+\varepsilon u(t)$, and

$$S(t)V_0=V(t)=\left(\bar{u}(t),\bar{v}(t)\right)^T,$$

set

$$\varphi(t)=S(t)U_0-S(t)V_0=U(t)-V(t)=\left(\omega(t),y(t)\right)^T,$$

where $y(t)=\omega_t(t)+\varepsilon\omega(t)$,

$$\varphi_t(t)+H(U)-H(V)=0, \qquad (4.6)$$

and

$$\varphi(0)=U_0-V_0. \qquad (4.7)$$

Lemma 4.2 (see[17]) (Lipschitz property) For any $U_0,V_0\in B,T\ge 0,$ we have

$\|S(t)U_0-S(t)V_0\|_{V_1}^2\le e^{\delta t}\|U_0-V_0\|_{V_1}^2$.

Proof: Taking the inner product of the equations (4.5) with $\varphi(t)$ in V_1, we find that

$$\frac{1}{2}\frac{d}{dt}\|\varphi(t)\|^2+\left(H(U)-H(V),\varphi(t)\right)=0, \qquad (4.8)$$

similar to Lemma 4.1, we get

$$\left(H(U)-H(V),\varphi(t)\right)_{V_1}\ge\delta_4\|\varphi(t)\|_{V_1}^2+\delta_5\|\nabla y(t)\|_{V_1}^2, \qquad (4.9)$$

substituting (4.9) into (4.8), we have

$$\frac{d}{dt}\|\varphi(t)\|^2+2\delta_4\|\varphi(t)\|^2+2\delta_5\|\nabla y(t)\|^2\le 0, \tag{4.10}$$

further, we have

$$\frac{d}{dt}\|\varphi(t)\|^2\le -2\delta_4\|\varphi(t)\|^2, \tag{4.11}$$

by using Gronwall inequality, we obtain

$$\|\varphi(t)\|^2\le e^{-2\delta_4}\|\varphi(0)\|^2=e^{\delta t}\|\varphi(0)\|^2, \tag{4.12}$$

so we have

$$\|S(t)U_0-S(t)V_0\|_{V_1}^2\le e^{\delta t}\|U_0-V_0\|_{V_1}^2.$$

We introduce the operator $A=-\Delta$ from $D(A)$ to H with domain

$$D(A)=\{u\in H\,|\,Au\in H\}=\{u\in H^2\,|\,u|_{\partial\Omega}=\nabla u|_{\partial\Omega}=0\},$$

Obviously, A is an unbounded self-adjoint positive operator and the inverse A^{-1} is compact. And thus there exists an orthnormal basis $\{\omega_i\}_{i=1}^{\infty}$ of H consisting of eigenvectors of A, such that

$$Aw_i=\lambda_i w_i,$$
$$0<\lambda_1\le\lambda_2\le\cdots\le\lambda_i\to+\infty,$$

$\forall N$ denote by

$$P=P_n:H\to span\{w_1,\cdots,w_N\},$$

the projector

$$Q=Q_N=I-P_N,$$

In the following, we will use

$$\|Au\|=\|\Delta u\|\ge\lambda_{m+1}\|u\|,\forall u\in Q_m\left(H^2(\Omega)\cap H_0^1(\Omega)\right),$$
$$\|Q_m u\|\le\|u\|,u\in H.$$

Lemma 4.3 $\forall U_0,V_0\in B,$ let $Q_{m_0}(t)=Q_{m_0}(U(t)-V(t))=Q_{m_0}\varphi(t)=(w_{m_0},y_{m_0})^T$, then

$$\left\|\varphi_{m_0}(t)\right\|_{V_1}^2 \le \left(e^{-2k_1 t} + \frac{c_2 \lambda_{m_0+1}^{-\frac{1}{2}}}{2k_1 + k} e^{kt} \right) \left\|\varphi(0)\right\|^2 .$$

Proof: Taking $Q_{m_0}(t)$ in (4.6), we get

$$\varphi_{m_{0t}}(t) + Q_{m_0}\left(H(U) - H(V)\right) = 0, \tag{4.13}$$

taking the inner product of (4.3) with $\varphi_{m_0}(t)$ in V_1, we find that

$$\frac{1}{2}\frac{d}{dt}\left\|\varphi_{m_0}(t)\right\|^2 + \delta_4 \left\|\varphi_{m_0}(t)\right\|^2 + \delta_5 \left\|\nabla y_{m_0}(t)\right\|^2 \le 0, \tag{4.14}$$

$$\frac{d}{dt}\left\|\varphi_{m_0}(t)\right\|^2 \le -2\delta_4 \left\|\varphi_{m_0}(t)\right\|^2, \tag{4.15}$$

by using Gronwall inequality, we obtain

$$\left\|\varphi_{m_0}(t)\right\|^2 \le e^{-2\delta_4 t} \left\|\varphi(0)\right\|^2 . \tag{4.16}$$

Lemma 4.4 (Discrete squeezing property) For any $U_0, V_0 \in B$, if
$\left\|P_{m_0}\left(S(T^*)U_0 - S(T^*)V_0\right)\right\|_{V_1} \le \left\|(I - P_{m_0})\left(S(T^*)U_0 - S(T^*)V_0\right)\right\|_{V_1}$, then $\left\|\left(S(T^*)U_0 - S(T^*)V_0\right)\right\|_{V_1} \le \frac{1}{8}\left\|U_0 - V_0\right\|_{V_1}$.

Proof: If $\left\|P_{m_0}\left(S(T^*)U_0 - S(T^*)V_0\right)\right\|_{V_1} \le \left\|(I - P_{m_0})\left(S(T^*)U_0 - S(T^*)V_0\right)\right\|_{V_1}$,

then

$$\begin{aligned}
&\left\|S(T^*)U_0 - S(T^*)V_0\right\|^2 \le \left\|(I - P_{m_0})\left(S(T^*)U_0 - S(T^*)V_0\right)\right\|_{V_1}^2 \\
&+ \left\|P_{m_0}\left(S(T^*)U_0 - S(T^*)V_0\right)\right\|_{V_1}^2 \\
&\le 2\left\|(I - P_{m_0})\left(S(T^*)U_0 - S(T^*)V_0\right)\right\|_{V_1}^2 \\
&\le 2e^{-2\delta_4 T^*}\left\|U_0 - V_0\right\|^2 ,
\end{aligned} \tag{4.17}$$

let T^* be large enough

$$e^{-2\delta_4 T^*} \le \frac{1}{128}, \tag{4.18}$$

substituting (4.17) into (4.18), we have

$$\left\|\left(S(T^*)U_0 - S(T^*)V_0\right)\right\|_{V_1} \le \frac{1}{8}\left\|U_0 - V_0\right\|_{V_1} . \tag{4.19}$$

Lemma 4.4 is proved.

Theorem 4.3 $(u_0, v_0) \in V_k, k = 1,2$, $f \in H$, $v = u_t + \varepsilon u$, then the initial boundary value problem (1.1)-(1.3) the solution semi-group $S(t)$ has a $(V_2,V_1)-$ exponential attractor on B,

$$\mathrm{M}=\bigcup_{0\le t\le T^*} S(t)\left(\mathrm{A}\cup\left(\bigcup_{j=1}^{\infty}\bigcup_{k=1}^{\infty} S\left(T^*\right)^j\left(E^{(k)}\right)\right)\right),$$

and the fractal dimension is satisfied

$$d_f(\mathrm{M})\le N_0\max\left\{1,\frac{\ln\left(16L_B+1\right)}{\ln 2}\right\}.$$

Proof: According to Theorem 4.1, Lemma 4.2, Lemma 4.4, Theorem 4.2 are easily proven.

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