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Symbolic Analytical Expressions for the Solution of Hyperbolic form of Kepler's Equation

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Abstract

In this paper, symbolic analytical expressions for the solution of hyperbolic form of Kepler's equation will be established .Mathematica procedure for the expressions is also established to together with some of its output.

Keywords Space dynamics; hyperbolic form of Kepler's equation; symbolic computations

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1- INTRODUCTION

The hyperbolic form of Kepler's equation plays an important role in dynamical astronomy. Many instances of hyperbolic orbits occur in the solar system and recently, among the artificial satellites, lunar and solar probes. .

The hyperbolic orbits not only exist naturally, but can also be used to solve some critical orbital situations [1].

Due to the importance of the hyperbolic form of Kepler's equation as mentioned briefly in the above, it now urgent needed to establish accurate expressions for its solution. Towards this goal the present paper is devoted.

To achieve this goal, we established symbolic analytical expressions for the solutions, because, the analytical formulae are usually offering much deeper insight into the nature of the problems to which they refer On the other hand, these expressions are obtained from utilizing exact theorems for their formulations. Mover, the symbolic analytical expressions could also be implemented for digital computations. Finally a Mathematica procedure for the expressions is also established together with some of its output.

2-BASICFORMULATIONS

2-1 Hyperbolic form of Kepler's equation

The position–time relation in hyperbolic orbits is known as Kepler's equation for the hyperbolic case and is written as:

$$M = e\sinh H - H\,, \qquad (1)$$

Where

$$M = \sqrt{\frac{\mu}{(-a)^3}}(t-\tau) = n(t-\tau). \qquad (2)$$

The quantity M, called the mean anomaly H the hyperbolic eccentric anomaly, e the eccentricity of the orbit $(e \succ 1)$, μ the gravitational parameter, n the mean motion a the semi-major axis of the orbit ($a \prec 0$), t the time, and τ is the time of passage through pericenter.

2-2 Lagrange Expansion Theorem

Consider the functional equation

$$y = x + \alpha\,\varphi(y)\;, \qquad (3)$$

where α is to be considered a small parameter – originally identified with a planetary eccentricity .Then y could be expanded in terms of x and α as

$$y = x + \sum_{n=1}^{\infty}\frac{\alpha^n}{n!}\frac{d^{n-1}}{dx^{n-1}}[\varphi(x)]^n\,. \qquad (4)$$

Lagrange's series is, of course, the Taylor series representation of the root of the functional equation $y - x - \alpha\,\varphi(x) = 0$ Sufficient conditions for a unique root are obtained by a direct application of Rouche's theorem for analytical function of a complex variable [2].

3- ANALYTICAL SOLUTION FOR HYPERBOLIC FORM OF KEPLER'SEQUATION

Write the hyperbolic form of Kepler's Equation (1) as:

$$\sinh H = \frac{M}{e} + \frac{1}{e}H\,, \qquad (5)$$

with,

$$y = \sinh H \quad ; \quad x = \frac{M}{e} \quad ; \quad \alpha = \frac{1}{e} \quad ; \quad \Phi(y) = \sinh^{-1} y$$

Lagrange's expansion theorem of Equation (4) yields:

$$\sinh H = \frac{M}{e} + \sum_{n=1}^{\infty}\frac{1}{n!}\left(\frac{1}{e}\right)^n\frac{d^{n-1}}{dx^{n-1}}[\sinh^{-1}x]^n\,. \qquad (6)$$

Up to the fourth order in $(1/e)$, we can write the last equation as:

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$$\sinh H=\frac{M}{e}+\frac{1}{e}[\sinh^{-1}x]+\frac{1}{2e^2}\frac{d}{dx}[\sinh^{-1}x]^2+\frac{1}{6e^3}\frac{d^2}{dx^2}[\sinh^{-1}x]^3+\frac{1}{24e^4}\frac{d^3}{dx^3}[\sinh^{-1}x]^4$$

(7)

Now let us evaluate each term of the equation

- $\frac{1}{e}\sinh^{-1}x=\frac{1}{e}\sinh^{-1}\frac{M}{e}=\frac{A}{e}$ (8-1)

- $\frac{1}{2e^2}\frac{d}{dx}[\sinh^{-1}x]^2=\frac{1}{2e^2}\frac{2\sinh^{-1}x}{\sqrt{1+x^2}}=\frac{1}{e^2}\frac{Ae}{\sqrt{M^2+e^2}}=\frac{A}{eB}$ (8-2)

- $T_1=\frac{1}{6e^3}\frac{d^2}{dx^2}[\sinh^{-1}x]^3 \xrightarrow{\text{Let } Q=[\sinh^{-1}x]^3} Q'=\frac{3[\sinh^{-1}x]^2}{\sqrt{1+x^2}}=\frac{3eA^2}{B}\Rightarrow$

$$Q''=3e\frac{2AA'B-B'A^2}{B^2}.$$

Where the prime is the derivatives w. r. t. x. Then,

$$A'=\frac{1}{\sqrt{1+x^2}}=\frac{e}{B},$$

$$B'=\frac{ex}{\sqrt{1+x^2}}=\frac{Ne}{B}.$$

Then,

$$Q''=\frac{3e}{B^2}\left\{2eA-\frac{MeA^2}{B}\right\}=\frac{3Ae^2}{B^2}\left\{2-\frac{MA}{B}\right\}\Rightarrow$$

$$T_1=\frac{A}{2eB^2}\left\{2-\frac{MA}{B}\right\} \qquad \text{(8-3)}$$

- $\frac{1}{24e^4}\frac{d^3}{dx^3}[\sinh^{-1}x]^4 \xrightarrow{\text{Let } G=[\sinh^{-1}x]^4} G'=\frac{4[\sinh^{-1}x]^3}{\sqrt{1+x^2}}=\frac{4eA^3}{B}\Rightarrow$

$$G''=4e\frac{3A^2A'B-B'A^3}{B^2}=\frac{4e}{B^2}\left(3A^2\frac{e}{B}B-A^3\frac{Me}{B}\right)=\frac{4A^2e^2}{B^2}\left[3-\frac{AM}{B}\right]=C_1+C_2$$

Where

$$C_1=\frac{12e^2A^2}{B^2} \quad ; \quad C_2=-\frac{4e^2MA^3}{B^3}$$

then

$$G'''=C_1'+C_2'$$

$$\because \ln C_1=\ln 12+2\ln A+2\ln e-2\ln B\Rightarrow\frac{C_1'}{C_1}=\frac{2A'}{A}-\frac{2B'}{B}\Rightarrow$$

$$C_1'=\frac{24A^2e^3}{B^3}\left[\frac{1}{A}-\frac{M}{B}\right]$$

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Let $T = MA^3e^2/B^3 \Rightarrow \ln T = 3\ln A + \ln M + 2\ln e - 3\ln B$

Since,

$$M = xe \Rightarrow M' = e$$

Then,

$$T' = \frac{MA^3e^2}{B^3}\left\{3\frac{A'}{A} + \frac{M'}{N} - 3\frac{B'}{B}\right\} = \frac{MA^3e^2}{B^3}\left\{\frac{3e}{BA} + \frac{e}{M} - \frac{3Me}{B^2}\right\}$$

Hence,

$$C_2' = \frac{-4Me^3A^3}{B^3}\left\{\frac{3}{BA} + \frac{1}{M} - \frac{3M}{B^2}\right\} \Rightarrow$$

$$\frac{1}{24e^4}\frac{d^3}{dx^3}[\sinh^{-1}x]^4 = \frac{1}{24e^4}G''' = \frac{1}{24e^4}\{C_1' + C_2'\} =$$

$$= \frac{A^2}{eB^3}\left[\frac{1}{A} - \frac{M}{B}\right] - \frac{A^3M}{6eB^2}\left[\frac{3}{AB} + \frac{1}{M} - \frac{3M}{B^2}\right] \Rightarrow$$

$$\frac{1}{24e^4}\frac{d^3}{dx^3}[\sinh^{-1}x]^4 = \frac{A\ M}{6eB^3}\left[6 - A^2 - \frac{9AM}{B} + \frac{3A^2M^2}{B^2}\right] \qquad (8\text{-}4)$$

Using Equations (8) into Equation (7) we get:

$$\sinh H = \frac{M}{e} + \frac{A}{e} + \frac{A}{eB} + \frac{A}{2eB^2}\left(2 - \frac{AM}{B}\right) + \frac{A}{6eB^3}\left(6 - A^2 - \frac{9AM}{B} + \frac{3A^2M^2}{B^2}\right) + \cdots$$

By using Equation (5) for the left hand side of the above equation, we get the Analytical Solution for Hyperbolic form of Kepler's Equation on the form

$$H = A + \frac{A}{B} + \frac{A}{2B^2}(2 - \frac{AM}{B}) + \frac{A}{6B^3}(6 - A^2 - \frac{9AM}{B} + \frac{3A^2M^2}{B^2}) + \cdots \qquad (9)$$

Where,

$$A = \sinh^{-1}\frac{M}{e} = \ln\frac{M+B}{e} \qquad (10)$$

And,

$$B = \sqrt{M^2 + e^2} \qquad (11)$$

4- MATHEMATICA PROCEDURE

Equation (2) could be written as:

$$y = \sum_{j=1}^{\infty} G_j A^j \ , \qquad (12)$$

Where,

$$A = \sinh^{-1}\left(\frac{M}{e}\right) \ .$$

The following procedure computes the G's coefficients.

4-1 Mathimatica procedure: KeplerHypSeries

- **Purpose**

To find analytical expression for the coefficient G_j of $\sinh^{-1}\left(\frac{M}{e}\right)$ in the m terms series solution for Kepler's hyperbolic orbits equation.

Input

1- m (integer) the maximum number of terms for the series solution of Kepler's equation for hyperbolic orbits

2- s (integer) $\leq m$ is number of the required term of the series.

- **Output**

G's coefficients

- **Needed procedures**

None

- **List of the procedure**

$$KeplerHypSeries[m\_, s\_] := [\{\ \}, \Phi = ArcSinh[x]; Q = Table[D[\Phi^n, \{x, n-1\}], \{n, 1, m\}]; y = \sum_{n=1}^{m} \frac{1}{n!} * \left(\frac{1}{e}\right)^{n-1} * \frac{Q[[n]]}{.\left\{x \to \frac{m}{e}, ArcSinh[x] \to A\right\}}; P_i := Coefficient[y, A^i]//FullSimplify;\ Do[Print[G_i, " = ", P_j].[j, 1, s\}]$$

4-2 List of G's coefficients

$$G_1=\frac{(1+e^2+M^2)\left(1+(e^2+M^2)^2\right)\left(1+(e^2+M^2)^4\right)\left(M^4+e\left(e+\sqrt{1+\frac{M^2}{e^2}}\right)\right)}{(e^2+M^2)^8}$$

$$G_2=-\frac{1}{2(e^2+M^2)^8}\Big(M(105+78(e^2+M^2)+55(e^2+M^2)^2+36(e^2+M^2)^3+21(e^2+M^2)^4+10(e^2+M^2)^5+3(e^2+M^2)^6$$

$$-e\sqrt{1+\frac{M^2}{e^2}}\left(-91-66(e^2+M^2)-45(e^2+M^2)^2-28(e^2+M^2)^3-15(e^2+M^2)^4-6(e^2+M^2)^5-(e^2+M^2)^6\right)\Big)\Big)$$

$$G_3=\frac{1}{6e(e^2+M^2)^8\sqrt{1+\frac{M^2}{e^2}}}\Big(-e^{14}5005M^2-2e^{12}(5+2M^2)-4e^{13}\sqrt{1+\frac{M^2}{e^2}}-e^{11}(20+9M^2)\sqrt{1+\frac{M^2}{e^2}}+e^9\sqrt{1+\frac{M^2}{e^2}}(-56+5M^2+15M^4)$$

$$+2e^7\sqrt{1+\frac{M^2}{e^2}}(-60+77M^2+110M^4+35M^6)+e^6(-165+294M^2+490M^4+250M^6+25M^8)+$$

$$2e^5\sqrt{1+\frac{M^2}{e^2}}(-110+315M^2+399M^4+215M^6+45M^8)+M^4(2717+1320M^2+546M^4+175M^6+35M^8+2M^{10})+$$

$$eM^2\sqrt{1+\frac{M^2}{e^2}}(3731+1925M^2+870M^4+322M^6+85M^8+11M^{10})$$

$$+e^3\sqrt{1+\frac{M^2}{e^2}}(-364+1705M^2+1620M^4+910M^6+320M^8+51M^{10})$$

$$-e^2(-455+2431M^2+2475M^4+1554M^6+665M^8+165M^{10}+11M^{12})-e^{10}\left(35+3M^2(5+M^2)\right)$$

$$+e^8\left(-84+5M^2(7+M^2)(1+2M^2)\right)+2e^4\left(-143+M^2\left(495+M^2(9+M^2)(77+42M^2+12M^4)\right)\right)\Big)$$

$$G_4 = \frac{1}{24} M(\frac{1}{(e^2+M^2)^9}\left(15M^2\left(-12376 - 5005(e^2+M^2) - 1716(e^2+M^2)^2 - 462(e^2+M^2)^3 - 84(e^2+M^2)^4 - 7(e^2+M^2)^5 - 8008e\sqrt{1+\frac{M^2}{e^2}} - 3003(e^2+M^2)\sqrt{1+\frac{M^2}{e^2}} - 924e(e^2+M^2)^2\sqrt{1+\frac{M^2}{e^2}} - 210e(e^2+M^2)^3\sqrt{1+\frac{M^2}{e^2}} - 28e(e^2+M^2)^4\sqrt{1+\frac{M^2}{e^2}} - e(e^2+M^2)^5\sqrt{1+\frac{M^2}{e^2}}\right) + \frac{1}{(e^2+M^2)^8}(42315 + 19305(e^2+M^2) + 7590(e^2+M^2)^2 + 2349(e^2+M^2)^3 + 525(e^2+M^2)^4 + 55(e^2+M^2)^5 + 29029e\sqrt{1+\frac{M^2}{e^2}} + 12375e(e^2+M^2)\sqrt{1+\frac{M^2}{e^2}} 4410e(e^2+M^2)^2\sqrt{1+\frac{M^2}{e^2}} + 1190e(e^2+M^2)^3\sqrt{1+\frac{M^2}{e^2}} + 195e(e^2+M^2)^4\sqrt{1+\frac{M^2}{e^2}} + 9e(e^2+M^2)^5\sqrt{1+\frac{M^2}{e^2}}\right))$$

In conclude the present paper, we stress that, symbolic analytical expressions for the solution of hyperbolic form of Kepler's equation are established .Mathematica procedure for the expressions is also established to together with some of its output.

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