Explicit Analytic Solution of Vibration Equation for large domain by mean of the Elzaki projected Differential Transform Method

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Abstract

The aim of this paper is to present a reliable and efficient algorithm Elzaki projected differential transform method (EPDTM) to obtain the explicit solution of vibration equation for a very large membrane with given initial conditions. By using initial conditions, explicit series solutions for six different cases have been derived for the fast convergence of the solution. Numerical results show the reliability, efficiency and accuracy of Elzaki projected differential transform method (EPDTM). Numerical results for the six different cases are presented graphically.

Keywords

Elzaki projected differential transform method; explicit series solution; vibration equation; large membrane; initial condition

Academic Discipline And Sub-Disciplines

Applied Mathematics

Subject Classification

35B40, 35Q55, 76D05

1. Introduction

In many problems arises in area of science and engineering for large membranes, vibration analysis plays an important role to determine the properties and behavior of vibrations. Vibration arises in music, acoustics membranes, microphones, speaker and numerous other devices. Human tissues and eardrum also shows vibrational characteristics and hearing aid devices are designed after understanding the vibrational behavior of membranes. Linear combination of the modes of structure can be used to explained vibrations. Alternatively propagation of wave travelling in a membrane structure, vibration can also cause the destruction of membrane structure in engineering, so characteristics of vibration of membrane and its dynamic response under the effect of external force become a great important scientific issue and number of researchers studied propagation, transmission and reflection of vibrations, like Tapaswini and Chakraverty studied non probabilistic solution of vibration equation using ADM [10], Yidrim studied the solution of vibration equation of a large membrane using HPM [7], Mohyud-din and Yildrim studied and analyzed the fractional vibrational equation for large membrane [9], further can studied in literature. In this paper we apply Elzaki projected differential transform method (EPDTM) [1, 2, 3] to solve the vibration equation and different cases has been discussed, numerical and graphical results are found with the help of Maple. In section 2, basic idea of Elzaki transform and projected differential transform method is explained. Solution of the problem can be studied in section 3 and some results and conclusion are discussed in section 4.

Basic idea of the method

1.1 Elzaki Transformation

Elzaki transform was introduced by Tarig. M. Elzaki in [3]. From the classical Fourier integral, Like Sumudu transform, Laplace transform and Fourier transform, Elzaki transform is used to simplify the process of solving ordinary and partial differential equations in the time domain. Mathematical formulation of Elzaki transformation is as follows:

$$A = \{\ f(t)\ :\ \exists M, k_1, k_2 > 0, |f(t)| < Me^{\frac{|t|}{k}}, if\quad t \in (-1)^j * [0, \infty)\} \tag{1}$$

For a given function in set A, the constant M must be finite number k_1, k_2 maybe finite or infinite. Elzaki transform is denoted by E (.) and defined by the integral equation,

$$E[f(t)] = T(v) = v\int_0^\infty f(t)\ e^{\frac{-t}{v}} dt, \qquad t > 0, k_1 \le v \le k_2, \tag{2}$$

Where variable 'v' is used in the transformation to the factor the variable't' in the argument of the function.

2.2 Projected Differential Transform Method

The basic idea of Projected Differential Transform Method (PDTM) was given by Tarig. Elzaki in [1, 2], by letting f (u_1, $u_{2,\dots}$, u_n) is defined as

$$f(u_1, u_2, \dots u_n) = \frac{1}{k!} [\frac{\partial^k f(u_1, u_2, \dots u_n)}{\partial u_n^k}]_{u_n = 0} \tag{3}$$

such that $f(u_1, u_2, \dots u_n)$ is the given function and $f(u_1, u_2, \dots u_{n-1}, k)$ is the projected transformed function, and the Inverse differential transform of $f(u_1, u_2, \dots u_{n-1}, k)$ is defined as

$$f(u_1, u_2, \dots u_n) = \sum_{k=0}^\infty f(u_1, u_2, \dots u_{n-1}, k)(u - u_0)^k \tag{4}$$

Some fundamental theorems [1] is given below

(1) If $f(u_1, u_2, \dots u_n) = \phi(u_1, u_2, \dots u_n) \pm \psi(u_1, u_2, \dots u_n)$

then $z(u_1, u_2, \dots u_{n-1}, k) = \phi(u_1, u_2, \dots u_{n-1}, k) \pm \psi(u_1, u_2, \dots u_{n-1}, k)$

(2) If $f(u_1, u_2, \dots u_n) = c\phi(u_1, u_2, \dots u_n)$

then $f(u_1, u_2, \dots u_{n-1}, k) = c\phi(u_1, u_2, \dots u_{n-1}, k)$

(3) If $f(u_1, u_2, \dots u_n) = \frac{d\phi(u_1, u_2, \dots u_n)}{du_n}$

then $f(u_1, u_2, \dots u_{n-1}, k) = (k+1)\phi(u_1, u_2, \dots u_{n-1}, k+1)$

(4) If $f(u_1, u_2, ... u_n) = \dfrac{d^n \phi(u_1, u_2, ... u_n)}{du_n^n}$

then $f(u_1, u_2, ... u_{n-1}, k) = \dfrac{k+n)!}{k!} \phi(u_1, u_2, ... u_{n-1}, k+n)$

(5) If $f(u_1, u_2, ... u_n) = \phi(u_1, u_2, ... u_n) \ \psi(u_1, u_2, ... u_n)$

then $f(u_1, u_2, ... u_{n-1}, k) = \sum\limits_{m=0}^{k} \phi(u_1, u_2, ... u_{n-1}, m) \psi(u_1, u_2, ... u_{n-1}, k-m)$

(6) If $f(u_1, u_2, ... u_n) = \phi_1(u_1, u_2, ... u_n) \phi_2(u_1, u_2, ... u_n) ..... \phi_n(u_1, u_2, ... u_n)$

then

$$f(u_1, u_2, ... u_{n-1}, k) = \sum\limits_{k_{n-1}=0}^{k} \sum\limits_{k_{n-2}=0}^{k_{n-1}} .... \sum\limits_{k_2=0}^{k_3} \sum\limits_{k_1=0}^{k_2} \phi_1(u_1, u_2, ... u_{n-1}, k_1) \phi_2(u_1, u_2, ... u_{n-1}, k_2 - k_1)$$

$$.... \phi_{n-1}(u_1, u_2, ... u_{n-1}, k_{n-1} - k_n) \phi_n(u_1, u_2, ... u_{n-1}, k - k_{n-1})$$

(7) If $f(u_1, u_2, ... u_n) = u_1^{q_1} u_2^{q_2} u_3^{q_3} ...., u_n^{q_n}$

then $f(u_1, u_2, ... u_{n-1}, k) = \delta(u_1, u_2, ... u_{n-1}, q_n - k) = \{ \begin{matrix} 1 & q_n = k \\ 0 & q_n \neq k \end{matrix}$

2. Solution of Vibration equation using EPDTM

Consider an open disk of radius 'x' centered at origin representing a shape of 'still' drum head. Due to circular geometry of disk we use cylindrical co-ordinates so the mode of vibration of radially symmetric circular drum having radius 'x', then the function φ does not depend on angular displacement ''θ'' , so the vibration equation simplifies to the equation

$$\frac{1}{c^2} \frac{\partial^2 \phi}{\partial t^2} = \frac{\partial^2 \phi}{\partial x^2} + \frac{1}{x} \frac{\partial \phi}{\partial x}, \qquad x \geq 0, t \geq 0 \qquad (5)$$

with initial conditions

$$\phi(x, 0) = f(x), \qquad (6)$$

and

$$\phi_t(x, 0) = cg(x), \qquad (7)$$

Where φ(x, t) represents the displacement of finding a particle at the point 'x' in the instant t, c is the wave velocity of free vibration. To solve Eq. (5) by Elzaki projected differential transform method, first we apply the Elzaki transform on both sides

$$E[\phi_{tt}] = E[c^2 \phi_{xx} + \frac{c^2}{x} \phi_x],$$

$$\frac{1}{v^2} E(\phi) - \phi(0) - v\phi'(0) = E[c^2 \phi_{xx} + \frac{c^2}{x} \phi_x],$$

$$E(\phi) - v^2 f(x) - v^3 cg(x) = v^2 E[c^2 \phi_{xx} + \frac{c^2}{x} \phi_x],$$

$$E(\phi) = v^2 f(x) + v^3 cg(x) + v^2 E[c^2 \phi_{xx} + \frac{c^2}{x} \phi_x],$$

by applying the inverse Elzaki transform

$$\phi(x,t) = f(x) + tcg(x) + E^{-1}[v^2 E(c^2 \phi_{xx} + \frac{c^2}{x}\phi_x)], \qquad (8)$$

now by applying projected differential transform method we have the following equation.

$$\phi(x, m+1) = E^{-1}[v^2 E(c^2 A_{m+1} + \frac{c^2}{x}B_{m+1})], \qquad \phi(x,0) = f(x) + ctg(x), \qquad (9)$$

where $A_{m+1} = \dfrac{\partial^2 \phi(x,m)}{\partial x^2}$ and $B_{m+1} = \dfrac{\partial \phi(x,m)}{\partial x}$,

$$\phi(x,1) = E^{-1}[v^2 E(c^2 A_1 + \frac{c^2}{x}B_1)],$$

$$\phi(x,2) = E^{-1}[v^2 E(c^2 A_2 + \frac{c^2}{x}B_2)],$$

$$\phi(x,3) = E^{-1}[v^2 E(c^2 A_3 + \frac{c^2}{x}B_3)],$$

and so on.

3.1 Particular cases

Case I:

when $f(x) = x$ and $g(x) = 1$

so by Eq. (9),

$$\phi(x,0) = x + ct$$

by using the above equation we can find the values of $\phi(x,1)$, $A_1 = 0$, $B_1 = 1$

$$\phi(x,1) = E^{-1}[v^2 E(c^2(0) + \frac{c^2}{x}(1))],$$
$$= E^{-1}[v^2 E(\frac{c^2}{x})],$$
$$= E^{-1}[v^2(\frac{c^2}{x}v^2)],$$
$$= E^{-1}[(\frac{c^2}{x}v^4)],$$
$$= \frac{c^2}{x}\frac{t^2}{2!},$$
$$= \frac{c^2 t^2}{2x},$$

Now $A_2 = \dfrac{c^2 t^2}{x^3}$, $\qquad B_2 = -\dfrac{c^2 t^2}{2x^2}$

so $\phi(x,2) = \dfrac{c^4 t^4}{24x^3}$,

$$A_3 = \frac{c^4 t^4}{2x^5}, \qquad B_2 = -\frac{c^4 t^4}{8x^4},$$

$$\phi(x,3) = \frac{c^6 t^6}{80x^5},$$

and so on. Therefore,

$$\phi(x,t) = \phi(x,0) + \phi(x,1) + \phi(x,2) + \phi(x,3) + ...,$$

$$= x + ct + \frac{c^2 t^2}{2x} + \frac{c^4 t^4}{24x^3} + \frac{c^6 t^6}{80x^5} + ...,$$

$$= x[1 + c(\frac{t}{x}) + \frac{c^2}{2}(\frac{t}{x})^2 + \frac{c^4}{24}(\frac{t}{x})^4 + \frac{c^6}{80}(\frac{t}{x})^6 + ...], \tag{10}$$

The above series will be convergent for $|t/x| << 1$ i. e. for a small range of time and large membrane

Case II:

when $f(x) = x^2$ and $g(x) = 1$

$$\phi(x,0) = x^2 + ct,$$

using the above equations we have,

$A_1 = 2$ and $B_2 = 2x$

$$\phi(x,1) = E^{-1}[v^2 E(c^2(2) + \frac{c^2}{x}(2x))],$$
$$= E^{-1}[v^2 E(2c^2 + 2c^2)],$$
$$= E^{-1}[v^2(4c^2 v^2)],$$
$$= E^{-1}[(4c^2 v^4)],$$
$$= 4c^2 \frac{t^2}{2!},$$
$$= 2c^2 t^2,$$

now $A_2 = 0$ and $B_2 = 0$

$$\phi(x,2) = 0,$$

so we can write,

$$\phi(x,n) = 0, \qquad n \geq 2$$

Therefore solution is of the form

$$\phi(x,t) = \phi(x,0) + \phi(x,1) + \phi(x,2) + \phi(x,3) + ...,$$

$$= x^2 + ct + 2c^2 t^2, \tag{11}$$

Case III:

when $f(x) = x^3$ and $g(x) = x^2$

$$\phi(x,0) = x^3 + ctx^2,$$

using the above equation we have

$$A_1 = 6x + 2ct, \text{ and } B_1 = 3x^2 + 2ctx$$

$$\phi(x,1) = \frac{9}{2} xc^2 t^2 + \frac{2}{3} c^3 t^3,$$

Now $A_2 = 0$ and $B_2 = \frac{9}{2} c^2 t^2$

so $\phi(x,2) = \frac{3}{8x} c^4 t^4,$

and so on. Therefore,

$$\phi(x,t) = \phi(x,0) + \phi(x,1) + \phi(x,2) + \phi(x,3) + ...,$$

$$= x^3 + ctx^2 + \frac{9}{2}xc^2t^2 + \frac{2}{3}c^3t^3 + \frac{3}{8x}c^4t^4 + ...,$$

$$= x^3[1 + c(\frac{t}{x}) + \frac{9c^2}{2}(\frac{t}{x})^2 + \frac{2c^3}{3}(\frac{t}{x})^3 + \frac{3c^4}{8}(\frac{t}{x})^4 + ...,] \tag{12}$$

The above series will be convergent for $|t/x| \ll 1$. i. e. for a small range of time and large membrane

Case IV:

When $f(x) = x^2$ and $g(x) = x$

$$\phi(x,0) = x^2 + ctx,$$

by using the above value, we have

$$A_1 = 2, \text{ and } B_1 = 2x + ct$$

$$\phi(x,1) = 2c^2t^2 + \frac{1}{6x}c^3t^3,$$

now $A_2 = \dfrac{c^3t^3}{3x^3}$, and $B_1 = -\dfrac{c^3t^3}{6x^3}$

$$\phi(x,2) = \frac{1}{120x^3}c^5t^5,$$

$$\phi(x,3) = \frac{1}{560x^5}c^7t^7,$$

and so on. Thus,

$$\phi(x,t) = \phi(x,0) + \phi(x,1) + \phi(x,2) + \phi(x,3) + ...,$$

$$= x^2 + ctx + 2c^2t^2 + \frac{1}{6x}c^3t^3 + \frac{1}{120x^3}c^5t^5 + \frac{1}{560x^5}c^7t^7 + ...,$$

$$= x^2[1 + c(\frac{t}{x}) + 2c^2(\frac{t}{x})^2 + \frac{c^3}{6}(\frac{t}{x})^3 + \frac{c^5}{120}(\frac{t}{x})^5 + \frac{c^7}{560}(\frac{t}{x})^7 + ...], \tag{13}$$

As Case I and III the above series is also convergent for $|t/x| \ll 1$.

Case V:

When $f(x) = x^2$ and $g(x) = x^2$

$$\phi(x,0) = x^2 + ctx^2,$$

from the above equation we have,

$$A_1 = 2 + 2ct, \text{ and } B_1 = 2x + 2xct,$$

$$\phi(x,1) = 2c^2t^2 + \frac{2}{3}c^3t^3,$$

$$A_2 = 0, \text{ and } B_2 = 0,$$

$$\phi(x,2) = 0,$$

so we can write,

$$\phi(x,n) = 0, \qquad n \geq 2$$

thus we have the solution of the form,

$$\phi(x,t) = \phi(x,0) + \phi(x,1) + \phi(x,2) + ...,$$

$$= x^2 + ctx^2 + 2c^2t^2 + \frac{2}{3}c^3t^3 \qquad (14)$$

Case VI:

when $f(x) = \sqrt{x}$ and $g(x) = \dfrac{1}{\sqrt{x}}$

$$\phi(x,0) = \sqrt{x} + \frac{ct}{\sqrt{x}},$$

by using the above values, we have

$$A_1 = -\frac{1}{4x^{3/2}} + \frac{3ct}{4x^{5/2}}, \text{ and } B_1 = \frac{1}{2x^{1/2}} - \frac{ct}{2x^{3/2}},$$

$$\phi(x,1) = \frac{c^2t^2}{8x^{3/2}} + \frac{1}{24x^{5/2}}c^3t^3,$$

now

$$A_2 = -\frac{15c^2t^2}{32x^{7/2}} + \frac{35c^3t^3}{96x^{9/2}}, \text{ and } B_2 = -\frac{3c^2t^2}{16x^{5/2}} - \frac{5c^3t^3}{48x^{7/2}},$$

$$\phi(x,2) = \frac{3c^4t^4}{128x^{7/2}} - \frac{5c^5t^5}{384x^{9/2}},$$

Similarly, with the help of A_3 and B_3 we find $\phi(x,3)$,

$$\phi(x,3) = \frac{49c^6t^6}{5120x^{11/2}} + \frac{135c^7t^7}{21504x^{13/2}}$$

$$\phi(x,t) = \phi(x,0) + \phi(x,1) + \phi(x,2) + ...,$$

$$= \sqrt{x} + \frac{ct}{\sqrt{x}} + \frac{c^2t^2}{8x^{3/2}} + \frac{1}{24x^{5/2}}c^3t^3 + \frac{3c^4t^4}{128x^{7/2}} + \frac{5c^5t^5}{384x^{9/2}} + \frac{49c^6t^6}{5120x^{11/2}} + \frac{135c^7t^7}{21504x^{13/2}} + ...,$$

$$= \sqrt{x}[1 + c(\frac{t}{x}) + \frac{c^2}{8}(\frac{t}{x})^2 + \frac{c^3}{24}(\frac{t}{x})^3 + \frac{3c^4}{128}(\frac{t}{x})^4 + \frac{5c^5}{384}(\frac{t}{x})^5$$
$$+ \frac{49c^6}{5120}(\frac{t}{x})^6 + \frac{135c^7}{21504}(\frac{t}{x})^7 + ...], \qquad (15)$$

as of case I, III and IV the above series is also convergent for $|t/x| << 1$.

For Cases I, III, IV and VI, convergence ratio of the series t/x is to be small. for Case I and VI, displacement is inversely proportional 'x' and directly proportional to 't'. But in Case IV displacement is directly proportional to both 'x' and 't'. In Case II and V, the series consist of finite number of terms, in these cases $\phi(x,t)$ does not depend on the ratio t/x. In both cases displacement increases with the increase in x and t for a fixed value of c = 2. FIGURE (4). (10) depicts that with the increase in t and c displacements increases for a fixed value of x = 25. Rate of increase of displacement is fast in Case V than in Case II.

Conclusion

The Elzaki projected differential transform method (EPDTM) is very powerful tool in order to find the solution of various linear and nonlinear problems, showing its application for vibration of very large membrane. Elzaki Projected differential transform method can be used to solve the physical and engineering problem both analytically and numerically. EPDTM also gives rapidly converging solutions. Numerical results also show the higher degree of accuracy of method.

Acknowledgements

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FIGURES

FIGURE 1: Plot of ф (x, t) with respect to x and t at c = 8 for Case **I**
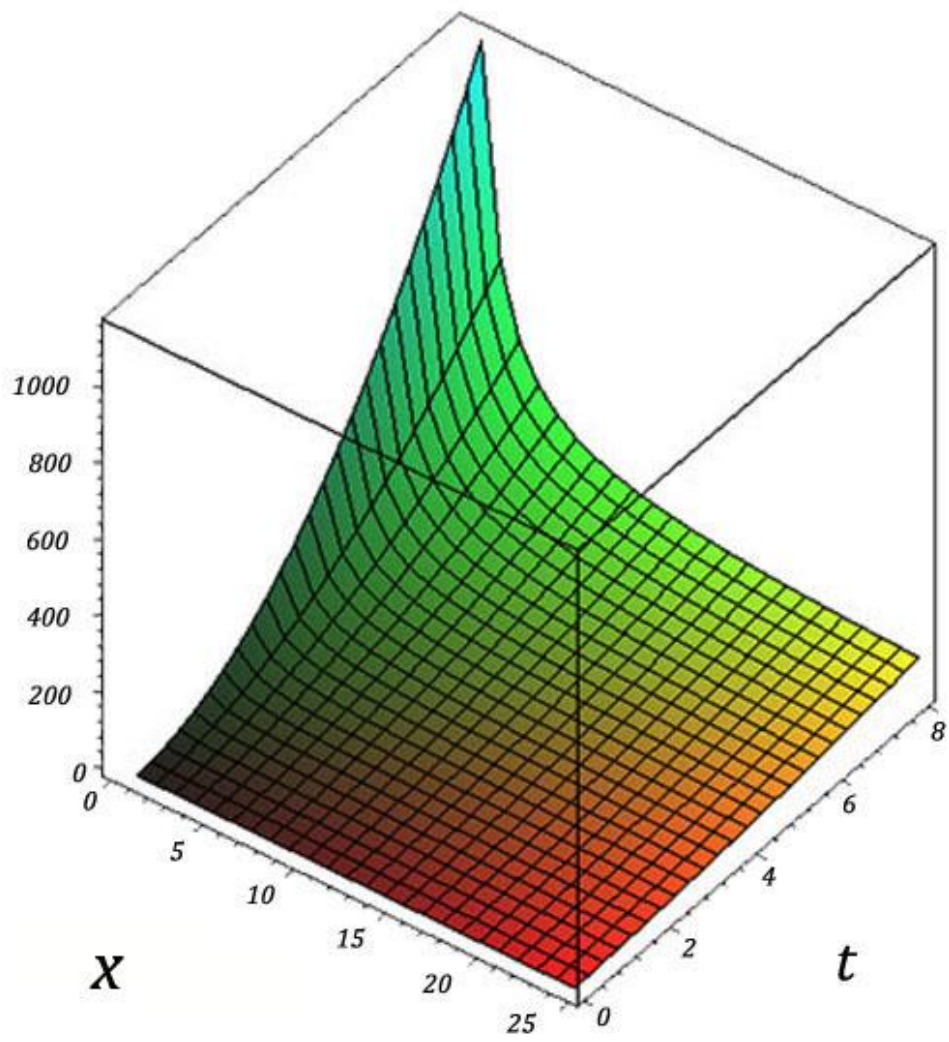
FIGURE 2: Plot of ф (x, t) vs. t for different values of c at x = 25 for Case I

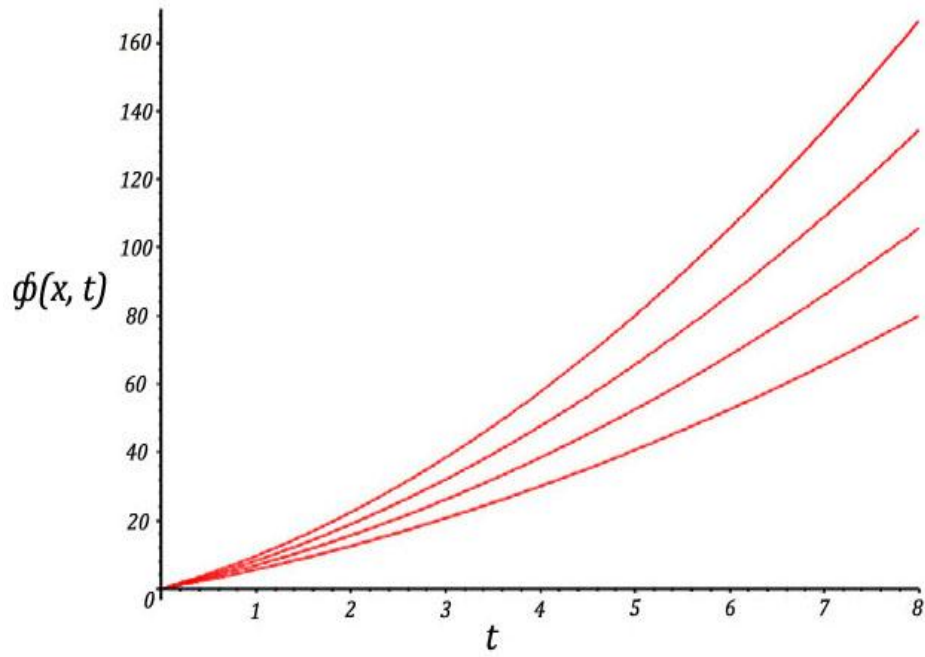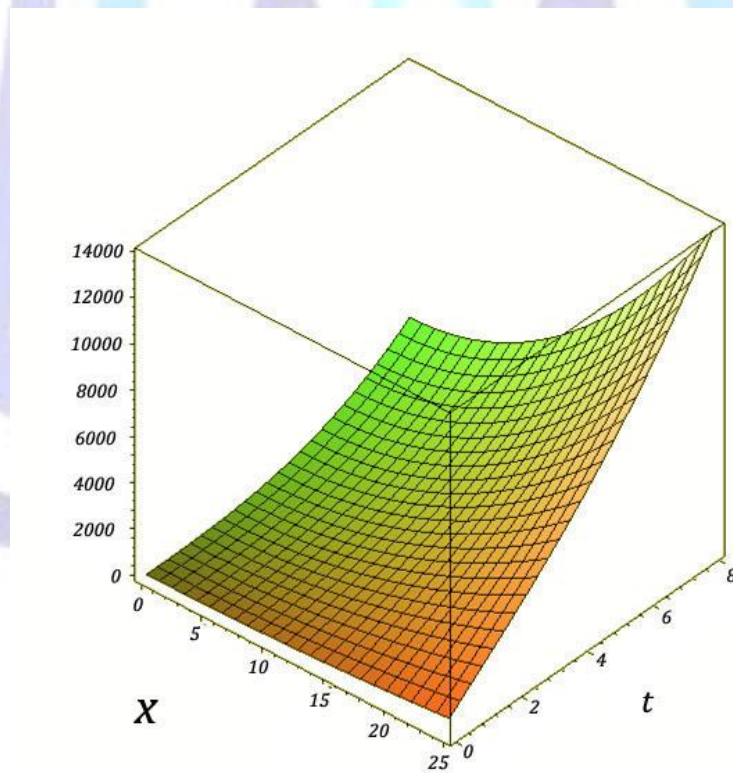**FIGURE 3:** Plot of ф (x, t) with respect to x and t at c = 8 for Case II

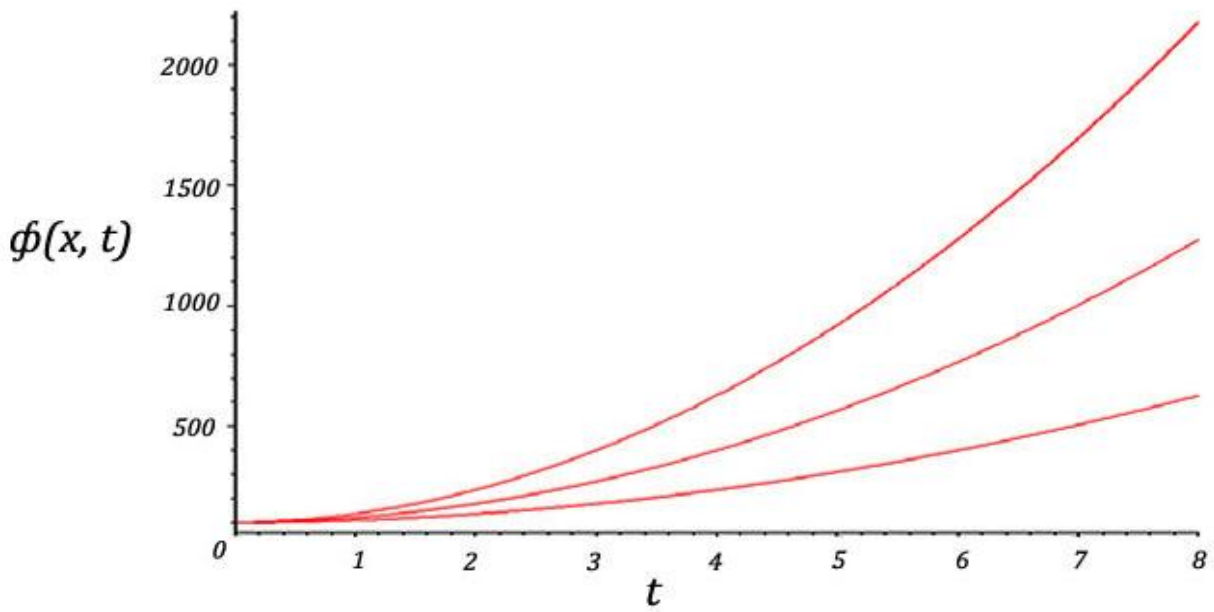FIGURE 4: **Plot of ф (x, t) vs. t for different values of c at x = 25 for Case II**

FIGURE 5: Plot of φ (x, t) with respect to x and t at c = 4 for Case III

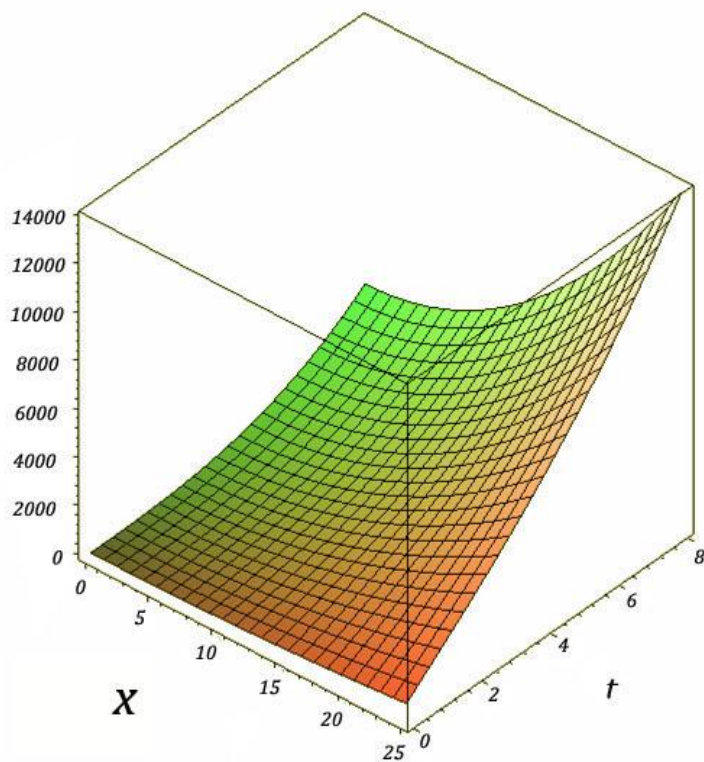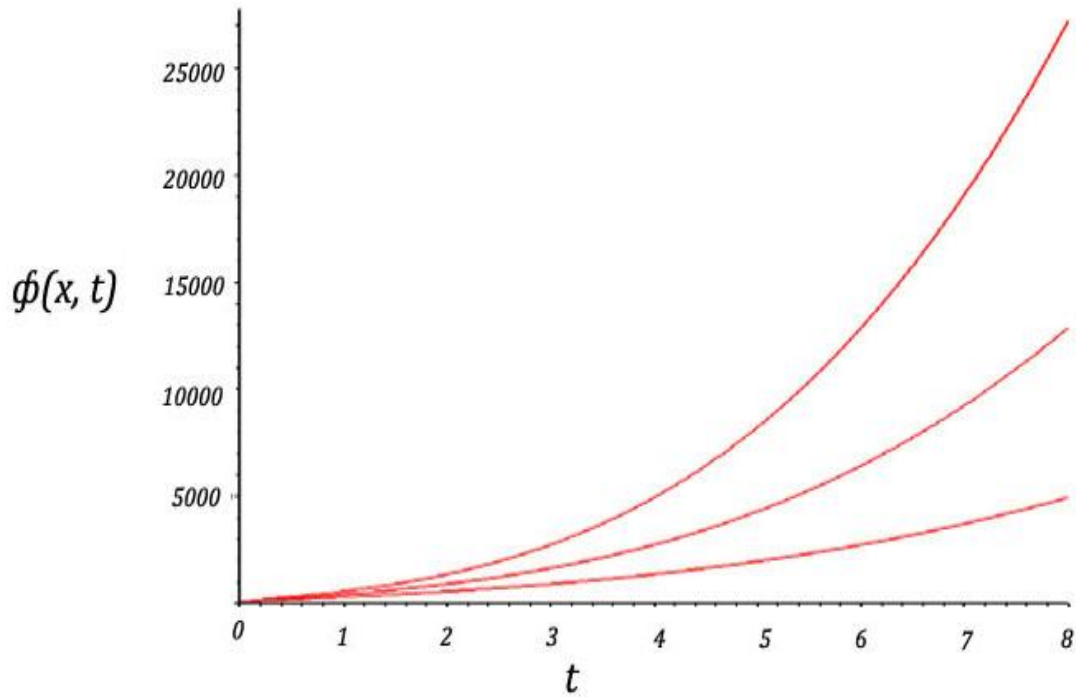**FIGURE 6:** Plot of φ (x, t) vs. t for different values of c (c = 2, 3, 4) at x = 25 for Case III

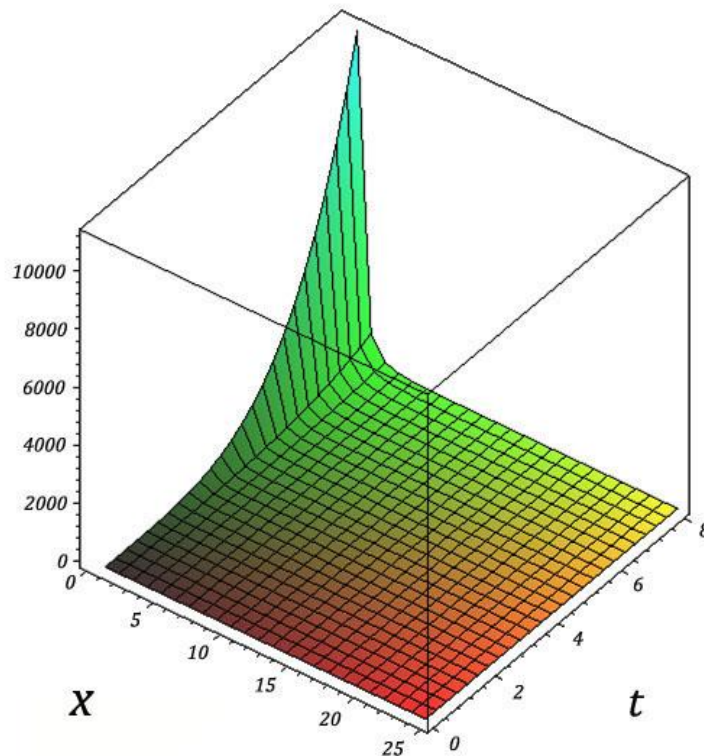FIGURE 7: **Plot of ф (x, t) with respect to x and t at c = 8 for Case IV**

FIGURE 8: *Plot of ф (x, t) vs. t for different values of c at x = 10 for Case IV*

FIGURE 1: **Plot of ф (x, t) with respect to x and t at c = 4 for Case V**

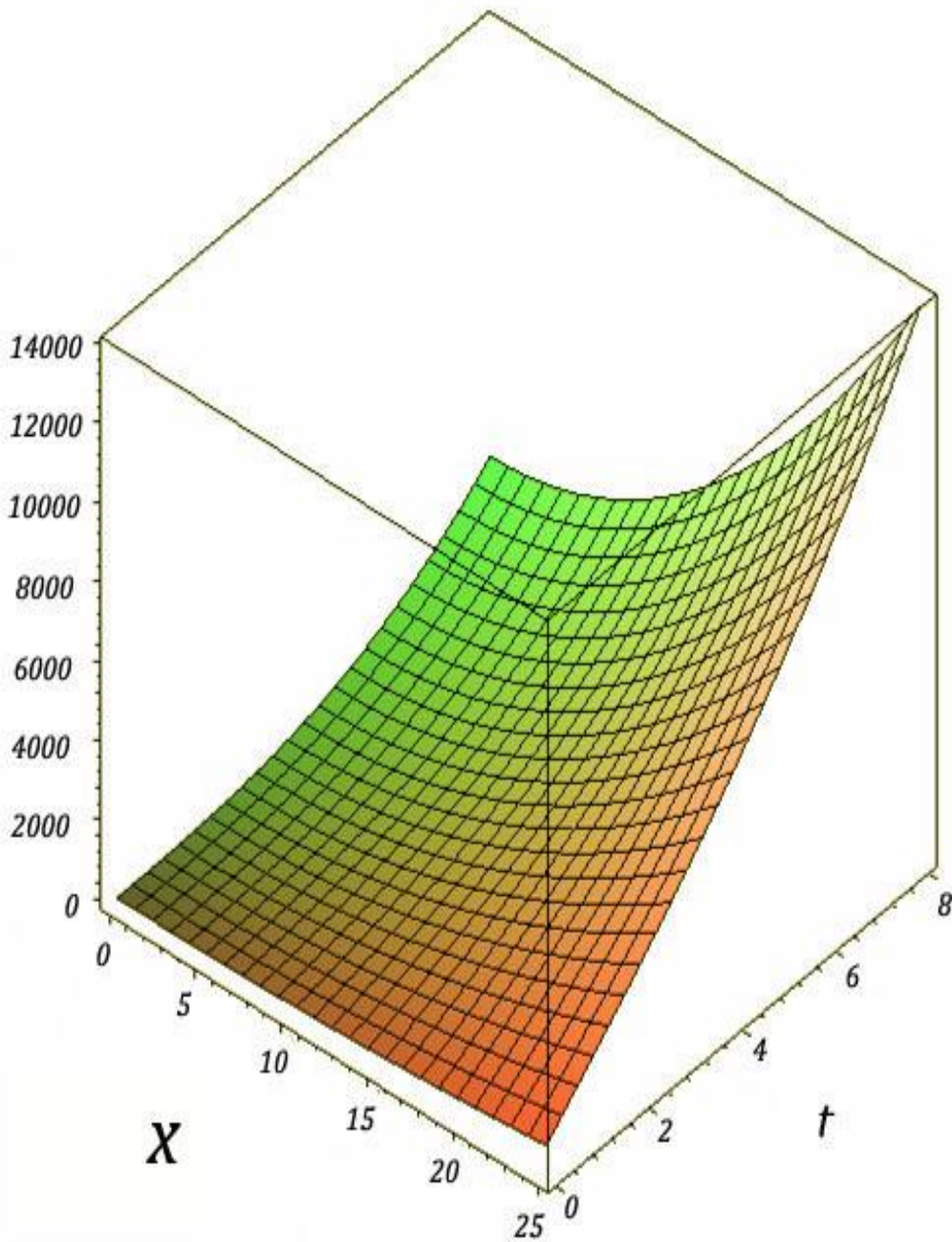**FIGURE 10:** Plot of ф (x, t) vs. t for different values of c (c = 2, 3, 4) at x = 25 for Case V

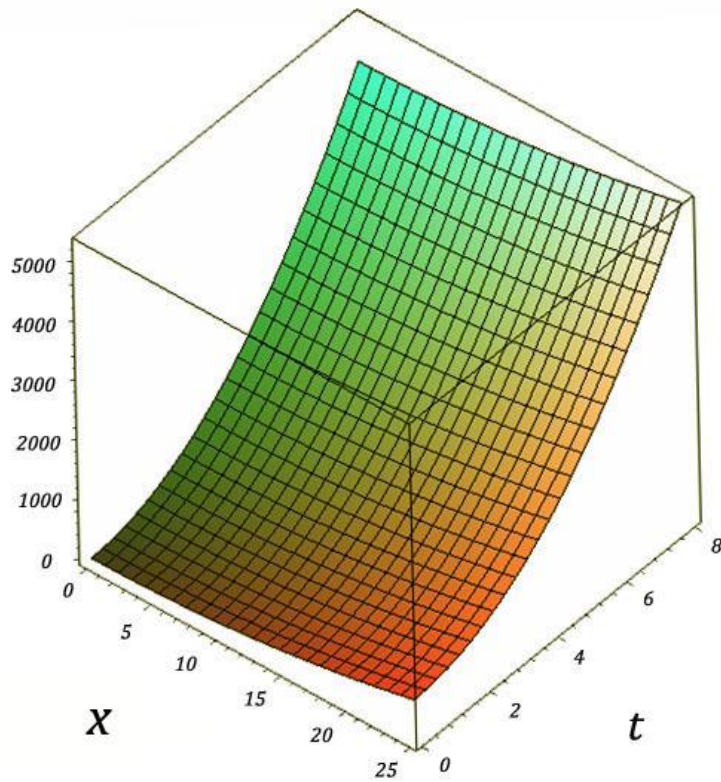FIGURE 11: **Plot of ф (x, t) with respect to x and t at c = 8 for Case VI**

FIGURE 12: Plot of ф (x, t) vs. t for different values of c at x = 10 for Case VI