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COMMON FIXED POINT THEOREM IN INTUITIONITIC FUZZY METRIC SPACES

Anil Rajput[1], Abha Tenguria[2] , Varsha Mandwariya[3] and D. P. Agrawal4
[1] Department of Mathematics, CSA Govt. PG College, Sehore MP India
[2]Department of Mathematics, Govt. MLB, PG College, Bhopal , India
[3] Department of Mathematics, Sant Hirdaram Girls College, Bhopal, India
4 Department of Mathematics, Govt. Science College, Gwalior, India

Abstract: Fixed point is an important branch of analysis to enhance its literature the prime .The object of this paper is to prove the common fixed point theorems for six self mapping taking the pair of maps as coincidentally commutating and compatible in an intuitionistic Fuzzy Metric Space. Our result is an extended and generalized result of Kumar et al. [11].

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1 INTRODUCTION:

Fixed point theory is an important area of functional analysis. The study of common fixed point of mappings satisfying contractive type conditions has been a very active field of research. In 1965 the concept of fuzzy set was introduced by Zadeh [17]. Deng [2], Erceg [3], Kaleva [9], Kramosil and Michalek [10] built the fuzzy metric spaces in various ways. George and Veermani [6] modified the notion of fuzzy metrics spaces introduced by Kramosil and Michalek [9] in order to get a Hausdorff topology. Vasuki [16] obtained the fuzzy version of common fixed point theorem which had extra conditions; in fact, he proved a fuzzy common fixed point theorem by a strong definition of Cauchy sequence. The commutativity condition of mappings was further replaced by a weaker type of notion viz., weakly commuting mapping as introduced by sessa [14] .Several common fixed point theorems have been proved for such mapping by many authors viz., sessa etal.[15].Atanassov [1] introduction and studied the concept of intuitionistic fuzzy sets as a generalization of fuzzy sets. In 2004. Park [12] defined the notion of intuitionistic fuzzy metric space with the help of continuous t-norm and continuous t-conorms as a generalization of fuzzy metric space due to Kramosil and Michalek [10].In this paper we prove a common fixed point theorem in intuitionistic fuzzy metric space taking the pair of map as compatible mapping along with the condition of coincidently commutating.

2. Definitions and preliminaries:

2.1 Definition: A binary operation$*$:[0,1]×[0,1] →[0,1] is a continuous t-norm if $*$ satisfies the following conditions:

(i) * is commutative and associative;

(ii) * is continuous;

(iii) a$*$1=a for all a ∈ [0, 1];

(iv) a$*$ b ≤ c $*$ d, whenever a ≤ c and b ≤ d, for all a, b, c, d ∈ [0, 1].

2.2 Definition: A binary operation⋄ : [0,1]×[0,1] →[0,1] is a continuous t-conorm if ⋄ satisfies the following conditions:

(i) ⋄ is commutative and associative;

(ii) ⋄ is continuous;

(iii) a⋄0=a for all a ∈ [0, 1];

(iv) a⋄ b ≤ c ⋄ d, whenever a ≤ c and b ≤ d, for all a, b, c, d ∈ [0, 1].

2.3 Definition: A 5-tuple (X, M, $N, *, \diamond$) is said to be an intuitionistic fuzzy metric space if X is an arbitrary set, $*$ is a continuous T-norm, ⋄ is a continuous T-conorm and M, N are fuzzy sets on X^2 × (0, ∞) satisfying the following conditions, for all x, y, z ∈X, s, t > 0,

(i) M(x, y, t) + N(x, y, t) ≤ 1

(ii) M(x, y, t) = o

(iii) M(x, y, t) = 1 if and only if x = y;

(iv) M(x, y, t) = M(y, x, t) ≠ 0 for t ≠ 0;

(v) M(x, y, t) $*$ M(y, z, s) ≤ M(x, z, t + s);

(vi) M(x, y, ·): [0,∞) → [0, 1] is continuous.

(vii) $\lim_{t\to\infty} M(x\,,y\,,t) = 1$

(viii) N(x, y, 0) = 1

(ix) N(x, y, t) = 0 if and only if x = y;

(x) N(x, y, t) = N(y, x, t) ≠ 0 for t ≠ 0;

(xi) N(x, y, t) ⋄ N(y, z, s) ≥ N(x, z, t + s);

(xii) N(x, y, ·): [0,∞) → [0, 1] is continuous.

(xiii) $\lim_{t\to\infty} N(x,\ y,\ t) = 0$

Then (M, N) is called an intuitionistic fuzzy metric on X. The functions M(x, y, t) and N(x, y, t) denote the degree of nearness and the degree of non-nearness between x and y with respect to t, respectively.

2.4 Definition: Let (X, M, $N, *, \diamond$) be an intuitionistic fuzzy metric space. Then a sequence $\{x_n\}$ is said to be

(i) Convergent to a point x∈X if

$$\lim_{n\to\infty} M(x_n,\ x,\ t) = 1 \text{ and } \lim_{n\to\infty} N(x_n,\ x,\ t) = 0 \qquad \text{For all } t > 0$$

(ii) Cauchy sequence if

$\lim_{n\to\infty} M(x_{n+p}, x_n, t) = 1$ and $\lim_{n\to\infty} N(x_{n+p}, x_n, t) = 0$ For all t > 0 and p > 0

2.5 Definition: A sequence $\{x_n\}$ in an intuitioistic fuzzy metric space (X, M, N, $*$, $\diamond$) is called complete if and only if every Cauchy sequence in X is convergent.

2.6 Definition: Let S and T be self mapping of an intuitionistic fuzzy metric space (X, M, N, $*$, $\diamond$). Then a pair (S, T) is said to be commuting if

M (ST$x, TSx, t$) =1 and N (ST$x, TSx, t$) =0

For all x∈X and t > 0

2.7 Definition: Let S and T be self mapping of an intuitionistic fuzzy metric space (X, M, N, $*$, $\diamond$). Then a pair (S, T) is said to be weakly commuting if

M (ST$x, TSx, t$) ≥ M (Sx, Tx, t) and N (ST$x, TSx, t$) ≤ N (Sx, Tx, t)

For all x∈X and $t > 0$

2.8 Definition: Let S and T be self mapping of an intuitionistic fuzzy metric space (X, M, N, $*$, $\diamond$). Then a pair (S, T) is said to be compatible if

$\lim_{n\to\infty} M(STx_n, TSx_n, t) = 1$ and $\lim_{n\to\infty} N(ST\,x_n, TSx_n, t) = 0$

For all t > 0, whenever $\{x_n\}$ is sequence in X such that $\lim_{n\to\infty} Sx_n = \lim_{n\to\infty} Tx_n = u$ for some u∈X.

Result

Let (X, M, N, $*$, $\diamond$) be a complete intuitionistic fuzzy metric space and let A, B, S, T, L and M self –mappings of X satisfying the following conditions:

L(X) ⊂ ST(X) and M(X) ⊂ AB(X) (1)

M (Lx, My,t) ≥ Φ (min {M (ABx, STy, t), M (ABx, Lx, t), M (STy, My, t)})

(2)

N (Lx, My,t) ≤ Φ (max {N (ABx, STy, t), N (ABx, Lx, t), N(STy, My, t)})

For all x, y ∈ X, t > 0 where Φ: [0, 1] → [0, 1] is continuous function with $\Phi(s)>s$ whenever

$0 < s < 1$.Then for any arbitrary point x_0 ∈ X, by (1), we choose a point x_1 ∈ X such that $Lx_0 = STx_1$ and for this pointx_1, there exists a point x_2 ∈ X such that ABx_2 =Mx_1 and so on. Continuing in this way, we can construct a sequence $\{z_n\}$ in such that

STx_{2n+1} = Lx_{2n}= z_{2n}, $ABx_{2n+2} = Mx_{2n+1} = z_{2n+1}$ for n= 0, 1, 2… (3)

Firstly we prove the following lemma.

2.9 Lemma: Let A, B, S and T be self mapping of an intuitionistic fuzzy metric space (X, M, N, $*$, $\diamond$) satisfying the condition (1) and (2). Then the sequence $\{z_n\}$ defined by (3) is a Cauchy sequence in X.

Proof: For $t > 0$

M (z_{2n}, z_{2n+1}, t) = M (Lx_{2n}, Mx_{2n+1}, t)

$\geq \Phi$ (min{M (ABx_{2n}, STx_{2n+1}, t), M (ABx_{2n}, Lx_{2n}, t), M (STx_{2n+1}, Mx_{2n+1}, t)})

$= \Phi$ (min{M(z_{2n-1}, z_{2n}, t), M (z_{2n-1}, z_{2n}, t), M (z_{2n}, z_{2n+1}, t)})

$$> \begin{cases} M(z_{2n-1}, z_{2n}, t) & \text{if } M(z_{2n-1}, z_{2n}, t) < M(z_{2n}, z_{2n+1}, t) \\ M(z_{2n}, z_{2n+1}, t) & \text{if } M(z_{2n-1}, z_{2n}, t) \geq M(z_{2n}, z_{2n+1}, t) \end{cases}$$

(4)

And

N(z_{2n}, z_{2n+1}, t) = N(Lx_{2n}, Mx_{2n+1}, t)

$\leq \Phi$ (max{N(ABx_{2n}, STx_{2n+1}, t), N(ABx_{2n}, Lx_{2n}, t), N(STx_{2n+1}, Mx_{2n+1}, t)})

$$= \Phi\,(\max\{N(z_{2n-1}, z_{2n}, t), N(z_{2n-1}, z_{2n}, t), N(z_{2n}, z_{2n+1}, t)\})$$

$$< \begin{cases} N(z_{2n-1}, z_{2n}, t) & \text{if } N(z_{2n-1}, z_{2n}, t) > N(z_{2n}, z_{2n+1}, t) \\ N(z_{2n}, z_{2n+1}, t) & \text{if } N(z_{2n-1}, z_{2n}, t) \leq N(z_{2n}, z_{2n+1}, t) \end{cases}$$

(5)

As $\Phi(s) > s$ for $0 < s < 1$. Thus $\{M(z_{2n}, z_{2n+1}, t), n \geq 0\}$ is an increasing sequence and $\{N(z_{2n}, z_{2n+1}, t), n \geq 0\}$ is decreasing sequence of positive real numbers in [0, 1] and therefore tends to a limit $l \leq 1$. We asserts that $l = 1$. If not, $l < 1$ which on letting $n \to \infty$ in (4) and (5) one gets $l \geq l(\Phi) > l$ a contradiction yielding there by $l = 1$. Therefore for every $n \in$ N, using analogous arguments one can shows that $\{M(z_{2n+1}, z_{2n+2}, t), n \geq 0\}$ and $\{N(z_{2n}, z_{2n+1}, t), n \geq 0\}$ is a sequence of positive real numbers in [0, 1] whichs tends to a limit $l = 1$. Therefore for every $n \in$ N

M $(z_n, z_{n+1}, t) >$ M (z_{n-1}, z_n, t) and $\lim_{t\to\infty}$ M $(z_n, z_{n+1}, t) = 1$

N $(z_n, z_{n+1}, t) <$ N (z_{n-1}, z_n, t) and $\lim_{t\to\infty}$ N $(z_n, z_{n+1}, t) = 0$

Now for any positive integer p

M $(z_n, z_{n+p}, t) \geq$ M $(z_n, z_{n+1}, \frac{t}{p}) * \ldots *$ M $(z_{n+p-1}, z_{n+p}, \frac{t}{p})$

N $(z_n, z_{n+p}, t) \leq$ N $(z_n, z_{n+1}, \frac{t}{p}) \diamond \ldots \diamond$ N $(z_{n+p-1}, z_{n+p}, \frac{t}{p})$

Since $\lim_{n\to\infty}$ M $(z_n, z_{n+1}, t) = 1$ and $\lim_{n\to\infty}$ N $(z_n, z_{n+1}, t) = 0$ for $t > 0$, it follows that

$\lim_{n\to\infty}$ M $(z_n, z_{n+p}, t) \geq 1 * 1 * 1 * \ldots * 1 = 1$

And

$\lim_{n\to\infty}$ N $(z_n, z_{n+p}, t) \geq 1 \diamond 1 \diamond 1 \diamond \ldots \diamond 1 = 1$

Which shows that $\{z_n\}$ is Cauchy sequence in X

Now we prove our main result as follows:

3 Main Results:

3.1 Theorem: Let A, B, S, T, Land M be six self –mappings of an intuitionistic fuzzy metric space (X, M, N, $*$, $\diamond$) satisfying the condition:

M $(Lx, My, t) \geq \Phi$ (min {M (ABx, STy, t), M (ABx, Lx, t), M (STy, My, t)})

For all x, y ∈ X, t > 0 where Φ: [0, 1] → [0, 1] is continuous function with $\Phi(s) > s$ whenever

$0 < s < 1$.If L(X) ⊂ ST(X) and M(X) ⊂ AB(X) and one of A(X), B(X), S(X), T(X), L(X) and M(X) is complete subspace of X ,then

(i) L and AB have a point of coincidence,

(ii) M and ST have a point of coincidence.

(iii) LB=BL, AB=BA, ST=TS and MT=TM

Moreover, if the pairs (L, AB) and (M, ST) are coincidentally commuting and compatible pair then A, B, S, T, L and M have a unique common fixed point.

Proof: Let x_0 be an arbitrary point in X. Then following arguments of Fisher [4], one can construct sequences $\{x_n\}$ and $\{z_n\}$ in X such that

STx_{2n+1} = Lx_{2n} = z_{2n} and $ABx_{2n+2} = Mx_{2n+1} = z_{2n+1}$

Then due to Lemma 2.9, $\{z_n\}$ is Cauchy sequence in X.

Now suppose that AB(X) is complete subspace of X, then the subsequence Lx_{2n}, STx_{2n+1}, Mx_{2n+1}, ABx_{2n+2} converges to z.

Since (L, AB) is compatible

Therefore M(LABx_n, $ABLx_n$, t) = 1

M (Lz, ABz, t) = 1

Lz = ABz

Taking $x = z$ and $y = x_{2n+1}$ in (2), we get (for t > 0)

M(Lz, Mx_{2n+1}, t)≥ Φ {min(M(ABz, STx_{2n+1}, t), M(ABz, Lz, t), M(STx_{2n+1}, Mx_{2n+1}, t))}

M(Lz, z, t) ≥ Φ {min(M(Lz, z, t), M(Lz, Lz, t), M(z, z, t))}

M(Lz, z, t) ≥ Φ {min(M (Lz, z), 1, 1)}

≥ Φ (M (Lz, z, t))

> M (Lz, z, t) a contradiction.

and

N(Lz, Mx_{2n+1} ,t) ≤ Φ {max(N(ABz, STx_n, t), N(ABz, Lz, t), N(STx_n, Mx_n, t))}

N(Lz, z ,t) ≤ Φ {max(N(Lz, z, t), N(Lz, Lz, t), N(z, z, t))}

N(Lz, z ,t) ≤ Φ {max(N (Lz, z, t), 0, 0)}

≤ Φ (N (Lz, z, t))

< N (Lz, z, t) a contradiction.

Therefore Lz = z

$Lz = ABz = z$

Taking $x = Bz$ and $y = x_n$ in (2), we get (for t > 0)

M (LBz, Mx_n ,t) ≥ Φ {min (M ($ABBz, STx_n, t$), M($ABBz, LBz, t$), M(STx_n, Mx_n, t))}

M (LBz, z ,t) ≥ Φ {min(M ($ABBz, z, t$), M($ABBz, LBz, t$), M(z, z, t))}

and

N (LBz, Mx_n ,t) ≤ Φ {max(N($ABBz, STx_n, t$), N($ABBz, LBz, t$), M(STx_n, Mx_n, t))}

N (LBz, z ,t) ≤ Φ {max(N($ABBz, z, t$), N($ABBz, LBz, t$), N(z, z, t))}

From equation (iii) LBz = B (Lz) = Bz

M(Bz, z ,t) ≥ Φ {min(M(Bz, z, t), 1, 1)}

≥ Φ (M (Bz, z, t))

> M (Bz, z, t) a contradiction.

and

N(Bz, z ,t) ≤ Φ {max(N (Bz, z, t), N (Bz, Bz, t), 1})

≤ Φ {max(N(Bz, z, t), 0, 0)}

≤ Φ (N (Bz, z, t))

< N (Bz, z, t) a contradiction.

Therefore Bz = z

$Lz = Az = Bz = z$

Similarly

Taking $x = x_{2n+1}$ and $y = z$ in (2), we get (for t > 0)

M(Lx_{2n+1}, Mz ,t)≥ Φ {min(M(ABx_{2n+1}, STz, t), M(ABx_{2n+1}, Lx_{2n+1}, t), M(STz, Mz, t))}

M(z, Mz ,t) ≥ Φ {min(M(z, Mz, t), M(z, z, t), M (Mz, Mz, t))}

M(z, Mz ,t) ≥ Φ {min(M (Mz, z, t), 1, 1)}

≥ Φ (M (Mz, z, t))

> M (Mz, z, t) a contradiction.

and

N(Lx_{2n+1}, Mz ,t) ≤ Φ {max(N(ABx_{2n+1}, STz, t), N(ABx_{2n+1}, Lx_{2n+1}, t), N(STz, Mz, t))}

N(z, Mz ,t) ≤ Φ{(max(N(z, Mz, t), N(z, z, t), N(Mz, Mz, t))}

N(z, Mz ,t) ≤ Φ {ma(N (Mz, z, t), 0, 0)}

≤ Φ (N (Mz, z, t))

< N (Mz, z, t) a contradiction.

Therefore Mz = z

Again Taking $x = x_n$ and $y = Tz$ in (2), we get (for t > 0)

M(Lx_n, MSz ,t) ≥ Φ {min(M($ABx_n, STTz, t$), M(ABx_n, Lx_n, t), M($STTz, MTz, t$))}

M(z, MSz ,t) ≥ Φ {min(M($z, STTz, t$), M(z, z, t), M(STTz, MTz, t))}

and

$N(Lx_n, MSz, t) \leq \Phi\{\max(N(ABx_n, STTz, t), N(ABx_n, Lx_n, t), M(STTz, MTZ, t))$

$N(z, MSz, t) \leq \Phi\{\max(N(z, STTz, t), N(z, z, t), N(STTz, MTz, t))\}$

$MT = TM$ and $ST = TS$

$MTz = T(Mz) = Tz$

$(ST)Tz = T(STz) = Tz$

$M(z, z, t) \geq \Phi\{\min(M(z, Tz, t), 1, M(Tz, Tz, t))\}$

$\geq \Phi\{\min(M(z, Tz, t), 1, 1)\}$

$\geq \Phi(M(z, Tz, t))$

$> M(z, Tz, t)$ a contradiction.

$N(z, z, t) \leq \Phi\{\max(N(z, Tz, t), 1, N(Tz, Tz, t))\}$

$\leq \Phi\{\max(N(z, Tz, t), 0, 0)\}$

$\leq \Phi(N(z, Tz, t))$

$< N(z, Tz, t)$ a contradiction.

Therefore $Tz = z$

$Mz = STz = z$

$Mz = Sz = Tz = z$

Thus we have Lz = Az = Bz = Mz = Sz = Tz = z

Thus z is the common fixed point of L, A, B, M, S and T.

Uniqueness: Finally we prove that A, B, S, T, L and M have a unique common fixed point.

Let r be another common fixed point of A, B, S, T, L and M.

From (3) we have

$M(r, z, t) = M(Lr, Mz, t)$

$\geq \Phi\{\min(M(ABr, STz, t), M(ABr, Lr, t), M(STz, Mz, t))\}$

$\geq \Phi\{\min(M(r, z, t), M(r, r, t), M(z, z, t))\}$

$= \Phi\{\min(M(r, z, t), 1, 1)\}$

$= \Phi\, M(r, z, t) > M(r, z, t)$ a contradiction unless $M(r, z, t) = 1$ for all $t > 0$ $i.e\ r = z$

and

$N(r, z, t) = N(Lr, Mz, t)$

$\leq \Phi\{\max(N(ABr, STz, t), N(ABr, Lr, t), N(STz, Mz, t))\}$

$\leq \Phi\{\max(N(r, z, t), N(r, r, t), N(z, z, t))\}$

$= \Phi\{\max(N(r, z, t), 0, 0)\}$

$= \Phi\, N(r, z, t) > N(r, z, t)$ a contradiction unless $N(r, z, t) = 0$ for all $t > 0$ $i.e\ r = z$

Hence z is the unique common fixed point of A, B, S,T, L and M in X.

References:

[1]. Atanassov K.T., intuitionistic fuzzy sets, Fuzzy Sets and System, 20, No. 1(1986), 87-96, doi: 10.2307/2318216.

[2]. Deng Z K, Fuzzy pseudo-metric spaces, *J. Math. Anal. Appl*. 86 (1982), 74-95.

[3]. Erceg M A, Metric spaces in fuzzy set theory, *J. Math. Anal. Appl*. 69 (1979), 205-230.

[4]. Fisher B, Common fixed points of four mappings, Bull. Inst. Math. Acad. Sinica 11(1983), 103-113.

[5]. Grabiec M, Fixed point in fuzzy metric space, Fuzzy sets Systems 27(1988),385-389.

[6]. George A & Veeramani P, On some results in fuzzy metric spaces, Fuzzy Sets and Systems, 64(1994), 395-399.

[7]. Imdad M and Ali J, common fixed point theorem, Math. Comm. 11(2006),153-163.

[8]. Jungck G and B. E. Rhoades, “Fixed Point Theorems for Occasionally Weakly compatible Mappings”, Fixed Point Theory, 7(2006), 287-296.

[9]. Kaleva O & Seikkala S, On fuzzy metric spaces, *Fuzzy Sets and Systems*, 12 (1984), 215- 229.

[10]. Kramosil I& Michalek J, Fuzzy metric and statistical metric spaces, *Kybernetica*, 15 (1975), 326-334.

[11]. Kumar M, Kumar P and Kumar S, Some common fixed point theorems in generalized metric spaces, volume 2013, Article ID 719324,7pages.

[12]. Park J.H., intuitionistic fuzzy metric space, Chaos, solitons Fractals, 22. No. 5 (2004), 1039-1046, doi:10.1016/j.chaos.2004.02.051

[13]. Schweizer B and A. Sklar, Statistical metric spaces, Pacific J. Math. 10(1960), 313-334.

[14]. Seesa S, On a weak commutativity condition in fixed point considerations, Publ. Inst. Math. 32 (46)(1982), 149-153.

[15]. Sessa S, Mukerjee, R N & Som, T, A common fixed point theorem for weakly commuting mappings, Math. Japonica 31(2) (1986), 235-245.

[16]. Vasuki R, Common fixed points for R-weakly commuting maps in fuzzy metric spaces, IndianJ.Pure.Appl.Math.30(1999), 419.

[17]. Zadeh L A, Fuzzy Sets, Information and Control, 8(1965) 338.