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Fully Pseudo Stable Systems

Mehdi S. Abbas[1] and Mustafa A. Aziz[2]
Department of Mathematics, College of Science Mustansiriyah University, Baghdad, Iraq.
1. m.abass@uomustansiriyah.edu.iq
2. Mustafa12345abass@yahoo.com

Abstract: In this work, the notion of full pseudo-stability has been introduced and studied, which is a generalization of full stability. We obtain a characterization of full pseudo-stability analogous to that of full stability. Certain class of subsystems which inheret this property have been considered. Finally, we studied the completely pseudo-injective systems and the relation between it and fully pseudo-stable systems.

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1. INTRODUCTION

Throughout the paper S represents a semigroup with non-zero identity 1(monoid). A right S-system M_S is a set M together with a map(written multiplicatively) from M × S into M satisfying m(ab) = (ma)b and m1 = m for all m ∈ M and a, b ∈ S. A non-empty subset N of an S-system M_S is a subsystem if NS ⊂ N, it is clear that xS = {xs | s ∈ S} where x in M_S is a subsystem of M called a principal subsystem. The right (resp. left) annihilator of a subset X of M (resp. X of S) is denoted by $r_S(X)$ (resp. $l_M(X)$) which is defined by $r_S(X) = \{(s, t) \in S \times S \mid xs = xt$ for all $x \in X\}$ (resp. $l_M(X) = \{(m, n) \in M \times M \mid xm = xn$ for all $x \in X\}$) and for a subset Y of M × M (resp. Y of S × S) is defined by $r_S(Y) = \{s \in S \mid ms = ns\ \forall\ (m, n) \in y\}$ (resp. $l_M(Y) = \{m \in M \mid ms = mt\ \forall\ (s, t) \in Y\}$). Let M_S and K_S be two S-systems. A mapping $\alpha : M_S \to K_S$ is called an S-homomorphism, if α(ms) = α(m)s for all m ∈ M, s ∈ S. An S-homomorphism $\alpha : M_S \to K_S$ is called an S-isomorphism if α is bijective. In this case we say that M_S and K_S are isomorphic and write $M_S \cong K_S$ [4].Let M_S be an S-system. Then $xS \cong S/r_S(x)$ for each x in M_S.A non-zero subsystem N of an S-system M_S is called essential (or large) in M_S if for each S-homomorphism $\alpha: M_S \to K_S$, where K_S is any S-system, with restriction to N_S is a monomorphism, α itself is a monomorphism [2]. If N is essential in M_S, then we say that M_S is essential extension of N. We denote this situation by $N_S \subseteq' M_S$. And a non-zero M_S is called reversible if each non-zero subsystem of M_S is essential in M_S. A subsystem N of M_S is called stable if α(N) ⊆ N for each S-homomorphism α of N into M_S. In case each subsystem of M_S is stable, then M_S is called fully stable. A monoid S is fully stable if it is fully stable S-system [7]. The stability condition can be reduced to the elements of the system, so an S-system M_S is fully stable if and only if each principal subsystem is stable. Also an S-system M_S is fully stable if and only if each principal subsystem satisfies the double annihilator condition, namely $l_M(r_S(x)) = xS$ for each x in M_S. An S-system M_S is called injective if for any monomorphism $\alpha : A_S \to B_S$ of S-systems A_S, B_S and any homomorphism $\mu : A_S \to M_S$, there exists a homomorpism $\beta : B_S \to M_S$ such that μ = βα [2]. Any maximal essential extension of an S-system M_S is called an injective envelope of M_S it is unique up to isomorphism (denoted by $E(M_S)$) [2].A. M. Lopez in [8] introduced quasi-injective systems, an S-system M_S is quasi injective if each S-homomorphism of a subsystem of M_S into M_S is a restriction of some S-endomorphism of M_S. We mentioned here generalization of quasi-injective system which is relevant to our work. An S-system M_S is called pseudo-injective if each S-monomorphism of a subsystem of M_S into M_S extends to an S-endomorphism of M_S [5]. The above concept motivate to consider systems in which all subsystems are stable under monomorphisms, termed fully pseudo stable systems. We show that an S-system M is fully pseudo-stable if and only if $r_S(x) = r_S(y)$ implies that xS = yS for each x, y in M_S. Also we prove that the injective envelope of fully pseudo-stable systems is fully pseudo-stable. An S-system M_S is said to be completely pseudo-injective if every subsystem of M_S is pseudo-injective. We show that every fully pseudo-stable systems is completely pseudo-injective.

2. Fully Pseudo Stable Systems

We start with the following generalized concept of full stability and we give basic general facts.

Definition 2.1: Let M_S be an S-system. A subsystem N of M_S is said to be pseudo-stable, if μ(N) ⊆ N for each S-monomorphism $\mu : N \to M_S$. M_S is called fully pseudo-stable system if each subsystem of M_S is pseudo-stable. A monoid S is called fully pseudo-stable if it is fully pseudo-stable S-system.

It is clear that every stable subsystem is pseudo-stable and hence every fully stable S-system is fully pseudo-stable. The property of pseudo stability of subsystems can be reduced to elements, so it is easy to see that an S-system M_S is fully pseudo-stable if and only if each principal subsystem of M_S is pseudo-stable. Every subsystem of fully pseudo-stable system is fully pseudo-stable.

In the following proposition we give a simpler form of fully pseudo-stable systems which is more usable than the definition.

Proposition 2.2: The following are equivalent for an S-system M_S.

1. M_S is fully pseudo-stable.
2. Every subsystem of M_S is fully pseudo-stable.
3. Every 2-generated subsystem of M_S is fully pseudo-stable.
4. If N, K are subsystems of M_S and N≅ K, then N = K.
5. $r_S(x) = r_S(y)$ implies that xS = yS for some x, y in M_S.

Proof: (1) ⟹ (2) and (2) ⟹ (3) are clear.

(3) ⟹ (1) Suppose N is a subsystem of M_S and $\alpha : N \to M$ is an S-monomorphism. Let n be an element of N and let K = nS∪α(n)S. Let $\beta = \alpha|_{nS} : nS \to M$. Then, clearly, α(n) = β(n). By assumption, K is fully pseudo-stable and so α(n) ∈ nS. It follows that M_S is fully pseudo-stable.

(1) ⟹ (4) If N, K are two subsystems of M_S and $\alpha : N \to K$ is an S-monomorphism, then K = α(N) ⊆ N. Since $\alpha^{-1} : K \to N$ is also S-isomorphism, then $N = \alpha^{-1}(K) \subseteq K$. Hence N = K.

(4) ⟹ (5) Suppose $r_S(x) = r_S(y)$ for some x, y ∈ M_S. Define α :xS → yS by α(xs) = ys for every s ∈ S. Clearly, α is a well-defined isomorphism and so xS = yS.

(5) ⟹ (1) Let N be any subsystem of M_S and $\alpha : N \to M$ is an S-monomorphism. Let n ∈ N, then $r_S(n) = r_S(\alpha(n))$ and hence α(n) ∈ α(n)S = nS⊆ N. Consequently, N is pseudo-stable.

Examples and Remarks 2.3:

1- It follows from proposition (2.2) that an S-system M_S is fully pseudo-stable if and only if for each subsystem N of M_S and S-monomorphism$\alpha : N \to M$, we have $\alpha(N) = N$.

2- Recall that an element x of a semigroup S is left (resp. right) zero if xy = x (resp. yx = x) for all y ∈ S. S is called left (resp. right) zero semigroup, if every element of S is left (resp. right) zero. Let S = {a, b, c, e} with a, b are left zero, ca = cb = cc = a, e is the identity, it is clear that S is a monoid. Consider S as S-system, then cS = {a, c} is pseudo-stable subsystem of S_S which is not stable.

3- We say that a subsystem N of S-system M_S satisfies Baer's*m*-criterion if for each S-monomorphism$\alpha : N \to M$ there exists an element s in S such that $\alpha(x) = xs$ for each x in N. It is an easy matter to see that an S-system M_S is fully pseudo-stable if and only if each principal subsystem of M_S satisfies Baer's*m*-criterion.

Next, we consider conditions under which full pseudo stability versus full stability. First an equivalence relation ρ on an S-system M_S is congruence, if m ρ m' implies that (ms) ρ (m's) for m, m' ∈ M, s ∈ S.

Lemma 2.4: Let M_S be an S-system where S is a commutative monoid and ρ be a congruence on S. Then

$l_M(\rho) \cong Hom_S(S / \rho, M)$.

Proof: Let$\alpha : l_M(\rho) \to Hom_S(S / \rho, M)$be defined by $\alpha(m)([s]_\rho) = ms$ for each $m \in l_M(\rho)$, it is clear that α is S-homomorphism. Also define $g : Hom_S(S / \rho, M) \to l_M(\rho)$ by $g(f) = f([1]_\rho)$ for each $f \in Hom(S/\rho, M)$. Then g is S-homomorphism. Now, for each f ∈ Hom(S / ρ, M) we have:

$(\alpha \circ g)(f)([s]_\rho) = \alpha(g(f))([s]_\rho) = \alpha(f([1]))([s]_\rho) = f([1]_\rho)s = f([1]_\rho s) = f([1.s]_\rho) = f([s]_\rho)$ (i.e. $\alpha \circ g = I_{Hom(S/\rho,\ M)}$). Also, for each $m \in l_M(\rho)$ we have:

$(g \circ \alpha)(m) = g(\alpha(m)) = \alpha(m)([1]_\rho) = m.1 = m$, then $g \circ \alpha = I_{lM(\rho)}$. Hence α is S-isomorphism.

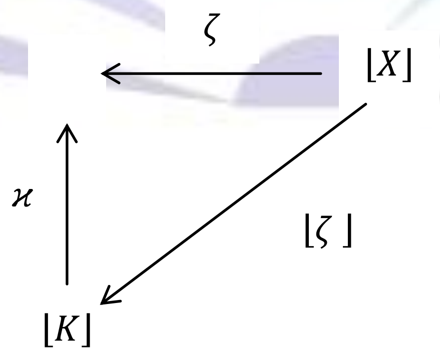
Proposition2.5: An S-system M_S is fully stable if and only if M_S is fully pseudo-stable and $xS \cong Hom(xS, M_S)$ for each x in M_S.

Proof: If M is a fully stable S-system, then $l_M(r_S(x)) = xS$ for each x in M_S. So by lemma(2.4),$xS = l_M(r_S(x)) \cong Hom(S/r_S(x), M) \cong Hom(xS, M)$. Conversely, for each $x \in M_S$ we have $xS \cong Hom(xS, M_S) \cong Hom(S/r_S(x), M_S) \cong l_M(r_S(x))$, then by proposition (2.2) implies that$xS = l_M(r_S(x))$. Thus M_S is fully stable.

In module theory, the injective envelope of fully pseudo-stable module over Noetherian ring is fully pseudo-stable [6]. In this part we study the injective envelop of fully pseudo-stable systems. First we need to recall some categorical concepts.

Definition 2.6 [4]: Let **C**be a concrete category. For A ∈**C** by $\lfloor A \rfloor$ ∈**Set** (category of sets) denote the underlying set of A. For $f \in Mor_C(A_1, A_2)$, $A_1, A_2 \in$**C** by $\lfloor f \rfloor : \lfloor A \rfloor \to \lfloor A \rfloor$ denoted the mapping in **Set** underlying *f*. Now $\lfloor - \rfloor$: **C** → **Set** defined as indicated is a covariant functor which is called the forgetful functor from **C** into **Set**. In particular, we have the forgetful functor$\lfloor - \rfloor$: S-system → **Set**.

Definition 2.7[4]: Let **C**be a concrete category. The object *K* ∈**C** is called a cofreein **C**, if there existsI∈**Set** and a mapping $\varkappa : \lfloor K \rfloor \to I$ such that the following universal propertyis valid. For every X∈**C**and everymapping $\zeta : \lfloor X \rfloor \to I$there exists exactly one $\zeta^* \in Mor_C(X, K)$ such that thefollowing diagram in **Set**is commutative



Recall that any cofree right S-system is isomorphic to asystem of the form $X^S = \{f \mid f : S \to X\}$ where X is a non-empty set and $(fs)(t) = f(st)$ for every s,t∈ S and every cofree is injective system [4].

Theorem 2.8: If M_S is a right fully pseudo-stable S-system, then

the right system M^S is fully pseudo-stable.

Proof: Let $\alpha \in M^S$, and θ: $\alpha S \to M^S$ be an S-monomorphism. Then $\alpha(t) \in M$ for each $t \in S$. Define $\alpha' : \alpha(t)S \to M$ by $\alpha'(\alpha(t)s) = \theta(\alpha t)(s)$.

If $\alpha(t)s_1 = \alpha(t)s_2 \Leftrightarrow \alpha(ts_1) = \alpha(ts_2) \Leftrightarrow \theta(\alpha(ts_1)) = \theta(\alpha(ts_2)) \Leftrightarrow \alpha'(\alpha(t)s_1) = \alpha'(\alpha(t)s_2)$, this shows that α' is well-defined and injective mapping.

$\alpha'((\alpha(t)s_1)s_2) = \alpha'(\alpha(ts_1)s_2) = \theta(\alpha ts_1)(s_2) = \theta(\alpha(t))s_1(s_2) = \alpha'(\alpha(t)s_1)(s_2)$ and hence $\alpha'(\alpha(t)s_1) = \alpha'(\alpha(t))s_1$. This shows that α' is S-homomorphism. So by full pseudo-stability of M_S, one has $\alpha'(\alpha(t)S) \subseteq \alpha(t)S$. There exists $s_1 \in S$ such that $\alpha'(\alpha(t)) = \alpha(t)s_1$. Thus $\theta(\alpha t)(1) = \alpha'(\alpha(t)) = \alpha(t)s_1 = \alpha(ts_1) \in \alpha S$ so $\theta(\alpha t) \in \alpha S$. This shows that M^S is fully pseudo-stable.

Theorem 2.9: Every fully pseudo-stable system has fully pseudo-

stable injective envelope.

Proof: Let M_S be a fully pseudo-stable system. M_S can be embedded in injective fully pseudo-stable system M^S [4] . If E(M) is the injective envelope of M, then by the definition of injective envelope , it is a minimal injective system which conteains M_S. So E(M) is a subsystem of M^S and hence E(M) is fully pseudo-stable.

3. Completely pseudo-injective systems

Definition 3.1: An S-system M_S is called completely pseudo-injective if each subsystem of M_S is pseudo-injective. A semigroup S is called right completely self pseudo-injective, if each right ideal is pseudo-injective S-system. It is clear that every completely pseudo-injective is pseudo-injective system, in fact the injective envelop of non pseudo-injective system is pseudo-injective system which is not completely pseudo-injective.

Example3.2: If S is a left zero semigroup. Then S_S is completely pseudo-injective. Since for each subsystem N_S of S_S and for each S-monomorphismα : $K_S \to N_S$, where K_S is a subsystem of N_S. Define $\beta : N \to N$ by:

$$\beta(n) = \begin{cases} \alpha(n) & if\ n \in K \\ a & if\ n \notin K \end{cases}$$

it is clear that β is S-homomorphism and extension of α. Then N is pseudo-injective subsystem of S_S. Hence S_S is completely pseudo-injective S-system.

Recall that an S-system M_S is multiplication if each subsystem of M_S of the form MI for some ideal I of S [9]. This is equivalent to saying that every principal subsystem is of this form. Multiplication systems and fully pseudo-stable systems are independent, since Z is multiplication Z-system which is not fully pseudo-stable and consider S = {a, b, c}with the product as in the table:

.	a	b	c
a	a	b	c
b	b	a	c
c	c	b	c

Then (S, .) is fully pseudo-stable S-system which is not multiplication.

In the following proposition, we consider conditions under which pseudo-injective system being completely pseudo-injective.

Proposition 3.3: Every multiplication pseudo-injective system is completely pseudo-injective.

Proof: Let M_S be a multiplication pseudo-injective S-system, N be a subsystem of M_S and $\alpha : K \to N$ be an S-monomorphism where K be any subsystem of N. By pseudo-injectivity of M_S implies that α can be extended to an S-endomorphism β of M_S. Since M_S is multiplication S-system, then there exists an ideal A of S such that N = MA. Hence $\beta(N) = \beta(MA) = \beta(M)A \subseteq MA = N$ and hence $\beta|_N : N \to N$ is an extension of α.

Now, we ask the following question. Is there a relation between the fully pseudo-stable system and completely pseudo-injective?, the following theorem and its corollary gives the answer positively.

Theorem 3.4: Every fully pseudo-stable system is pseudo-injective.

Proof: Let M_S be a fully pseudo-stable S-system. It was proved in theorem 2.9, that the injective envelope E(M) of M is fully pseudo-stable S-system. Let N be a subsystem of M_S and $\alpha : N \to M$ be an S-monomorphism. Then $\alpha : N \to N$ by hypothesis. There is an S-homomorphism $\beta : E(M) \to E(M)$ which extends α. Full pseudo-stability of E(M) implies that $\beta'(= \beta|_M) : M \to M$ is an extension of α. Thus M_S is pseudo-injective.

Corollary 3.5: Every fully pseudo-stable system is completely pseudo-injective.

Proof: Let M_S be a fully pseudo-stable S-system and N be a subsystem of M_S. It is clear that N is a fully pseudo-stable subsystem. Then by above theorem we have N is pseudo-injective subsystem. Hence M_S is completely pseudo-injective The convers of above corollary may not be true in general. For example, let S be a left zero semigroup, then S as S-system is completely pseudo-injective S-system (example 3.2) but it is not fully pseudo-stable S-system.

We knew from example 3.2 the (completely) pseudo-injective S-system need not be fully pseudo-stable S-system. Is there a condition make the above true? The answer in the following concept.

A subsystem N of an S-system M_S is called fully invariant if $\alpha(N) \subseteq N$ for every endomorphism α of M_S. M_S is called duo if every subsystem of M_S is fully invariant [10].

Proposition 3.6: Every pseudo-injective duo S-system is fully pseudo-stable.

Proof: Let M_S be a pseudo-injective duo S-system, N be a subsystem of M_S, and $\alpha : N \to M$ be any S-monomorphism. Then α can be extended to an S-endomorphism$\beta : M \to M$. Now, $\alpha(N) = \beta(N) \subseteq N$.Thus M_S is fully pseudo-stable.

4. Fully Pseudo Stable Extension

In this part we can raise the question: Is for every proper subsystem of any system there exists proper pseudo-stable subsystem contains it? First we introduce the following.

Definition 4.1: Let X_S, Y_S be two S-systems. Define the mono-trace of X_S in Y_S denoted by $m\text{-}tr_{Y_S}(X_S)$ by

$m\text{-}tr_Y(X_S) = \cup_{f : X \to Y} f(X_S)$, where f is S-monomorphism.

If N_S is a subsystem of M_S, then the mono-trace m-tr(N) in M is the mono-trace of N_S in M_S.

Examples 4.2:

(1) Let (N, ∗) be a monoid defined by $n * m = \max\{n, m\}$ for each n, m in N. Since the only N-momomorphism from any subsystem of N_N into N_N is the inclusion map. Hence $tr_N(K) = K$ for each subsystem K in N.

(2) Let S be a monoid in example (2.3)(3). Consider S as an S-system, then:$m\text{-}tr_S(\{a, b\}) = m\text{-}tr_S(\{a\}) = m\text{-}tr_S(\{b\}) = \{a, b\}$, $m\text{-}tr_S(\{a, b, c\}) = \{a, b, c\}$ and $m\text{-}tr_S(\{a, c\}) = \{a, c\}$.

we shall consider the pseudo-stable extension of S-system. First, let $M_S$be an S-system and N be a subsystem of M_S (not necessarily pseudo-stable). Define:

$m\text{-}tr_M(N) = < \alpha(x) : x \in N$ and $\alpha : N \to M$ is S-monomorphism $> =$

$\cup_{f : N \to M} f(N)$ where f is S-monomorphism,

the subsystem generated by all the images of elements of N under the S-monomorphism from N into M. Clearly $N \subseteq m\text{-}tr_M(N)$. In fact M is fully pseudo-stable S-system if and only if $N = m\text{-}tr_M(N)$ for each subsystem N of M_S this is equivalent to saying that $N = \cup\, \theta(N)$ for each subsystem N of M where the union is taken over all S-monomorphisms$\theta$from N into M.

Proposition 4.3: Let $N_S$be a subsystem of an S-system M_S. Then

1. $m\text{-}tr_M(N)$ is a pseudo-stable subsystem of M_S.

2. If M_S is pseudo-injective and N is essential subsystem of M_S, then $m\text{-}tr_M(N)$ is the smallest pseudo-stable subsystem of M_S containing N.

Proof: (1) let $\beta : m\text{-}tr_M(N) \to M$ be an S-monomorphism and $f(x) \in m\text{-}tr_M(N)$ where $x \in N$, then $\beta(f(x)) = (\beta \circ f)(x) \in m\text{-}tr_M(N)$, and hence $m\text{-}tr_M(N)$ is pseudo-stable.

(2) Let K be a pseudo-stable subsystem of M containing N and $f(x) \in m\text{-}tr_M(N)$. Pseudo-injectivity of M implies that there is an S-homomorphism $\beta : M \to M$ which extends f. Since f is S-monomorphism, then the definition of essential subsystem implies that β is S-monomorphism also. Now $x \in N \subseteq K$, thus $f(x) = \beta(x) \in K$, since K is pseudo-stable subsystem in M_S. Hence $m\text{-}tr_M(N) \subseteq K$.

Proposition 4.4: If N_S is an essential subsystem of a pseudo-injective S-system M_S, then the intersection of two pseudo-stable subsystems of M_S which are containing N_S is pseudo-stable.

Proof: Let A_1, A_2 be two pseudo-stable subsystems of M with $N \subseteq A_1$, $N \subseteq A_2$. To show that $A_1 \cap A_2$ is pseudo-stable. Let $\mu : A_1 \cap A_2 \to M$ be an S-monomorphism. Pseudo-injectivity of M implies that there is an S-homomorphism $\beta : M \to M$ which extend μ. Put $\beta_i = \beta|_{A_i} : A_i \to M$ for i=1, 2. Since N is essential and $\beta|_N = \alpha = \mu \circ i : N \to M$, then β is S-monomorphism, and hence β_i is S-monomorphism. Since A_1, A_2 are pseudo-stable subsystem of m, then for each a in $A_1 \cap A_2$ we have $\mu(a) = \beta_i(a) \in A_1 \cap A_2$. Hence $\mu(A_1 \cap A_2) \subseteq A_1 \cap A_2$.

Let M_S be an S-system and E(M) be the injective envelop of M. We can consider M as essential subsystem of E(M) [2]. We denote PS(M) the pseudo-stable extension of M in E(M) and call it the pseudo-stable envelop of M. It is clear that PS(M) is an essential extension of M [4]. On the other hand, in each injective envelop, the pseudo-stable envelope is unique, proposition 4.3. Since the injective envelop is unique up to isomorphism, then any two pseudo-stable envelops of M_S are isomorphic. Thus we have the following.

Theorem 4.5: Every S-system M_S has pseudo-stable envelope, and any two pseudo-stable envelops of M_S are isomorphic.

Proposition 4.6: If M_S is a reversible pseudo-injective S-system, then every pseudo-stable subsystem of M_S is pseudo-injective.

Proof: Suppose N_S is a pseudo-stable subsystem of a reversible pseudo-injective S-system M_S. Let K be a subsystem of N and $\alpha : K \to N$ be an S-monomorphism. Then $i \circ \alpha : K \to M$ is monomorphism where i is the inclusion map of N into M_S. Pseudo-injectivitiy of M_S implies that there is $\beta : M_S \to M_S$ which extends $i \circ \alpha$. Essential property of K gives that β is monomorphism and by pseudo-stablity of N, we have $\beta_1 = \beta|_N : N \to N$ is the extension of α. Hence N is pseudo-injective subsystem of M_S.

Let M_S be an S-system. The pseudo-injective envelope of M_S denoted by P(M), is defined as the minimal pseudo-injective extension of M_S, which is an essential extension of M_S.

Proposition 4.7: If M_S is a reversible S-system. Then for any S-system M, the pseudo-stable envelop PS(M) of M in E(M) is the pseudo-injective envelop of M, that is PS(M) = P(M).

Proof: PS(M) is the smallest pseudo-stable subsystem of E(M) containing M. Thus PS(M) is the smallest pseudo-injective system containing M, proposition 4.6. Further it is essential extension of M. Hence PS(M) is the pseudo-injective envelope of M i.e. PS(M) = P(M).

Proposition 4.8 : For every reversible pseudo-injective S-system M_S, the following are equivalent:

1. N_S is pseudo-stable subsystem of M_S;

2. N_S is pseudo-injective.

Proof: N_S is essential pseudo-stable subsystem of M_S, if and only if N_S = PS(N), proposition 4.3. proposition4.7 implies that PS(N) (and hence N_S) is the pseudo-injective envelope of N_S, if and only if N_S is pseudo-injec

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