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Existence Theorems For α(u,v)-monotone of nonstandard Hemivariational Inequality

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ABSTRACT:

In this paper, we consider the existence result of a nonstandard hemivariational inequalities with $\alpha(u,v)$-monotone mapping in reflexive and non reflexive Banachs space. Finally, we provide sufficient conditions for which that inequality has a solution in the case of unbounded sets, via the fixed point and KKM theorems.

KEYWORDS

Hemivariational inequalities; existence of solutions; KKM mapping; fixed point theorem'Clarke's generalized gradient; α(u, v)- monotone mappings.

MSC

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INTRODUCTION

The theory of hemivariational inequalities introduced by Panagiotopoulos in the early eighties of the 20th century (see [20, 22]). Indeed, theory of hemivariational inequalities may be considered as an extension of the theory of variational inequalities [15]. The existence of a solution of a hemivariational inequality, a generalized hemivariational inequality, and other related problems have became a basic research topic which continues to attract the attention of researchers in applied mathematics as well as in engineering sciences, since the main ingredient used in the study of these inequalities is the notion of Clarke subdifferential of a locally Lipschitz functional (see for instance [1,5,6,9,12,17,18, 19] and the references therein).

In (1999) Panagiotopoulos, Fundo and Radulescu [21] extended the classical results of [10], by proving a number of versions of theorems of Hartman-Stampacchia's type for the case of hemivariational inequalities on compact or on closed and convex subsets, in infinite and finite dimensional Banach spaces. In (2002)[8] L.Gasinski studied a hyperbolic hemivariational inequality with a nonlinear, pseudomonotone operator depending on the derivative of an unknown function and a linear, monotone operator depending on an unknown function. In (2007) [3] Carl, S., and Le, V.K., provided existence results for multivalued quasilinear elliptic problems of hemivari- ational inequality type with measure data. In (2009)[5] N. Costea,V. Radulescu studied existence results for hemivariational inequalities with relaxed $\eta-\alpha$ monotone mappings on bounded, closed and convex subsets in reflexiveBanach spaces. Moreover, K. Teng. In (2013)[24] establ-ished the existence of two nontrivial solutions of a class of nonlocal hemivariational inequality-es depending on two parametric.His methods are based on critical point theory for non-differentiable functional. In this paper, we introduce a new class of monotone operator namely α(u, v) monot- one mappings and prove the existence of solutions for non-standard hemivariational inequality with α(u, v)- monotone mappings in Banach and reflexive Banach spaces, via fixed point theorem and the KKM technique. In addition, we provide sufficient conditions for existence solute-on for the case of unbounded sets. The results presented in this paper improve and extend some corresponding results of several authors.[1], [5], [3],[8].

Definition 0.1. For functions $\mathbf{A}: \mathbf{X} \to \mathbf{2}^{\mathbf{X}^*}$ **and** $\boldsymbol{\alpha}$: $\mathbf{dom\ A} \times \mathbf{dom\ A} \to \mathbf{R}$, T is said to be α(x, y)-monotone if for each x, y ∈ dom A, such that $< A(\mathbf{x}) - \mathbf{A}(\mathbf{y}), \mathbf{x}-\mathbf{y} > \geq \boldsymbol{\alpha}(\mathbf{x}, \mathbf{y})$.

Throughout this paper, we assume that K be a nonempty subset of X. and α define as follows:

D ={α:α:dom A×dom A→R }is continuous function with respect to first variable such that

$\lim_{t\to 1^+} \frac{\alpha(u+t(v-u),v)}{1-t} = 0$ for each u, v ∈ K, and $\alpha(v,v) = 0$, for each v ∈ K.

In the last years, a number of authors have introduced several generalization of monotonicity such as pre-monotonicity, st-rong monotone, γ –para-monotone and θ-monotone.

Remark 0.2. (i) If $\boldsymbol{\alpha}(\mathbf{x},\mathbf{y}) = \mathbf{0}$, we obtain the concept of Minty monotonicity [16].

(ii)If $\boldsymbol{\alpha}(\mathbf{x},\mathbf{y}) = -\mathbf{min}\{\boldsymbol{\alpha}(\mathbf{x}), \boldsymbol{\alpha}(\mathbf{y})\}.\|\mathbf{x}-\mathbf{y}\|$, we obtain the concept of Pre-monotone operator [11].

(iii)If $\boldsymbol{\alpha}(\mathbf{x},\mathbf{y}) = \mathbf{r}\|\mathbf{x}-\mathbf{y}\|^2$ we obtain the concept of strong monotone operator [25].

(iv) If $\boldsymbol{\alpha}(\mathbf{x},\mathbf{y}) = \boldsymbol{\theta}(\mathbf{x},\mathbf{y}).\|\mathbf{x}-\mathbf{y}\|$ we obtain the concept of θ-monotone operator [14].

(v)If α(x, y)= $-\mathbf{c}\|\mathbf{x}-\mathbf{y}\|^{\gamma}$ we obtain the concept of γ-para-monotone operator[25].

(vi) If α(x,y) = $\mathbf{f}(\|\mathbf{x}-\mathbf{y}\|).\|\mathbf{x}-\mathbf{y}\|)$ we obtain the concept of strong monotone operator [13].

1. PRELIMINARIES .

For the convenience of the reader.We mention here some notations and properties that will be used during this paper.

Let X be a Banach space with dual space $X^*$and < . , . > denoted the pairing between X and X^*. $\mathbf{T}: \mathbf{X} \to \mathbf{L}^{\mathbf{p}}(\Omega, \mathbf{R}^{\mathbf{k}})$ will be a linear and compact operator where p ∈ (1,∞) and Ω is a bounded and open subset of R^N. We will denote by Tv = $\hat{v}$ and $\acute{p}$ the conjugated exponent of p. The Clarke's generalized gradient ∂f(x,y) of the locally Lipschitz map f(x, .) is defined by:

$$\partial \mathbf{f}(\mathbf{x},\mathbf{y}) = \{\mathbf{z} \in \mathbf{R}^{\mathbf{k}} : \mathbf{z}.\boldsymbol{h} \leq \boldsymbol{f}^{\mathbf{0}}(\mathbf{x},\mathbf{y};\ \mathbf{h}), \mathbf{for\ each\ h} \in \mathbf{R}^{\mathbf{k}}\},$$

Where the symbol "." means the inner product on R^k. The euclidian norm in R^k, k ≥ 1, resp. the duality pairing between a Banach space and its dual will be denoted by | ⋅ |, respectively < ., .>. We also denote by $\|.\|_p$ the norm in the space $L^p(\Omega, R^k)$ defined by:

$$\|\hat{\mathbf{v}}\|_{\mathbf{p}} = \left(\int_{\Omega} |\hat{\mathbf{v}}(\mathbf{x})|^{\mathbf{p}}\, \mathbf{dx}\right)^{\frac{1}{p}},\ \boldsymbol{p} \in (\mathbf{1}, \infty).$$

Let **f = f(x, y):** $\boldsymbol{\Omega} \times \mathbf{R}^{\mathbf{k}} \to \mathbf{R}$ be a Caratheodory function, locally Lipschitz with respect to the second variable which satisfies the following condition:

$$|\mathbf{z}| \leq \mathbf{C}\left(\mathbf{1} + |\mathbf{y}|^{\mathbf{p-1}}\right) \text{ for some C >0} \tag{1.1}$$

a.e. $\mathbf{x} \in \Omega$**, for all** $\mathbf{y} \in \mathbf{R^k}$ **and all** $\mathbf{z} \in \partial \mathbf{f}(\mathbf{x}, \mathbf{y})$**.**

We shall use the notation f^0(x, y; h) for the Clarke's generalized directional derivative (see e.g. [4]) of the locally Lipschitz mapping f(x, .) at the point y$\in R^k$ with respect to the direction h $\in R^k$, where x $\in \Omega$, i.e.,

$$\mathbf{f^0}(\boldsymbol{x}, \boldsymbol{y}; \boldsymbol{h}) = \underset{\mathbf{u} \to \mathbf{y}, \mathbf{t} \downarrow \mathbf{0}}{\mathbf{limsub}} \frac{\boldsymbol{f}(\mathbf{x}, \mathbf{u}+\mathbf{th}) - \boldsymbol{f}(\mathbf{x}, \mathbf{u})}{\boldsymbol{t}}.$$

Definition 1.1. [13] A mapping N: K $\to 2^E$ (by 2^E we understand the family of all the subsets of E) is said to be a KKM mapping if for any $\{u_1, u_2, \dots u_n\} \in$ k, $co\{u_1, u_2, \dots u_n\} \subset \cup_{i=1}^{n} N(u_i)$, where, $co\{u_1, u_2, \dots u_n\}$ is denotes the convex hull of $\{u_1, u_2, \dots u_n\}$.

Lemma 1.2. [7] Let K be a nonempty subset of a Hausdorff topological vector space E and N: K $\to 2^E$ be a KKM mapping. If N (u) is closed in E for each u $\in$ K and compact for some $u_0 \in K$, then $\cap_{u \in k} N(u) \neq \emptyset$.

Lemma 1.3. [23]. Let K be a nonempty convex subset of a Hausdorff topological vector space E. Let F : K $\to 2^K$ be a set valued map such that:

(C_1) For each v $\in$ K, F(v) is a nonempty convex subset of K.

(C_2) For each u $\in$ K, $F^{-1}(u)$ = {v $\in$ K: u $\in$ F (v)} contains an open set S_u .

(C_3) $\cup_{u \in k} S_u = K$.

(C_4) There exists a nonempty set U_0 contained in a compact convex subset U_1 of K such that L = $\cap_{u \in k} S_u^c$ is empty or compact, where S_u^c is the complement of S_u in K.

Then there exists a point $v_0 \in$ K in which $v_0 \in F(v_0)$.

Lemma 1.4.[21] Suppose that f satisfies the assumption (1.1) and X_1, X_2 are nonempty subsets of X, and T: X→ $L^p(\Omega ; R^K)$ is a linear compact operator. Then the mapping from $X_1 \times X_2$ to R defined by:

$$(v, u) \to \int_\Omega f^0\big(x, \hat{v}(x), \hat{u}(x) - \hat{v}(x)\big) dx \ .$$

is weakly upper semicontinuous.

Throughout this paper $\int_\Omega f^0(x, \hat{v}, \hat{u} - \hat{v}) dx$, will denote $\int_\Omega f^0\big(x, \hat{v}(x), \hat{u}(x) - \hat{v}(x)\big) dx$.

Definition 1.5.[2] Let A: K $\to X^*$ be mappings. A is called hemicontinuous at a point u in K if

$\lim_{t \to 0} < Au + th, v > = < Au, v >$ for each h , v $\in$ K and t $\in$ (0,1).

2. MAIN RESULTS

In order to prove the existence results. We will consider the following problem:

Find v $\in$ K, such that

$$< Au, u - v > + \int_\Omega f^0(x, \hat{v}, \hat{u} - \hat{v}) dx \geq \alpha(u, v) \quad \forall u \in K \tag{2.1}$$

Lemma 3.1 Let K be a nonempty, convex and subset of a reflexive Banach space X, Assume that:

(C_1)A: K $\to X^*$ be hemicontinuous and α(u,v)-monotone operator and continuous with respect to first variable.

(C_2) v→ <Az, u-v> is convex for each u, z $\in$ K.

(C_3) $\alpha(v, v) = 0$, for each u,v $\in$ K.

Then v $\in$ K is a solution of (2.1) iff it solves the following hemivariational inequality problem.

Find v $\in$ k such that

$$< Av, u - v > + \int_\Omega f^0(x, \hat{v}, \hat{u} - \hat{v}) dx \geq 0 \quad \forall u \in k \tag{2.2}$$

Proof. Assume that v is a solution of (2.1) and fix u $\in$ K, letting $z = u +$ t(v-u), t $\in$ (0,1], then z$\in$ K. It follows from (2.1),

$$< Az, z - v > + \int_\Omega f^0(x, \hat{v}, \hat{z} - \hat{v}) dx \geq \alpha(z, v) = \alpha((u + t(v - u), v) \tag{2.3}$$

Take left side, by $(C_1), (C_2)$ *and convexity of the mapping* $\hat{u} \to f^0(x, \hat{v}, \hat{u})$,

$$< Az, z - v > + \int_\Omega f^0(x, \hat{v}, \hat{z} - \hat{v}) dx \leq t < Az, v - v > + (1 - t) < Az, u - v >$$

$$+t\int_{\Omega} f^0(x,\hat{v},\hat{v}-\hat{v})dx + (1-t)\int_{\Omega} f^0(x,\hat{v},\hat{u}-\hat{v})dx.$$

So, by (2.3)

$$(1-\mathrm{t})\left[< Az, u-v > + \int_{\Omega} f^0(x,\hat{v},\hat{u}-\hat{v})dx\right] \geq \alpha((u+t(v-u),v) \quad (2.4)$$

$$\left[< Az, u-v > + \int_{\Omega} f^0(x,\hat{v},\hat{u}-\hat{v})dx\right] \geq \frac{\alpha((u+t(v-u),v)}{(1-t)} \quad (2.5)$$

Letting t→ 1^+, $using\ (C_1)-(C_3)$, v solves (2.2).

Conversely, assume that v is a solution of (2.2).From definition of α(u,v)-monotone operator,

$$< Au - Av, u - v \geq \alpha(u,v) \quad (2.6).$$

Hence by (2.2) and (2.6) we have solution of (2.1).This completes the proof. □

Theorem. 2.2 Let K be a nonempty, convex and closed subset of reflexive Banach space X. Assume that:

(C_1) $\alpha(v,v) = 0$, $for\ each\ u, v \in K$.

(C_2) $v \to < Az, u-v >$ $is\ convex\ and\ lower\ semicontinuous$ $\forall\, u, z \in K$.

(C_3) A: K→ X^* is hemicontinuous map and α(u,v)-monotone operator such that α is continuous with respect to first variable.

(C_4) For any sequence $\{v_n\} \subset X$ in which $v_n \to v$, then $\limsup_n \alpha(u, v_n) \geq \alpha(u,v)$.

Then there exists at least one solution for (2.1).

Proof: By contradiction, assume that (2.1) has no solution. Then for each v ∊ K, there exists u ∊ K,

$$< Au, u-v > + \int_{\Omega} f^0(x,\hat{v},\hat{u}-\hat{v})dx < \alpha(u,v) \quad (2.7).$$

Using Lemma (2.1), we get for each $v \in K$, there exist$s\ u \in K$,

$$< Av, u-v > + \int_{\Omega} f^0(x,\hat{v},\hat{u}-\hat{v})dx < 0\,. \quad (2.8).$$

Define two set valued mappings $F, S_u : K \to 2^K$ as follows:

$$F(v) = \{u \in K\colon < Av, u-v > + \int_{\Omega} f^0(x,\hat{v},\hat{u}-\hat{v})dx < 0\,\}$$

$$S_u(v) = \{v \in K\colon < Au, u-v > + \int_{\Omega} f^0(x,\hat{v},\hat{u}-\hat{v})dx < \alpha(u,v)\,\}$$

Our claim F satisfies the conditions of Lemma 1.3. Proof is divided into the following six steps.

Step 1: F (v) is a nonempty. This is directly from (2.8).

Step 2: F (v) is a convex for each v ∊ K.

Let v ∊ K be fixed, choose $r, m \in F(v), and\ z = (1-t)r + tm, where\ t \in$ [0,1]. By (C_2) and convexity of the mapping $\hat{u} \to f^0(x,\hat{v},\hat{u})$,

$$< Av, z-v > + \int_{\Omega} f^0(x,\hat{v},\hat{z}-\hat{v})dx \leq t < Av, m-v > + (1-\mathrm{t}) < Av, r-v >$$

$$+t\int_{\Omega} f^0(x,\hat{v},\widehat{m}-\hat{v})dx + (1-t)\int_{\Omega} f^0(x,\hat{v},\hat{r}-\hat{v})dx \quad (2.9)$$

Since $r, m \in F(v)$,

$$< Av, z-v > + \int_{\Omega} f^0(x,\hat{v},\hat{z}-\hat{v})dx \leq 0$$

Therefore, z ∊ F(v) and F(v) is convex for each v ∊ K.

Step 3: S_u^c is weakly closed.

Let $\{v_n\} \subset S_u^c$ be a sequence that converge weakly to v as n →∞. We will prove v ∊ S_u^c.

Since f satisfy conditions Lemma 1.4. The application$(v,u) \to \int_{\Omega} f^0(x,\hat{v},\hat{u}-\hat{v})dx$ is weakly upper semicontinuous, and by $(C_2)-(C_4)$.

$$\alpha(u,v) \leq \limsup_n \alpha(u, v_n)$$

$$\leq \limsup_n < Au, u-v_n > + \limsup_n \int_{\Omega} f^0(x,\widehat{v_n},\hat{u}-\hat{v})dx\,.$$

$$\leq -\liminf_n < Au, v_n - u > +\limsup_n \int_\Omega f^0(x, \widehat{v_n}, \hat{u} - \hat{v})dx$$
$$\leq - < Au, v - u > + \int_\Omega f^0(x, \hat{v}, \hat{u} - \hat{v})dx$$
$$\leq < Au, u - v > + \int_\Omega f^0(x, \hat{v}, \hat{u} - \hat{v})\, dx.$$

This means that $v \in S_u^c$.

Step4: $S_u \subseteq F^{-1}$(u).

Let $v \notin F^{-1}(u)$. This is equivalent to $u \notin F(v)$. Then

$$< Av, u - v > + \int_\Omega f^0(x, \hat{v}, \hat{u} - \hat{v})dx \geq 0.$$

And from definition of α(u,v)-monotonicity of A,

$$< Au, u - v > + \int_\Omega f^0(x, \hat{v}, \hat{u} - \hat{v})dx \geq \alpha(u, v).$$

Hence, $v \notin S_u$. This implis that $S_u \subseteq F^{-1}(u)$.

Step 5: $\bigcup_{u \in k} S_u = K$. It is enough to show that $K \subseteq \bigcup_{u \in k} S_u$.

For any v ∊ K, by (2.8), there exists u ∊ K such that

$$< Au, u - v > + \int_\Omega f^0(x, \hat{v}, \hat{u} - \hat{v})dx < \alpha(u, v)\,.$$

Therefore, $v \in S_u$.

Step 6: L $=\bigcap_{u \in k} S_u^c$ is empty or weakly compact.

Since L is the intersectin of weakly closed sets S_u^c , so, L is weakly closed. K is a nonempty, bounded closed and convex subset of a reflexive Banach space. So, K is weakly compact. Then L is also weakly compact. Thus, the conditions of Lemma 1.3 satisfiy in the weak topology. There is a fixed point $v_0 \in K$, i.e, $v_0 \in F(v_0)$ which implies that :

$$0 =< Av_0, v_0 - v_0 > + \int_\Omega f^0(x, \widehat{v_0}, \widehat{v_0} - \widehat{v_0})dx < 0.$$

This is a contradiction. Hence (2.1) has at least one solution. □

Theorem 2.3. Assume that the assumptions as in Theorem 2.2 hold without the hypothesis of boundedness of K. In addition, suppose that A(0) = 0 ∊ K and there exists r ≥ m> 1 such that

$$\frac{\alpha(0, v_n)}{\|v_n\|^r} \to \infty \;as\; \|v_n\| \to \infty.$$

Then the inequality problem (2.1) admits at least one solution.

Proof. Define $K_n = \{u \in K : \|u\| \leq n\}$. By Theorem 3.2, there exists $v_n \in K_n$,

$$< Au, u - v_n > + \int_\Omega f^0(x, \widehat{v_n}, \hat{u} - \widehat{v_n})dx \geq \alpha(u, v_n) \qquad \forall u \in K. \tag{2.10}$$

Claim 1.There exist $n_0 \in$ N such that $\|v_{n_0}\| < n_0$.

by contradiction , assume that $\|v_n\| = n$ for each $n \in [0, \infty)$. Setting u = 0 in (2.10), then

$$\alpha(0, v_n) \leq \int_\Omega f^0(x, \hat{v}, -\widehat{v_n})dx \qquad \forall u \in K \tag{2.11}$$

On the other hand, for any y, h ∊ R^k there exists $z \in \partial f(x, y)$ (see [4] Prop. 2.1.2), in which

$$f^0(x, y; h) = z.h = \max\{z.h : z \in \partial f(x, y)\}.$$

It follows from (1.1) that

$$|f^0(x, y; h)| \leq |z||h| \leq C(1 + |y|^{p-1})|h|.$$

Using Holders inequality and the fact that $T: X \to L^p(\Omega, R^k)$ is a linear and compact, so

$$\left|\int_\Omega f^0(x, \widehat{v_n}, -\widehat{v_n})dx\right| \leq (C_1\|\widehat{v_n}\|_m + C_2\, \|\widehat{v_n}\|_m^m) \leq (C_3\|v_n\| + C_4\, \|v_n\|^m) \tag{2.12}$$

For some suitable constants $C_1, C_2, C_3, C_4 > 0$ .Therefore, from (2.11) and (2.12),

$$\frac{\alpha(0, v_n)}{\|v_n\|^r} \leq (C_3\|v_n\|^{1-r} + C_4\, \|v_n\|^{m-r})\,.$$

Passing to the limit as n→ ∞ we reach a contradiction, since r ≥ m>1.

Claim 2. v_{n_0} solves inequality problem (2.1)

Since $\|v_{n_0}\| < n_0$, for each $u \in K$. Definie $t \in (0,1]$ as follows

$$t=\begin{cases} 1 & if \quad v_{n_0}=u \\ min\frac{n_0-\|v_{n_0}\|}{\|u-v_{n_0}\|} & otherwise \end{cases}.$$

We conclude $z = t\, v_{n_0} + (1-t)u_0$ satisfies z in K_{n_0}. From (2.10) and the positive homogeneity of the mapping $\hat{u} \to f^0(x,\hat{v},\hat{u})$. We get

$$\alpha(z, v_{n_0}) \leq < Az, z - v_{n_0} > + \int_\Omega f^0(x, \widehat{v_{n_0}}, \hat{z} - \widehat{v_{n_0}})dx .$$

So,

$$\alpha(\, t\, v_{n_0} + (1-t)u_0\, , v_{n_0}) \leq t[< Az, v_{n_0} - v_{n_0} > + \int_\Omega f^0(x, \widehat{v_{n_0}}, \widehat{v_{n_0}} - \widehat{v_{n_0}})dx \;]$$

$$+(1-t)[< Az, u_0 - v_{n_0} > + \int_\Omega f^0(x, \widehat{v_{n_0}}, \widehat{u_0} - \widehat{v_{n_0}})dx] \tag{2.13}$$

Letting t→ 0^+ in (2.13) and using the hemicontinuous of A,

$$\alpha(u_0 , v_{n_0}) \leq < Au_0 , u_0 - v_{n_0} > + \int_\Omega f^0(x, \widehat{v_{n_0}}, \widehat{u_0} - \widehat{v_{n_0}})dx .$$

That means v_{n_0} is a solution of (2.1).

Here, we will give the sufficient conditions for which Theorem 2.2 for the case X Banach space by using KKM technique.

Theorem. 2.4. Assume that K is a nonempty convex and compact subset of a Banach space X, and the assumptions $(C_1)-(C_4)$ in Theorem 2.2 hold. Then there exists at least one solution for (2.1).

Proof. Define two set valued mappings $N, M : K \to 2^K$ as follows :

$$N(u) = \{v \in k: \; < Av, u - v \geq + \int_\Omega f^0(x,\hat{v},\hat{u}-\hat{v})dx \geq 0\,\}$$

$$M(u) = \{v \in k: \; < Au, u - v \geq + \int_\Omega f^0(x,\hat{v},\hat{u}-\hat{v})dx \; \geq \alpha(u,v)\,\}$$

Obviously, $N(u), M(u)$ are nonempty for each $u \in k$. The proof is divided into the following four steps.

Step 1: N is a KKM mapping,

In fact, if N is not a KKM mapping, then there exist $\{u_1, u_2, \dots \dots u_n\} \subseteq K$ such that: $co\{u_1, u_2, \dots \dots u_n\} \not\subset \cup_{i=1}^n N(u_i)$. There exist a $u_0 \in co\{u_1, u_2, \dots \dots u_n\}$, $u_0 = \sum_{i=1}^n t_i u_i,\; t_i \geq 0, \forall\, i = 1,2, \dots n, \sum_{i=1}^n t_i = 1$, in which $u_0 \notin \cup_{i=1}^n N(u_i)$. By definition of N,

$$< Au_0, u_i - u_0 > + \int_\Omega f^0(x, \widehat{u_0}, \widehat{u_i} - \widehat{u_0})dx < 0 , i = 1,2, \dots\; n.$$

Since $\sum_{i=1}^n t_i = 1, t_i \geq 0 \;\forall\, i = 1,2, \dots . n$, so

$$0 > \left[< Au_0, u_i - u_0 \geq + \int_\Omega f^0(x, \widehat{u_0}, \widehat{u_i} - \widehat{u_0})dx\right] \sum_{i=1}^n t_i$$

$$= < Au_0, \sum_{i=1}^n t_i\, u_i - u_0 \geq + \int_\Omega f^0(x, \widehat{u_0}, \widehat{\sum_{i=1}^n t_i\, u_i} - \widehat{u_0})dx$$

$$= < Au_0, u_0 - u_0 \geq + \int_\Omega f^0(x, \widehat{u_0}, \widehat{u_0} - \widehat{u_0})dx = 0.$$

Which is contradiction and this implies that N is a KKM map.

Step 2: N(u) ⊆ M(u) for each u ∊ K. This satisfies from Lemma.2.1. It implies that M(u) is also a KKM map .

Step 3: M (u) is closed for each u ∊ K. This is satisfies (see Theorem 2.2, step 3).

Step 4: M (u) is compact for each u ∊ K.

Because M (u) is closed subset of compact K for each u ∊ K. Thus, M (u) is compact for each u ∊ K. According to Lemma 1.2. We have $\cap_{u \in K} M(u) \neq \emptyset$. It follows that there exists u ∊ K

$$< Au, u - v \geq + \int_\Omega f^0(x,\hat{v},\hat{u}-\hat{v})dx \geq \alpha(u,v).$$

Therefore, (2.1) has at least one solution. The proof is complete. □

Corollary. 2.5. Assume that K is a nonempty, convex subset of a Banach space X, and the assumptions $(C_1 - C_4)$ in Theorem 2.2 hold. In addition, assume that:

$C_5: \int_\Omega f^0(x, \hat{v}, \hat{0} - \hat{v})dx = 0.$

$C_6: \sigma = \{v \in K: \alpha(0, v) \leq 0\}$ is relative compact in which A(0) = 0 ∊ K.

Then there exists at least one solution for (2.1).

Proof. It suffices to prove M (u) is compact for some u ∊ K. Since

$$M(u) = \{v \in k: \ < Au, u - v \geq + \int_\Omega f^0(x, \hat{v}, \hat{u} - \hat{v})dx \ \geq \alpha(u, v) .$$

So,

$$M(u) \subseteq \{v \in k: \ < Au, u - v \geq \frac{\alpha(u,v)}{2}\} \cup \{\int_\Omega f^0(x, \hat{v}, \hat{u} - \hat{v})dx \ \geq \frac{\alpha(u,v)}{2}\} .$$

Since $A(0) = 0 \in K$, we get

$$M(0) \subseteq \{v \in k: \ < A(0), 0 - v \geq \frac{\alpha(0,v)}{2}\} \cup \{\int_\Omega f^0(x, \hat{v}, \hat{0} - \hat{v})dx \ \geq \frac{\alpha(0,v)}{2}\} .$$

Hence, M(0) is a closed subset of relative compact set σ. That means M (u) is compact for some u ∊ k.

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