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Oscillatory Behavior of High Order Nonlinear Mixed Type Difference Equations With a Nonlinear Neutral Term

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Abstract

This paper discusses the high order nonlinear neutral mixed type difference equation

$$\Delta^m \left[x(n) + p(n)h\left(x(\sigma(n))\right)\right] + q(n)f\left(x(\tau(n))\right) = 0, \quad n = 0, 1, 2, \ldots,$$

where $(p(n))$, $(q(n))$ are sequences of nonnegative real numbers, h, $f : \mathbb{R} \to \mathbb{R}$ are continuous and nondecreasing with $uh(u) > 0$, $uf(u) > 0$ for all $u \neq 0$, and $(\sigma(n))$ and $(\tau(n))$ are sequences of integers such that

$$\lim_{n\to\infty} \tau(n) = \lim_{n\to\infty} \sigma(n) = \infty.$$

Generally, the oscillatory behaviour of the solutions for this equation will be investigated. Especially, when m is even, the result obtained here has completed the oscillation studies related to the above equation. In addition, examples showing the accuracy of the results are given.

Keywords: Difference equation, high order, nonlinear, oscillation.

Mathematics Subject Classification: 39A10, 39A21.

Introduction

$$\Delta^m \left[x(n) + p(n)h\left(x(\sigma(n))\right)\right] + q(n)f\left(x(\tau(n))\right) = 0, \quad n = 0, 1, 2, \ldots$$

We consider the following higher order nonlinear neutral mixed type difference equation

$$\Delta^m \left[x(n) + p(n)h\left(x(\sigma(n))\right)\right] + q(n)f\left(x(\tau(n))\right) = 0, \quad n = 0, 1, 2, \ldots, \tag{1.1}$$

where Δ is the usual forward difference operator defined by $\Delta x(n) = x(n+1) - x(n)$, $(p(n))$ and $(q(n))$ are sequences of nonnegative real numbers, $h, f : \mathbb{R} \to \mathbb{R}$ are continuous and nondecreasing with $uh(u) > 0$, $uf(u) > 0$ for all $u \neq 0$, and $(\sigma(n))$ and $(\tau(n))$ are sequences of integers such that

$$\lim_{n\to\infty} \tau(n) = \lim_{n\to\infty} \sigma(n) = \infty. \tag{1.2}$$

Throughout this paper, we assume the following conditions to hold;

(H_1) $(p(n))$ is a real-valued sequence with $p(n) \geq 0$, $n \in \mathbb{N}$.

(H_2) $(q(n))$ is a real-valued sequence with $q(n) \geq 0$, $n \in \mathbb{N}$ and $(q(n))$ is not eventually identically zero.

(H_3) $h, f : \mathbb{R} \to \mathbb{R}$ are continuous and nondecreasing with $uh(u) > 0$, $uf(u) > 0$ for all $u \neq 0$.

A solution $(x(n))$ of (1.1) is called oscillatory, if the terms $x(n)$ of the sequence are neither eventually positive nor eventually negative. Otherwise, the solution is said to be nonoscillatory. An equation is oscillatory if all its solutions oscillate.

For a long time, the oscillatory behavior and the existence of nonoscillatory solutions of difference equations have been extensively studied, see, for example, papers [1–28] and references cited therein. Most of these papers concern the special case where the arguments are delayed or advanced, while a small number of these papers [see, 9] are dealing with the case where the arguments are mixed.

In [1], Agarwal et al. and in [27], Zhang obtained some oscillatory results for the following equation, which is a special case of equation (1.1)

$$\Delta^m \left[x(n) + p(n)x(n-k)\right] + q(n)f\left(x(n-l)\right) = 0, \quad n = 0, 1, 2, \ldots, \tag{1.3}$$

where $(p(n))$ and $(q(n))$ are real sequences with $0 \leq p(n) < 1$ and $q(n) \geq 0$ for $n \in \mathbb{N}$, and $k,\ l$ are fixed nonnegative integers, $f : \mathbb{R} \to \mathbb{R}$ is continuous with $uf(u) > 0$ for all $u \neq 0$.

In [12], Kaleeswari and in [13], Kaleeswari and Selvaraj investigated the oscillatory behaviour of solutions for the following equation,

$$\Delta^m \left[x(n) + p(n)x(\sigma(n))\right] + q(n)f\left(x(\tau(n))\right) = 0, \quad n = 0, 1, 2, \ldots, \tag{1.4}$$

where $(p(n))$ and $(q(n))$ are real sequences with $0 \leq p(n) < 1$ and $q(n) \geq 0$ for $n \in \mathbb{N}$, $f : \mathbb{R} \to \mathbb{R}$ is continuous with $uf(u) > 0$ for all $u \neq 0$, which is another special case of equation (1.1).

For $m = 1$, $p(n) \equiv 0$ and $f(x) = x$, Eq.(1.1) reduce to

$$\Delta x(n) + q(n)x(\tau(n)) = 0, \quad n = 0, 1, 2, \ldots. \tag{1.5}$$

When $\tau(n) \leq n$, (where they can be $\tau(n) = n$ for some $n \in \mathbb{N}$, or $\tau(n) \equiv n$ for $n \in \mathbb{N}$) and $\tau(n)$ is nondecreasing, in 1998, Zhang and Tian [24] proved that, if

$$\limsup_{n\to\infty} q(n) > 0 \quad \text{and} \quad \liminf_{n\to\infty} \sum_{j=\tau(n)}^{n-1} q(j) > \frac{1}{e}, \tag{1.6}$$

then all solutions of (1.5) are oscillatory, while, in 2020, the present author [16] proved that, if

$$\liminf_{n\to\infty} \sum_{j=\tau(n)}^{n-1} q(j) > \frac{1}{e}, \tag{1.7}$$

then all solutions of (1.5) are oscillatory, which is the best result in the literature when $\tau(n) \leq n$, $n \in \mathbb{N}$ and $\tau(n)$ is nondecreasing, involving only lower limit condition.

In 2006 Braverman and Karpuz [6] obtained that, if

$$\limsup_{n\to\infty} \sum_{j=\tau(n)}^{n} q(j) \prod_{i=\tau(j)}^{\tau(n)-1} \frac{1}{1-q(i)} > 1, \tag{1.8}$$

then all solutions of (1.5) are oscillatory, which is the best result in the literature when $\tau(n) \leq n$, $n \in \mathbb{N}$ and $\tau(n)$ is nondecreasing, involving only upper limit condition.

When $\tau(n) < n$, $n \in \mathbb{N}$ and $\tau(n)$ is nondecreasing, in 2006, W. Yan, Meng and J. Yan [21] improved the condition (1.7) with the following

$$\liminf_{n\to\infty} \sum_{j=\tau(n)}^{n-1} q(j) \left(\frac{j-\tau(j)+1}{j-\tau(j)} \right)^{j-\tau(j)+1} > 1, \tag{1.9}$$

which is the best result in the literature, involving only lower limit condition.

Finally, when $\tau(n) < n$, $n \in \mathbb{N}$ and $\tau(n)$ is nondecreasing, for the all solutions of (1.5) to be oscillatory, involving only upper limit condition, which is the best result obtained in the literature so far is given in [17] by Öcalan, which is as follows; if

$$\limsup_{n\to\infty} \sum_{j=\tau(n)}^{n} q(j) \left(\frac{j-\tau(j)+1}{j-\tau(j)} \right)^{j-\tau(j)+1} \prod_{i=\tau(j)}^{\tau(n)-1} \frac{1}{1-q(i)} > e, \tag{1.10}$$

then all solutions of (1.5) oscillate.

For $m = 1$ and $p(n) \equiv 0$, Eq. (1.1) reduce to

$$\Delta x(n) + q(n) f\left(x(\tau(n))\right) = 0, \quad n = 0, 1, 2, \ldots. \tag{1.11}$$

When $\tau(n) \leq n$, $n \in \mathbb{N}$ and $\tau(n)$ is nondecreasing, in 2018, Öcalan, Özkan and Yıldız [14] (See also [15]) gave the following result regarding (1.11).

THEOREM A. Assume that $\tau(n) \leq n$, $n \in \mathbb{N}$ and $\tau(n)$ is nondecreasing, (H_2) and (H_3) hold. Also, we suppose that

$$\limsup_{x\to 0} \frac{x}{f(x)} = M. \tag{1.12}$$

Thus, if

$$\liminf_{n\to\infty} \sum_{j=\tau(n)}^{n-1} q(j) > \frac{M}{e}, \text{ where } 0 \leq M < \infty, \tag{1.13}$$

or

$$\limsup_{n\to\infty} \sum_{j=\tau(n)}^{n} q(j) > M, \text{ where } 0 < M < \infty, \tag{1.14}$$

then all solutions of (1.11) oscillate.

For the oscillation results of some superlinear and sublinear equations, which are important special cases of equation (1.1), see papers [5, 7, 10, 11, 18, 19, 20, 22, 23, 26].

In this study, we obtain some interesting results for the oscillation of the solutions of equation (1.1) by considering the $p(n)$ and $q(n)$ coefficients more broadly than the conditions studied in the literature. Moreover, the results obtained here are new when $h(x) = x$ and $f(x) = x$.

We need following lemmas proved in [2].

Lemma 1. *(Discrete Kneser's Theorem) Let $z(n)$ be defined for $n \geq a$, and $z(n) > 0$ with $\Delta^m z(n)$ of constant sign for $n \geq a$ and not identically zero. Then, there exists an integer j, $0 \leq j \leq m$ with $(m+j)$ odd for $\Delta^m z(n) \leq 0$, and $(m+j)$ even for $\Delta^m z(n) \geq 0$, such that*

$$j \leq m-1 \text{ implies } (-1)^{j+i} \Delta^i z(n) > 0, \quad \text{for all } n \geq a, \;\; j \leq i \leq m-1,$$

and

$$j \geq 1 \text{ implies } \Delta^i z(n) > 0, \quad \text{for all large } n \geq a, 1 \leq i \leq j-1.$$

Lemma 2. *Let $z(n)$ be as in Lemma 2.1 and bounded. Then*

$$(-1)^{i+1}\Delta^{m-i} z(n) > 0, \quad \text{for all large } n \geq a,\ 1 \leq i \leq m-1,$$

and

$$\lim_{n\to\infty} \Delta^i z(n) = 0,\ 1 \leq i \leq m-1.$$

The following lemma is an extension of the discrete analogue of known results in [8, Theorem 5.1.1]; it can also be found in [4, Lemma 6.2.2] and [25, Theorem 1]. The proof is immediate.

Lemma 3. *Let $(q(n))$ be a sequence of nonnegative real numbers, $(\tau(n))$ be a nondecreasing sequence of integers such that $\tau(n) \leq n$, and $f : \mathbb{R} \to \mathbb{R}$ be a continuous nondecreasing function with $uf(u) > 0$ for all $u \neq 0$. If the first order delay inequality*

$$\Delta y(n) + q(n) f\left(y(\tau(n))\right) \leq 0, \quad n = 0, 1, 2, \ldots,$$

has an eventually positive solution, then so does the delay equation

$$\Delta y(n) + q(n) f\left(y(\tau(n))\right) = 0, \quad n = 0, 1, 2, \ldots.$$

Equation (1.1) with m is even

In this section, when m is even, we obtain new result for the oscillation of the solutions of equation (1.1) under the assumptions $(H_1) - (H_3)$.

The following Theorem is the best result obtained so far in the literature when m is even for equation (1.1).

Theorem 1. *Let m be even. Assume that $\sigma(n)$, $\tau(n)$ are any delay or advanced arguments and (1.2), $(H_1) - (H_3)$ hold. Then, every solution of (1.1) oscillates.*

Proof. Let $(x(n))$ be a nonoscillatory solution of (1.1), with $x(n) > 0$ eventually. Set

$$z(n) = x(n) + p(n) h\left(x(\sigma(n))\right). \tag{2.1}$$

So, we get $z(n) \geq x(n) > 0$ and

$$\Delta^m z(n) = -q(n) f\left(x(\tau(n))\right) \leq 0,\ n \geq n_0. \tag{2.2}$$

Also, it is clear from Lemma 1.1 that $\Delta^i z(n)$ is eventually constant sign for $i = 1, 2, 3, ..., m-1$, $\Delta^{m-1} z(n) > 0$ and nonincreasing, and there are two possible cases where $\Delta z(n) < 0$ or $\Delta z(n) > 0$. We claim that $\Delta z(n) < 0$. If $\Delta z(n) > 0$, then $\lim_{n\to\infty} z(n) = a$, where $0 < a \leq \infty$. Summing both sides of (1.1) from $n_1 \geq n_0$ to N repeatedly m-times, we get

$$z(N+m) - z(n_1) + \sum_{j_1=n_1}^{N} \sum_{j_2=j_1}^{N} \cdots \sum_{j_m=j_{m-1}}^{N} q(j_m) f\left(x(\tau(j_m))\right) = 0. \tag{2.3}$$

Letting $N \to \infty$ in (2.3), we have

$$a - z(n_1) + \sum_{j_1=n_1}^{\infty} \sum_{j_2=j_1}^{\infty} \cdots \sum_{j_m=j_{m-1}}^{\infty} q(j_m) f\left(x(\tau(j_m))\right) = 0. \tag{2.4}$$

Here, since $\Delta z(n) > 0$, we get $a > z(n_1)$. Thus, by (H_1) and since $f(x) > 0$, we obtain a contradiction to (2.4). So, our claim is true, that is $\Delta z(n) < 0$. Thus, in Lemma 1.1, we must have $j = 0$. On the other hand, we know from Lemma 1.1 that $(m+j)$ odd for $\Delta^m z(n) \leq 0$. This is a contrediction to m is even and $j = 0$. The proof is complete. $\square$

Example 1. *Consider the fourth order difference equation*

$$\Delta^4 \left[x(n) + 2nx^{\frac{1}{3}}(n+1)\right] + \frac{n}{n+1}x^{\frac{5}{3}}(n-3) = 0, \quad n \geq 0. \tag{2.5}$$

Clearly no paper in the literature answers this. However, all conditions of Theorem 2.1 are satisfied, and hence all solutions of equation (2.5) are oscillatory.

Equation (1.1) with m is odd

In this section, when m is odd, we obtain new results for the oscillation of the solutions of equation (1.1) for the different possible cases of $(p(n))$ and $(q(n))$.

Theorem 2. *Let m be odd. Assume that $\sigma(n)$, $\tau(n)$ are any delay or advanced arguments, (1.2), $(H_1)-(H_3)$ and $\limsup_{n\to\infty} q(n) > 0$ hold. Then, every solution of (1.1) either oscillates, or every nonoscillatory solution of (1.1) tends to zero as $n \to \infty$.*

Proof. Let $(x(n))$ be a nonoscillatory solution of (1.1), with $x(n) > 0$ eventually, and assume further that $x(n)$ does not tend to zero as $n \to \infty$. By (2.1) and (2.2) , it is clear from Lemma 1.1 that $\Delta^i z(n)$ is eventually constant sign for $i = 1, 2, 3, ..., m-1$. Moreover, by Lemma 1.1 that $\Delta^{m-1}z(n) > 0$ and nonincreasing. Thus, summing (1.1) from $n_1 \geq n_0$ to ∞, we get

$$\sum_{n=n_1}^{\infty} \Delta^m z(n) = -\sum_{n=n_1}^{\infty} q(n) f\left(x(\tau(n))\right),$$

or

$$0 < \Delta^{m-1}z(n_1) - L = \sum_{s=n_1}^{\infty} q(n) f\left(x(\tau(n))\right),$$

where $0 \leq L := \lim_{n\to\infty} \Delta^{m-1}z(n) < \infty$. Since $\sum_{n=n_1}^{\infty} q(n)f\left(x(\tau(n))\right) < \infty$, we have $\lim_{n\to\infty} q(n)f\left(x(\tau(n))\right) = 0$. So, in the view of $\limsup_{n\to\infty} q(n) > 0$, we obtain $\lim_{n\to\infty} f\left(x(\tau(n))\right) = 0$. Therefore, from our assumption about f, we get that $\lim_{n\to\infty} x(n) = 0$, which is a contradiction to $x(n) \nrightarrow 0$. The proof is complete. □

Theorem 3. *Let m be odd. Assume that $\sigma(n)$ is any delay or advanced argument, $\tau(n) \leq n$ and nondecreasing, (1.2), $(H_1)-(H_3)$ hold. If all solutions of the following difference equation*

$$\Delta y(n) + q(n) f\left[c(\tau(n)) y(\tau(n))\right] = 0, \quad n = 0, 1, 2, \ldots, \tag{3.1}$$

where $(c(n))$ is a sequence such that $\lim_{n\to\infty} c(n) = 0$, are oscillatory, then every solution of (1.1) oscillates.

Proof. Let $(x(n))$ be a nonoscillatory solution of (1.1) with $x(n) > 0$ eventually. By (2.1), we have $\Delta^m z(n) = -q(n)f\left(x(\tau(n))\right) \leq 0$, $n \geq n_0$. It is clear from Lemma 1.1 that $\Delta^i z(n)$ is eventually constant sign for $i = 1, 2, 3, ..., m-1$, $\Delta^{m-1}z(n) > 0$ and nonincreasing. Since $\left(\Delta^{m-1}z(n)\right)$ is bounded, we can find a sequence $(c(n))$ such that $\lim_{n\to\infty} c(n) = 0$ and

$$x(n) \geq c(n)\Delta^{m-1}z(n), \quad n \geq n_1 \geq n_0. \tag{3.2}$$

So, by (1.1), (2.1) and (3.2), and since f is nondecreasing, we have

$$\Delta^m z(n) + q(n) f\left[c(\tau(n))\left(\Delta^{m-1}z(\tau(n))\right)\right] \leq 0, \ n \geq n_1. \tag{3.3}$$

Letting $y(n) = \Delta^{m-1}z(n) > 0$. Thus, from (3.3), we obtain

$$\Delta y(n) + q(n) f\left(c(\tau(n)) y(\tau(n))\right) \leq 0, \ n \geq n_1, \tag{3.4}$$

which means that inequality (3.4) has an eventually positive solution. Thus, we know from Lemma 1.3 that equation (3.1) has also a positive solution. This contradicts to our assumption. The proof is complete. □

Theorem 4. *Let m be odd. Assume that $\sigma(n)$, $\tau(n)$ are any delay or advanced arguments, (1.2), $(H_1)-(H_3)$, $\lim_{n\to\infty} q(n) = 0$, $\limsup_{n\to\infty} p(n) = \infty$ hold. Then, every solution of (1.1), which does not converge to zero as $n \to \infty$, is oscillatory. (In other words, there is no any nonoscillatory solution tends to zero).*

Proof. Let $(x(n))$ be a nonoscillatory solution of (1.1) with $x(n) > 0$ eventually, and assume that $x(n)$ does not tend to zero as $n \to \infty$. By (2.1) and (2.2) , it is clear from Lemma 1.1 that $\Delta^i z(n)$ is eventually constant sign for $i = 1, 2, 3, ..., m-1$, $\Delta^{m-1} z(n) > 0$ and nonincreasing, and there are two possible cases where $\Delta z(n) < 0$ or $\Delta z(n) > 0$. From (2.1) and our assumptions, we obtain

$$\lim_{n\to\infty} z(n) = \infty \text{ and } \Delta z(n) > 0.$$

On the other hand, we know from the proof of Theorem 2.1 that $\Delta z(n) < 0$. This is a contradiction and the proof is complete. □

Theorem 5. *Let m be odd. Assume that $\sigma(n)$, $\tau(n)$ are any delay or advanced arguments, (1.2), $(H_1)-(H_3)$, $\lim_{n\to\infty} q(n) = 0$ hold. Then, every unbounded solution of equation (1.1) oscillates.*

Proof. Let $(x(n))$ be a unbounded nonoscillatory solution of (1.1) with $x(n) > 0$ eventually. By (2.1) and (2.2), it is clear from Lemma 1.1 that $\Delta^i z(n)$ is eventually constant sign for $i = 1, 2, 3, ..., m-1$, $\Delta^{m-1} z(n) > 0$ and nonincreasing, and there are two possible cases where $\Delta z(n) < 0$ or $\Delta z(n) > 0$. By (2.1), and since $(x(n))$ is unbounded, we have $\lim_{n\to\infty} z(n) = \infty$ and $\Delta z(n) > 0$. On the other hand, we know from the proof of Theorem 2.1 that $\Delta z(n) < 0$. This is a contradiction and the proof is complete. □

Theorem 6. *Let m be odd. Assume that $\sigma(n)$ is any delay or advanced argument, $\tau(n) \le n$ and nondecreasing, (1.2), $(H_1)-(H_3)$, $\lim_{n\to\infty} q(n) = 0$, $\liminf_{n\to\infty} p(n) < \infty$ hold. If all solutions of the following difference equation*

$$\Delta y(n) + q(n) f\left(y(\tau(n))\right) = 0, \quad n = 0, 1, 2, \dots \tag{3.5}$$

are oscillatory, then every bounded solution of equation (1.1), which does not converge to zero as $n \to \infty$, is oscillatory.

Proof. Let $(x(n))$ be a bounded nonoscillatory solution of (1.1) with $x(n) > 0$ eventually. By (2.1) and (2.2), it is clear from Lemma 1.1 that $\Delta^i z(n)$ is eventually constant sign for $i = 1, 2, 3, ..., m-1$, $\Delta^{m-1} z(n) > 0$ and nonincreasing, and there are two possible cases where $\Delta z(n) < 0$ or $\Delta z(n) > 0$. Moreover, we know from the proof of Theorem 2.1 that $\Delta z(n) < 0$ and $(z(n))$ is bounded. On the other hand, since $(x(n))$ is bounded and does not converge to zero as $n \to \infty$, we get $0 < \limsup_{n\to\infty} x(n) < \infty$. Moreover, since $\liminf_{n\to\infty} p(n) < \infty$, we obtain $\lim_{n\to\infty} x(n) = a$, where $0 < a < \infty$. So, we know from Lemma 1.2 that

$$\lim_{n\to\infty} \Delta^{m-1} z(n) = 0.$$

Thus, since $\lim_{n\to\infty} x(n) = a$ and $\lim_{n\to\infty} \Delta^{m-1} z(n) = 0$, we have

$$x(n) \ge \Delta^{m-1} z(n), \quad n \ge n_1 \ge n_0. \tag{3.6}$$

So, by (1.1), (2.1) and (3.6), and since f is nondecreasing, we have

$$\Delta^m z(n) + q(n) f\left[\Delta^{m-1} z(\tau(n))\right] \le 0, \ n \ge n_1. \tag{3.7}$$

Letting $y(n) = \Delta^{m-1} z(n) > 0$. Thus, from (3.7), we obtain

$$\Delta y(n) + q(n) f\left(y(\tau(n))\right) \le 0, \ n \ge n_1, \tag{3.8}$$

which means that inequality (3.8) has an eventually positive solution. Thus, we know from Lemma 1.3 that equation (3.5) has also a positive solution. This contradicts to our assumption. The proof is complete. □

Example 2. *Consider the third order difference equation*

$$\Delta^3\left[x(n)+\frac{n+1}{n}x(n+1)\ln\left(e+|x(n+1)|\right)\right]+nx(n-1)\ln\left(e+|x(n-1)|\right)=0,\ \ n\in\mathbb{N}. \tag{3.9}$$

Here, $p(n)=\frac{n+1}{n}$, $q(n)=n$, $\sigma(n)=n+1$, $\tau(n)=n-1$, $h(x)=f(x)=x\ln\left(e+|x|\right)$. *So, if we take* $c(n)=\frac{1}{n}$, *then*

$$f\left[c(\tau(n))y(\tau(n))\right]=f[\frac{1}{n-1}y(n-1)]=\frac{1}{n-1}y(n-1)\ln\left(e+\frac{|y(n-1)|}{n-1}\right). \tag{3.10}$$

Thus, from (3.1) and (3.10), we obtain

$$\Delta y(n)+\frac{n}{n-1}y(n-1)\ln\left(e+\frac{|y(n-1)|}{n-1}\right)=0. \tag{3.11}$$

Now, we consider the equation (3.11). Let $k(n)=\frac{n}{n-1}$, $\tau(n)=n-1$, $g(y)=y\ln\left(e+\frac{|y|}{n-1}\right)$. *So, from (1.12), we get*

$$\limsup_{y\to 0}\frac{y}{g(y)}=M=1.$$

Therefore, by (1.13), we have

$$\liminf_{n\to\infty}\sum_{j=\tau(n)}^{n-1}k(j)=\liminf_{n\to\infty}\sum_{j=n-1}^{n-1}\frac{j}{j-1}=1>\frac{1}{e}.$$

*Thus, we know from Theorem **A** that every solution of (3.11) is oscillatory. Thus, all conditions of Theorem 3.2 are satisfied, and hence all solutions of equation (3.9) are oscillatory. It should be noted that since the equation (3.9) is a mixed type and* $p(n)>1$ *for* $n\in\mathbb{N}$, *no result in the literature answers this equation.*

Example 3. *Consider the fifth order difference equation*

$$\Delta^5\left[x(n)+p(n)x\left(\left[\frac{3}{2}n\right]\right)\right]+\frac{1}{n+1}x\left(\left[\frac{n}{2}\right]\right)=0,\ \ n\in\mathbb{N}. \tag{3.12}$$

Here, $p(n)=\begin{cases}2n, & n \text{ is even}\\ \frac{n}{n+1}, & n \text{ is odd}\end{cases}$, $q(n)=\frac{1}{n+1}$, $\sigma(n)=\left[\frac{3}{2}n\right]$, $\tau(n)=\left[\frac{n}{2}\right]$, *where* $[.]$ *is the greatest integer function,* $h(x)=f(x)=x$. *Then from (3.5), we obtain*

$$\Delta y(n)+\frac{1}{n+1}y\left(\left[\frac{n}{2}\right]\right)=0. \tag{3.13}$$

Therefore, we see that, if $n=2k$, *then*

$$\sum_{j=\tau(n)}^{n-1}q(j)=\sum_{j=k}^{2k-1}q(j)=\sum_{j=k}^{2k-1}\frac{1}{j+1}=\frac{k}{2k}=\frac{1}{2}>\frac{1}{e},$$

and if $n=2k-1$, *then*

$$\sum_{j=\tau(n)}^{n-1}q(j)=\sum_{j=k-1}^{2k-2}\frac{1}{j+1}=\frac{k}{2k-1}>\frac{1}{2}>\frac{1}{e}.$$

Hence, by (1.7), we have

$$\liminf_{n\to\infty}\sum_{j=\tau(n)}^{n-1}q(j)\geq\frac{1}{2}>\frac{1}{e}.$$

Thus, we know from (1.7) that every solution of (3.13) is oscillatory. Thus, all conditions of Theorem 3.5 are satisfied, and hence all solutions of equation (3.12) are oscillatory. It should be noted that since the equation (3.12) is a mixed type and $\lim_{n\to\infty}p(2n)=\infty$, *no result in the literature answers this equation.*

Conclusions

In conclusion, our investigation has shed light on the oscillatory behavior of the high order nonlinear neutral mixed type difference equation. The obtained results expand our understanding of the dynamic properties of this equation and contribute to the broader field of difference equations. We hope that this work will inspire further research in this area, leading to new insights and advancements in the study of nonlinear difference equations.

Data Availability

Since no data sets were generated or analyzed in the course of this study, there is no data available for sharing. The research primarily involved theoretical analysis, literature review, and conceptual exploration, without involving the collection or utilization of specific data sets. As a result, there are no data files, supplementary materials, or datasets associated with this paper.

Conflicts of Interest

The authors declare that they have no conflicts of interest regarding the publication of this manuscript. There are no financial or non-financial interests or relationships that could have influenced the work presented in this paper. The authors affirm that the research findings and conclusions have been arrived at objectively and without bias.

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