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New types of almost contact metric submersions

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Abstract:
We introduce the concept of conjugaison in contact geometry. This concept allows to define new structures which are used as base space of a Riemannian submersion. With these new structures, we study new three types of almost contact metric submersions.

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1 Introduction

Almost contact metric subersions constitute a wide class of Riemannian submersions whose total space is an almost contact metric manifold. This study has been initiated by D. Chinea [2, 3]. Independetly from Chinea, Watson [12], also studied two types of such a class of submersions.

In the sense of Watson, submersions of type I are those whose base space is an almost contact metric manifold while those of type II have almost Hermitian manifold as base space. Thus, the study of Watson intertwines almost contact and almost Hermitian structures. It seems understanding to intertwine almost contact and almost paracontct structures via Riemannian submersions.

Note that in [6], Y. G*ü*nd*ü*zalp and B. Sahin have introduced the study of submersions considering that the total and the base space are almost paracontact structures.

In this paper, we introduce the concept of conjugaison in contact geometry. This concept allows to define new structures that we use as base space of Riemannian submersions. With these new structures, considering that the total space remains almost contact metric, we study new three types of almost contact submersions whose base space is:

(i) conjugated almost contact metric manifold;

(ii) almost paracontact metric manifold;

(iii) almost para-Hermitian manifold.

This paper is organized in the following way:

§2 is devoted to the preliminary on manifolds where a large part is reserved to new structures.

In §3 we give some examples of the new structures.

§4 deals with the study of various types of almost contact metric submersions where we degage fundamental properties, structure of the fibers and the base space. We then have shown that

If the base space is a conjugated almost contact or an almost paracontact manifold, then the fibres are almost Hermitian manifolds.

The above result shows that the under consideration manifolds have some common properties which force the fibres to lie in the class of almost Hermitian manifolds. It resembles to that of G*ü*nd*ü*zalp and Sahin [6, Prop 3.5].

In the same manner, another result states that

If the base space is an almost Hermitian or an almost para-Hermitian manifold, then the fibres are almost contact metric manifolds.

This result shows that almost Hermitian and almost para-Hermitian manifolds have some common properties which force the fibres to lie in the class of almost contact manifolds.

2 Preliminaries on manifolds

2.1 Almost Hermitian and para-Hermitian manifolds

By an almost Hermitian manifold, one understands a Riemannian manifold, (M, g), of even dimension $2m$, furnished with a tensor field , J, of type $(1, 1)$ satisfying the following two conditions:

(i) $J^2D = -D$, and

(ii) $g(JD, JE) = g(D, E)$, for all $D, E \in \Gamma(M)$.

The tensor field J is called almost complex structure. A differentiable manifold, equipped with an almost complex structure is called an almost complex manifold. The above condition (ii) means that the Riemannian metric g is compatible with the almost complex structure J. In this case, g is an almost Hermitian metric. Then, (M^{2m}, g, J), is an almost Hermitian manifold.

Any almost Hermitian manifold admits a differential 2-form, Ω, called the fundamental form or the Kähler form, defined by

$$\Omega(D, E) = g(D, JE).$$

A local J-basis of an open subset of M is

$$\{E_1, ..., E_m, JE_1, ..., JE_m\}.$$

Extending the Levi-Civita connection ∇ to all tensorial algebra of M, one obtains many tensor fields such as $\nabla_D J$, $\nabla_D \Omega$ and so on which occur in the defining relations of various classes of almost Hermitian manifolds obtained by Gray and Hervella in [5].

Let us recall some remarkable identities obtained by the use of the following known Koszul formula

$$2g(\nabla_E G, D) = E.g(G, D) + Gg(D, E) - Dg(E, G) - g(E, [G, D]) + g(G, [D, E]) + g(D, [E, G]), \quad (2.1)$$

$$(\nabla_D J)E = \nabla_D JE - J\nabla_D E, \quad (2.2)$$

$$(\nabla_D \Omega)(E, G) = g((\nabla_D J)E, G) = -g(E, (\nabla_D J)G), \quad (2.3)$$

$$3d\Omega(D, E, G) = \mathcal{G}\left\{(\nabla_D \Omega)(E, G)\right\}, \quad (2.4)$$

$$(\nabla_D \Omega)(E, G) = (\nabla_D \Omega)(JE, JG), \quad (2.5)$$

where $\mathcal{G}$ denotes the cyclic sum over D, E and G.

The codifferential, δ, of Ω is given by

$$\delta\Omega(D) = -\sum_{i=1}^{m} \{(\nabla_{E_i}\Omega)(E_i, D) + (\nabla_{JE_i}\Omega)(JE_i, D)\} \quad (2.6)$$

Let us recall that the Lee form of an almost Hermitian manifold is a 1-form θ, given by

$$\theta(D) = \frac{1}{m-1}\delta\Omega(JD). \quad (2.7)$$

The Nijenhuis tensor, N_J, of the almost complex structure J is a tensor field of type $(1, 2)$ given in [?, p. 63] by

$$N_J(D, E) = J^2[D, E] + [JD, JE] - J[JD, E] - J[D, JE]. \quad (2.8)$$

When $N_J(D, E) = 0$, the almost complex structure J is said to be integrable; in this case, the almost Hermitian manifold is called Hermitian.

Almost Hermitian structures have been completely classified by A. Gray and L.M. Hervella [5]. We just recall the defining relations of some classes which will be used in this study.

An almost Hermitian manifold (M^{2m}, g, J) is said to be :

(1) *Kählerian* if $d\Omega(D, E, G) = 0$ and $N_J = 0$, where N_J denotes the Nijenhuis tensor of J;

(2) *almost Kählerian* (or W_2-*manifold*) if $d\Omega(D, E, G) = 0$;

(3) *nearly Kählerian* (or W_1-*manifold*) if $(\nabla_D \Omega)(D, E) = 0$;

(4) W_3-*manifold* if $(\nabla_D \Omega)(E, G) - (\nabla_{JD}\Omega)(JE, G) = 0 = \delta\Omega$;

(5) *semi-Kählerian* (or $W_1 \oplus W_2 \oplus W_3 - manifold$) if $\delta\Omega = 0$;

(6) $W_1 \oplus W_3$-*manifold* if $(\nabla_D \Omega)(D, E) - (\nabla_{JD}\Omega)(JD, E) = 0 = \delta\Omega$;

(7) G_1-*manifold* if $(\nabla_D \Omega)(D, E) - (\nabla_{JD}\Omega)(JD, E) = 0$;

(8) *Hermitian* or ($W_3 \oplus W_4$-*manifold*) if $N_J = 0$ or equivalently

$$(\nabla_D \Omega)(E, G) - (\nabla_{JD}\Omega)(JE, G) = 0;$$

(9) a G_2-*manifold* or ($W_2 \oplus W_3 \oplus W_4$-*manifold*) if

$$\mathcal{G}\{(\nabla_D \Omega)(E, G) - (\nabla_{JD}\Omega)(JE, G)\} = 0 \text{ or } \mathcal{G}\{g(N_J(D, E), JG)\} = 0;$$

(10) *quasi Kählerian* or ($W_1 \oplus W_2$-*manifold*) if

$$(\nabla_D\Omega)(E,G) + (\nabla_{JD}\Omega)(JE,G) = 0;$$

(11) $W_2 \oplus W_3$-*manifold* if

$$\mathcal{G}\left\{(\nabla_D\Omega)(D,E) - (\nabla_{JD}\Omega)(JD,E)\right\} = 0 = \delta\Omega;$$

(12) *locally conformal almost Kähler* ($W_2 \oplus W_4$-*manifold*) if

$$d\Omega = \Omega\Lambda\theta \text{ or } \mathcal{G}\left\{(\nabla_D\Omega)(E,G) - \frac{1}{m-1}\Omega(D,E)\,\delta\Omega(JG)\right\} = 0;$$

(13) *locally conformal Kähler* (W_4-*manifold*) if

$$\begin{aligned}(\nabla_D\Omega)(E,G) &= \frac{-1}{2(m-1)}\left\{g(D,E)\delta\Omega(G) - g(D,G)\delta\Omega(E)\right\}\\ &+ \frac{-1}{2(m-1)}\left\{-g(D,JE)\delta\Omega(JG) + g(D,JG)\delta\Omega(JE)\right\};\end{aligned}$$

(14) $W_1 \oplus W_4$-*manifold* if

$$(\nabla_D\Omega)(D,E) = \frac{-1}{2(m-1)}\{g(D,D)\delta\Omega(E) - g(D,E)\delta\Omega(D) - g(JD,E)\delta\Omega(JD)\};$$

(15) $W_1 \oplus W_2 \oplus W_4$-*manifold* if

$$\begin{aligned}(\nabla_D\Omega)(E,G) + (\nabla_{JD}\Omega)(JE,G) &= \frac{-1}{m-1}\{g(D,E)\delta\Omega(G) - g(D,G)\delta\Omega(E)\}\\ &+ \frac{-1}{m-1}\{-g(D,JE)\delta\Omega(JG) + g(D,JG)\delta\Omega(JE)\};\end{aligned} \tag{2.9}$$

Let M^{2m} be a smooth manifold of even dimension $2m$. Consider an almost para complex structure J such that $J^2 = \mathbb{I}$, where $\mathbb{I}$ is the identity transformation. If there exists on M a metric tensor g such that $g(JD,JE) = -g(D,E)$, then the couple (g,J) is called an almost para complex metric structure (or an almost para-Hermitian metric). So, (M^{2m}, g, J) is an almost para-Hermitian manifold.

As in the case of almost Hermitian manifolds, the fundamental $2-$form Ω, of the structure (g,J) is given by $\Omega(D,E) = g(D,JE)$. If further, J is parallel along the Levi-Civita connection ∇, (meaning that $\nabla J = 0$), then the manifold is said to be para-Kählerian.

Let us note some remarkable classes of almost para-Hermitian structures susceptible to be used in this study.

Following Gray and Hervella [5], see also [11], an almost para-Hermitian manifold is called:

(1) para-Kählerian if $\nabla J = 0$;

(2) almost para- Kählerian if $d\Omega(D,E,G) = 0$;

(3) quasi para- Kählerian if $(\nabla_D\Omega)(E,G) + (\nabla_{JD}\Omega)(JE,G) = 0$;

(4) nearly para- Kählerian if $(\nabla_D\Omega)(D,E) = 0$.

Let M be a differentiable manifold of dimension $2m+1$. An almost contact structure on M is a triple (φ, ξ, η), where:

(1) ξ is a characteristic vector field,

(2) η is a 1−form such that $\eta(\xi) = 1$, and

(3) φ is a tensor field of type $(1, 1)$ satisfying

$$\varphi^2 = -I + \eta \otimes \xi,$$

where I is the identity transformation. If M is equipped with a Riemannian metric g such that

$$g(\varphi D, \varphi E) = g(D, E) - \eta(D)\eta(E),$$

then (g, φ, ξ, η) is called an almost contact metric structure. So, the quintuple $(M^{2m+1}, g, \varphi, \xi, \eta)$ is an almost contact metric manifold. As in the case of almost Hermitian manifolds, any almost contact metric manifold admits a fundamental 2−form, ϕ, defined by

$$\phi(D, E) = g(D, \varphi E).$$

If the tensor field φ is such that $\varphi^2 = -I - \eta \otimes \xi$, then (φ, ξ, η) is called a *conjugated almost contact structure*. If in addition, there exists a tensor metric g such that $g(\varphi D, \varphi E) = g(D, E) + \eta(D)\eta(E)$, then (φ, ξ, η, g) is a conjugated almost contact metric structure. So, the quintuple $(M^{2m+1}, \varphi, \xi, \eta, g)$ is a conjugated almost contact metric manifold.

As in the case of almost contact metric manifold, there exists a fundamental 2−form ϕ, defined by $\phi(D, E) = g(D, \varphi E)$.

Let us say something about the classification of these structures.

Let $(M^{2m+1}, \varphi, \xi, \eta, g)$ be a conjugated almost contact metric manifold and $\mathbb{R}$ the real line. Let us consider the manifold product $\bar{M} = M \times \mathbb{R}$. Vector fields on $\bar{M}$ are of kind $(D, a\frac{d}{dt})$ where $D \in \Gamma(M)$, a is a function on $\mathbb{R}$ and t is a real variable.

As in Oubiña [8], we define an almost Hermitian structure $(\bar{J}, \bar{g})$ by setting

$$\bar{J}(D, a\frac{d}{dt}) = (\varphi D - a\xi, \eta(D)\frac{d}{dt}), \tag{2.10}$$

$$\bar{g}((D, a\frac{d}{dt}), (E, b\frac{d}{dt})) = g(D, E) + ab. \tag{2.11}$$

Denoting by $\bar{\Omega}$ the Kähler form of $\bar{M}$, $\bar{\nabla}$ the Riemannian connection of $\bar{g}$ and $\bar{\delta}$ the codifferential, we have

Theorem 2.1. *Let $(\bar{M}, \bar{g}, \bar{J})$ be the manifold product of a conjugated almost contact metric manifold with the real line. Then:*

(i) $\bar{\Omega}((D, a\frac{d}{dt}), (E, b\frac{d}{dt})) = \phi(D, E) - b\eta(D) + a\eta(E)$;

(ii) $3d\bar{\Omega}((D, a\frac{d}{dt}), (E, b\frac{d}{dt}), (G, c\frac{d}{dt})) = 3d\phi(D, E, G) - 2\{cd\eta(D, E) + ad\eta(E, G) + bd\eta(G, D)\}$;

(iii) $\bar{\nabla}_{(D, a\frac{d}{dt})}(E, b\frac{d}{dt}) = (\nabla_D E, \{D(b) + a\frac{db}{dt}\}\frac{d}{dt})$;

(iv) $(\bar{\nabla}_{(D, a\frac{d}{dt})}\bar{J})(E, b\frac{d}{dt}) = ((\nabla_D \varphi)E - b\nabla_D \xi, (\nabla_D \eta)E\frac{d}{dt})$;

(v) $(\bar{\nabla}_{(D, a\frac{d}{dt})}\bar{\Omega})((E, b\frac{d}{dt}, (G, c\frac{d}{dt}) = (\nabla_D \Phi)(E, G) - c(\nabla_D \eta)E + b(\nabla_D \eta)G$;

(vi) $\delta\bar{\Omega}(D, a\frac{d}{dt}) = \delta\phi - a\delta\eta$.

Proof. See Oubiña [8] □

Following Theorem 2.1, one obtains a classification of conjugated almost contact metric structure (φ, ξ, η, g) from the Gray-Hervella classification [5] of almost Hermitian structures.

With this in mind, we say that a conjugated almost contact metric structure is: conjugated almost cosymplectic, conjugated cosymplectic, conjugated semi-cosymplectic or conjugated semi-cosymplectic normal, if and only if, the almost Hermitian structure $(\bar{g}, \bar{J})$ is almost Kähler, Kähler, semi-Kähler or a W_3-manifold respectively.

Theorem 2.2. *Let $(M^{2m+1}, \varphi, \xi, \eta, g)$ be a conjugated almost contact metric manifold. Then, it is:*

(a) conjugated almost cosymplectic if $d\phi = 0$ and $d\eta = 0$;

(b) conjugated cosymplectic if $\nabla\varphi = 0$;

(c) conjugated semi-cosymplectic if $\delta\phi = 0$ and $\delta\eta = 0$;

(d) conjugated semi-cosymplectic normal if $\delta\Phi = 0 = \delta\eta$ and
$(\nabla_D\varphi)E - (\nabla_{\varphi D}\varphi)\varphi E + \eta(E)\nabla_{\varphi D}\xi = 0$.

Proof. Consider (a). By Theorem 2.1 (ii), it is clear that if $d\phi = 0 = d\eta$, then $d\bar{\Omega} = 0$ wich shows that $(\bar{J}, \bar{g})$ is an almost Kählerian structure. Thus, M is a conjugated almost cosymplectic manifold.

Other statements are established in the similar way. □

Note that in the case of Sasakian geometry, illustrations are obtained by the use of the conform metric g^o defined by $g^o = e^{2\sigma}g$ where

$\sigma : M \times \mathbb{R} \longrightarrow \mathbb{R}$ such that $\sigma(x, t) = t$.

If the tensor field φ is such that $\varphi^2 = I - \eta \otimes \xi$, then (φ, ξ, η) is called an *almost paracontact structure* in the sense of Sato [9] and [10]. As in the case of almost contact metric manifolds, if the manifold is equipped with a metric tensor g such that $g(\varphi D, \varphi E) = -g(D, E) + \eta(D)\eta(E)$, then $(M^{2m+1}, \varphi, \xi, \eta, g)$ is said to be an almost paracontact metric manifold.

For some remarkable classes, we have the following defining relations.

An almost paracontact manifold is said to be:

(1) *normal* if $N_\varphi - 2d\eta \otimes \xi = 0$, where N_φ is the Nijenhuis tensor of φ.

(2) *para-contact* if $\phi = d\eta$,

(3) *para-K-contact* if it is para-contact and ξ is Killing,

(4) *para-cosymplectic* if $\nabla\eta = 0$ and $\nabla\phi = 0$,

(5) *almost para-cosymplectic* if $d\phi = 0$ and $d\eta = 0$,

(6) *para-weakly-cosymplectic* if $d\phi = 0$, $d\eta = 0$ and
$[R(D, E), \varphi] = R(D, E)\varphi - \varphi R(D, E) = 0$,

(7) *para-Sasakian* if $\phi = d\eta$ and M is para-normal,

(8) *quasi para-Sasakian* if $d\phi = 0$ and M is para-normal,

(9) *quasi para-K-cosymplectic* if

$(\nabla_D \varphi)E + (\nabla_{\varphi D}\varphi)\varphi E - \eta(E)(\nabla_{\varphi D}\xi) = 0;$

(10) *almost para-Kenmotsu* if $d\phi(D,E,G) = \frac{2}{3}\mathcal{G}\{\eta(D)\phi(E,G)\}$, where $\mathcal{G}$ denotes the cyclic sum over D, E, G;

(11) *para-Kenmotsu* if $d\phi(D,E,G) = \frac{2}{3}\mathcal{G}\{\eta(D)\phi(E,G)\}$, $d\eta = 0$ and $N^{(1)} = 0$;

(12) *quasi para-Kenmotsu* if

$(\nabla_D \phi)(E,G) + (\nabla_{\varphi D}\phi)(\varphi E, G) = \eta(E)\phi(G,D) + 2\eta(G)\phi(D,E)$
and $d\eta = 0$;

(13) *nearly para-Kenmotsu* if $(\nabla_D \varphi)D = -\eta(D)\varphi D$ and $d\eta = 0$.

3 Examples of conjugated almost contact manifolds

3.1 Examples of conjugated manifold

Let $(\mathbb{R}^3, g, \varphi, \xi, \eta)$ where $\eta = ydx - dz$; $\xi = -\frac{\partial}{\partial z}$;

$\varphi = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & y & 0 \end{pmatrix}$ and $g = \eta \otimes \eta + (dx)^2 + (dy)^2$.

Easily it is checked that $\eta \otimes \xi = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ -y & 0 & 1 \end{pmatrix}$;

and $\varphi^2 = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ -y & 0 & 0 \end{pmatrix} = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix} - \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ y & 0 & -1 \end{pmatrix}$

This gives $\varphi^2 = -I - \eta \otimes \xi$ which shows that (φ, ξ, η) is a conjugated almost contact structure.

3.2 Example of a conjugated almost paracontact manifold

Let $(\mathbb{R}^3, g, \varphi, \xi, \eta)$ where $\eta = ydx - dz$; $\xi = -\frac{\partial}{\partial z}$;

$\varphi = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ y & 0 & 0 \end{pmatrix}$ and $g = \eta \otimes \eta + (dx)^2 + (dy)^2$.

One can verifie that $\eta \otimes \xi = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ y & 0 & -1 \end{pmatrix}$; $\eta(\xi) = -1$

and $\varphi^2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ y & 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ y & 0 & -1 \end{pmatrix}$

We then derive that $\varphi^2 = I + \eta \otimes \xi$ which shows that (φ, ξ, η) is a conjugated almost paracontact structure.

4 Almost Contact Metric Submersions

In [7], O'Neill has defined a Riemannian submersion as a surjective mapping

$$\pi : M \longrightarrow B$$

between two Riemannian manifolds such that

(i) π is of maximal rank;

(ii) $\pi_{*|(Ker\pi_*)^{\perp}}$ is a linear isometry.

The tangent bundle $T(M)$, of the total space M, admits an orthogonal decomposition

$$T(M) = V(M) \oplus H(M),$$

where $V(M)$ and $H(M)$ denote respectively the vertical and horizontal distributions. We denote by $\mathcal{V}$ and $\mathcal{H}$ the vertical and horizontal projections respectively. A vector field X of the horizontal distribution is called a basic vector field if it is $\pi-$related to a vector field X_* of the base space B. Such a vector field means that $X_* = \pi_* X$.

On the base space, tensors and other objects will be denoted by a prime ' while those tangent to the fibres will be specified by a carret ˆ. Herein, vector fields tangent to the fibres will be denoted by U, V and W.

Let $(M^{2m+1}, g, \varphi, \xi, \eta)$ be an almost contact metric manifold. Consider that $(M'^{2m'+1}, g', \varphi', \xi', \eta')$ is a conjugated almost contact metric or an almost paracontact metric manifold. By an almost contact metric submersion, one understands a $(\varphi, \varphi')-$holomorphic map

$$\pi : M^{2m+1} \to M'^{2m'+1}$$

satisfying

(i) $\pi_* \varphi = \varphi' \pi_*$,

(ii) $\pi_* \xi = \xi'$.

When the base space is an almost para-Hermitian manifold, $(M'^{2m'}, g', J')$, we say that the submersion is $(\varphi, J)-$holomorphic map. In this case, we have $\pi_* \varphi = J' \pi_*$.

Various classes of Riemannian submersions are presented in [4].

4.1 Fundamental Properties

Now, we overview some of the fundamental properties of these submersions.

Proposition 4.1. *Let $\pi : M^{2m+1} \longrightarrow M'^{2m'+1}$ be an almost contact metric submersion. If the base space is a conjugated or an almost paracontact metric manifold, then:*

(a) $\pi^*\phi' = \phi$;

(b) $\pi^*\eta' = \eta$;

(c) $\eta(U) = 0$ *for all* $U \in V(M)$;

(d) $\mathcal{H}(\nabla_X\varphi)Y$ *is the basic vector field associated to* $(\nabla'_{X_*}\varphi')Y_*$ *if* X *and* Y *are basic.*

Proof. It is an adaptation of Watson [12]. □

Proposition 4.2. *Let $\pi : M^{2m+1} \longrightarrow M'^{2m'}$ be an almost contact metric submersion. If the base space is an almost para-Hermitian manifold, then*

(a) $\pi^*\Omega' = \phi$;

(b) $\eta(X) = 0$ *for all* $X \in H(M)$;

(c) $\mathcal{H}(\nabla_X\varphi)Y$ *is the basic vector field associated to* $(\nabla'_{X_*}J')Y_*$ *if* X *and* Y *are basic.*

Proof. See again Watson [12]. □

4.2 Structure of the fibres and the base space

Theorem 4.1. *Let $\pi : M^{2m+1} \longrightarrow M'^{2m'+1}$ be an almost contact metric submersion. If the base space M' is a conjugated almost contact manifold, an almost paracontact manifold or a conjugated almost paracontact metric manifold, then the fibres are almost Hermitian manifolds.*

Proof. It is clear that the dimension of the fibers is $2p$ if we put $m - m' = p$. Considering that $J = (\hat{\varphi})$ on the fibers, let us show that $J^2 = -I$. In fact if U is a vertical vector fields tangent to the fibers, then $J^2U = (\hat{\varphi})U$. On the other hand, $(\hat{\varphi})^2U = -U + (\hat{\eta})(U)$. Since $\hat{\eta}(U) = 0$ acording to Proposition 11 we deduce that $\hat{(\varphi)}^2U = -U$ which leads to $J^2U = -I$. and shows that J is an almost complex structure. It remains to establish the compatibility of J with the Riemannian structure $(\hat{g})$. Considere two vertical vector fields U and V tangent to the fibers, one has $(\hat{g})(JU, JV) = (\hat{g})(U, V)$ which shows the compatibility of g with $(\hat{g})$; thus $\hat{g}, J)$ is an almost Hermitian structure.

□

Theorem 4.2. *Let $\pi : M^{2m+1} \longrightarrow M'^{2m'}$ be an almost contact metric submersion whose base space is an almost conjugated para Hermitian manifold, then the fibers are almost contact metric manifolds.*

Proof. As in the above Theorem 4.1, it is clear that the dimension of the fibers is $2p + 1$ which shows that the fibers can be suspected to have an almost contact structure. □

Theorem 4.3. *Let $\pi : M^{2m+1} \longrightarrow M'^{2m'+1}$ be an almost contact metric submersion whose total space is a conjugated almost contact metric manifolfd. If it is conjugated almost cosymplectic (resp. conjugated cosymplectic, conjugated semi-cosymplectic, conjugated semi-cosymplectic normal) then the base space M' is respectively almost cosymplectic, cosymplectic, semi-cosymplectic or semi-cosymplectic normal..*

Proof. Let us consider the case where the total space is furnished with a conjugated cosymplectic structure. For the cases of manifolds defined by the use of codifferentials, $\delta\phi$ and $\delta\eta$;, such as conjugated semi-cosymplectic, we can refer to the equations of Chinea. Recal that, in [4], Cinea has established the following structure equations

Using the tensor A, Chinea [3] has defined an associated tensor A^* on horizontal vector fields by setting

$$A^*(X,Y) = A_X\varphi Y - A_{\varphi X}Y,$$

and has established the following structure equations

$$\delta\phi(U) = \delta\hat{\phi}(U) + \frac{1}{2}g(trA^*, U), \tag{4.1}$$

$$\delta\phi(X) = \delta\phi'(X_*) + g(H, \varphi X), \tag{4.2}$$

$$\delta\eta = \delta\eta' \circ \pi - g(H, \xi), \tag{4.3}$$

where, trA^* is the trace of A^*.

In the case under consideration, equations, equations 4.2 and 4.3 will be with helpfull.

□

Concerning the Lee forms ω and θ, we have the following.

Lemma 4.1. *Let $\pi : M^{2m+1} \longrightarrow M'^{2m'+1}$ be an almost contact metric submersion, then $\omega(U) = \frac{m-m'-1}{m}\theta(U)$ if and only if $trA^* = 0$.*

Proof. Let us recall that on the total space M^{2m+1}, the Lee form is

$$\omega(D) = \frac{1}{m}\left\{\delta(\varphi D) - \eta(D)\delta\eta\right\}$$

and the Lee form on the fibres is given by

$$\theta(U) = \frac{1}{m - m' - 1}\delta\hat{\phi}(\varphi U).$$

Now, using equation (4.1)

$$\begin{aligned} \omega(U) &= \frac{1}{m}\delta\phi(\varphi U) = \frac{1}{m}(\delta\hat{\phi}(\varphi U) + \frac{1}{2}g(trA^*, U)) \\ &= \frac{m - m' - 1}{m}\theta(U) + \frac{1}{2m}g(trA^*, U) \end{aligned}$$

.

Thus $\omega(U) = \frac{m-m'-1}{m}\theta(U)$ if and only if $trA^* = 0$. □

References

[1] Blair, D. E., Riemannian geometry of contact and symplectic manifolds, Progress in Mathematics, Vol.203, Birkhäuser, New York, 2002.

[2] Chinea, D., Almost contact metric submersions, *Rend. Circ. Mat. Palermo,* **34**(1985),89-104.

[3] Chinea, D., Almost contact metric submersions and structure equations, *Publicationes Mathematicae*, Debrecen **34** (1987), 207-213.

[4] Falcitelli, M., Ianus, S. and Pastore, A.M., "Riemannian submersions and related topics", World Sci. Pub.Co. 2004.

[5] Gray, A. and Hervella, L. M., The sixteen classes of almost Hermitian manifolds and their linear invariants, *Ann. Mat. Pura Appl.*, **123** (1980), 35-58.

[6] Gündüzalp, Y. and Sahin, B., Paracontact semi-Riemannian submersions, *Turk. J. Math.,* **37**(2013), 114-128.

[7] O'Neill, B., The fundamental equations of a submersion, *Michigan Math.J.*, **13** (1966), 459-469.

[8] Oubiña, J. A., New classes of almost contact metric structures, *Publicationes Mathematicae*, Debrecen, **32** (1985), 187-193.

[9] Sato, I., On a structure similar to the almost contact structure I, *Tensor, N.S.,* **30** (1976), 219-224.

[10] Sato, I., On a structure similar to the almost contact structure II, *Tensor, N.S.,* **31** (1977), 199-205.

[11] Shukla, S.S. and Shankar Verna, U.,Paracomplex Paracontact Pseudo-Riemannian Submersions, *Hindawi Publ. Corporation Geometry,* Vol. 2014, Article ID 616487.

[12] Watson, B.,The differential geometry of two types of almost contact metric submersions, in *The Math. Heritage of C.F. Gauss, (Ed. G.M. Rassias),* World Sci. Publ. Co. Singapore, 1991, pp. 827-861.