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Partition Theoretic Interpretation of Two Identities of Euler

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Abstract:

In this paper we have derived generating function for a restricted partition function. This is in conjunction two identities of Euler provides new partition theoretic interpretation of two identities of Euler.

1. Introduction , Definition and the Main Results

The following two " Sum –Product" Identities are known as Rogers – Ramanujan identities :

$$\sum_{n=0}^{\infty}\frac{q^{n^2}}{(q;q)_n}=\prod_{n=1}^{\infty}\frac{1}{(1-q^{5n-1})(1-q^{5n-4})}$$

$$\sum_{n=0}^{\infty}\frac{q^{n^2+n}}{(q;q)_n}=\prod_{n=1}^{\infty}\frac{1}{(1-q^{5n-2})(1-q^{5n-3})}$$

Where $|q|<1$ *and* $(q;q)_n$ *is* a rising factorial defined by

$$(a;q)_n=\prod_{n=0}^{\infty}\frac{1-aq^i}{1-aq^{n+i}}$$

If n is a positive integer , then obviously

$$(a;q)_n=(1-a)(1-aq)\left(1-aq^2\right)\dots\dots\left(1-aq^{n-1}\right)$$

$$(a;q)_\infty=(1-a)(1-aq)\left(1-aq^2\right)\dots\ \dots\ \dots$$

In this paper we give the partition theoretic interpretation of the following two identities of Euler :

$$\sum_{n=0}^{\infty}\frac{q^{n^2}}{(q^2;q^2)_n}=\left(-q;q^2\right)_\infty \qquad \dots\ \dots\ (1)$$

$$\sum_{n=0}^{\infty}\frac{q^{n(n+1)}}{(q^2;q^2)_n}=\left(-q^2;q^2\right)_\infty \qquad \dots\ \dots\ (2)$$

Theorem1 : For a positive integer k , let $A_k(n)$ denote the number of partition of n such that the smallest part (or the only part) is $\equiv k\ (mod2)$ and the difference between any two parts is

$\equiv 0(mod2) then$

$$\sum_{n=0}^{\infty}A_k(n)=\sum_{n=0}^{\infty}\frac{q^{n(n+k-1)}}{(q^2;q^2)_n}$$

Proof : Let $A_k(m,n)$ *denote the number of partitions of n enumerated by* $A_k(n)$ into m parts. We shall first show that

$$A_k(m,n) = A_k(m-1, n-k-2(m-1)) + A_k(m, n-2m) \quad \dots \dots \quad (3)$$

To prove the identity (4) we split the partitions enumerated by $A_k(m,n)$ *into two classes* :

(i) *those who have least part k*

(ii) *those who have least part greater than k*

For those whose smallest part is equal to k , we delete k and then subtract 2 from all the remaining parts . This produces a partition of $n-k-2(m-1)$ *into exactly* $m-1$ *parts*.

Those who have smallest part greater than k , we subtract 2 from each part that produced a partition of

$n-2m$ *into m parts. The transformations are invertible and thus we have*

$$A_k(m,n) = A_k(m-1, n-k-2(m-1)) + A_k(m, n-2m)$$

For $|q| < 1$ *and* $|zq| < 1$ *, let*

$$f_k(z,q) = \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} A_k(m,n) z^m q^n \quad \dots \dots \quad (4)$$

Substituting $A_k(m,n)$ *from* (4) *in* (3) *and then simplifying , we get*

$$f_k(z,q) = zq^k f_k(zq^2, q) + f_k(zq^2, q) \quad \dots \dots \quad (5)$$

Setting

$$f_k(z,q) = \sum_{n=0}^{\infty} \alpha(n,k;q) z^n$$ *and then comparing the coefficients of* z^n *on both sides of* (6) *we*

see that

$$\alpha(n,k;q) = \frac{q^{2n-1+k}}{1-q^{2n}} \alpha(n-1,k;q) \quad \dots \dots \quad (6)$$

Iterating (6), n times and observing that $\alpha(0,k;q) = 1$ *, we see that*

$$\alpha(n,k;q) = \frac{q^{n(n+k-1)}}{(q^2;q^2)_n}$$

$$\therefore \quad f_k(z,q) = \sum_{n=0}^{\infty} \alpha(n,k;q) z^n$$

$$= \sum_{n=0}^{\infty} \frac{q^{n(n+k-1)}}{(q^2;q^2)_n} z^n$$

$$= f_k(1,q)$$

$$= \sum_{n=0}^{\infty} \frac{q^{n(n+k-1)}}{(q^2;q^2)_n}$$

This completes the proof of Theorem1.

Particular Cases :

For k = 1 , theorem 1 reduces to the identity (1)

For k =2 , theorem reduces to the identity (2)

References :

1. L.J. Rogers , " Second memoir on the expansion of certain infinite products", Proc. Of London Math.Soc.vol 25 .pp 318 – 343 , 1894
2. P.A.MacMohan, Combinatory Analysis , Vol2 . Cambridge University Press, New York ,USA , 1916.
3. B. Gordon, " A combinatorial generalization of the Rogers – Ramanujan identities ," Amer.Jour. of Math. Vol . 83 .pp 393-399, 1961.
4. A.K.Agarwal and M.Goyal , "New Partition Theoritic Interpretations of Rogers – Ramanujan Identities," International Journal of combinatorics.Vol.2012,Article ID 409505 , Hindawi Publishing Corporation.
5. B.Gordon ,"Some continued fractions of the Rogers – Ramanujan type,"Duke Mathematical Journal ,Vol32,pp.741-748,1965.
6. W.G.Connor, "Partition theorems related to some identities of Rogers and Watson," Transactions of the American Mathematical Society,Vol.214,pp 95-111,1975
7. M.D.Hirschhorn, " Some partition theorems of the Rogers-Ramanujan type,"Journal of Combinatorial theory,vol.27,no.1,pp.33-37,1979.
8. A.K.Agarwal and G.E.Andrews, "Hook differences and lattice paths'" Journal of Statistical Planning and Inference,vol.14,no.1,pp.5-14,1986. [9] m.V.Subbarao and A.K. agarwal, "Further theorems of the Rogers – Ramanujan type theorems,"
9. Canadian Mathematical Bulletin,vol.31,no.2,pp210-214 , 1988. [10]D.M. Bressoud , " A new family of partition identities," Pacific Journal of Mathematics , vol.77,pp.71-74,1978.