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A Study of The Density Property in Module Theory

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Abstract

In this paper, there are two main objectives. The first objective is to study the relationship between the density property and some modules in detail, for instance; semisimple and divisible modules. The Addition complement has a good relationship with the density property of the modules as this importance is highlighted by any submodule N of M has an addition complement with Rad(M)=0. The second objective is to clarify the relationship between the density property and the essential submodules with some examples. As an example of this relationship, we studied the torsion-free module and its relationship with the essential submodules in module M.

Keywords: Dense Module, Semisimple Module, Essential Submodule, Divisible Module, Artinian Module.

1. Introduction

We say N is a dense in M if for all $0\neq x$, $y \in M$, $\exists\, r \in R \ni xr\neq\{0\}$ and $yr \in N$. Any submodule N is essential in M iff for all $0\neq y$ in M, the set $y.(y^{-1}N)\neq\{0\}$, equivalently, if there exists another nonzero submodule K such that $N\cap K\neq 0$ [11].

(**Baer's criterion**). An R-module M is an injective iff any morphism $I\rightarrow M$, where I is an ideal of R, can be extended to a morphism $R\rightarrow M$ [1].

An R-module M is an R-divisible if $rM = M$ for all $0 \neq r \in R$ and any abelian group D is divisible if $y \in D$ and $0 \neq n \in Z$, $\exists\, x \in D \ni nx = y$ ([2], [8]).

"It is clear that there is a strong relationship between the injective module and the density property, for more information about the injective module" [3]. Any ideal I of Z is a f. g. Z-module it is called fractional ideal and denoted by (FI), if for every maximal ideal I, I_i is a principal ideal over the ring R_i [9]. If $0\neq N\leq M$ is invertible, then M is a Dedekind module [10].

Recall that the singular submodule Z(M) of a module M is the set of $m \in M$ such that $mI = 0$ for some essential right ideal I of R, or equivalently, $r_R(m) \leq_{ess} R_R$. So for any module M there is defined a submodule Z(M) which consists of singular elements in M, i.e. elements annihilated by essential right ideals. The module M is a singular (resp. a nonsingular) according to whether Z(M) = M (resp. Z(M) = 0) [7]. Any submodule N of an R-module M is called a rational if $Hom_R(M/N, E(M)) = 0$, where E(M) is the injective hull of M [6].

In this paper, the focus was on showing basic and important results about the property of density and it's a relationship with other concepts in module theory.

2. Dense Property And Semisimple modules

Definition 2.1. Let R and S be two rings. Any module $_RM$ is called a dense in ($_RM\subseteq_{den}\, _SM$) iff for all $m_1, m_2, \ldots, m_n \in M$ and $s \in S$, there is $r \in R \ni sm_i = rm_i$, $i=1,\ldots,n$.

Theorem 2.2. Let $0\neq I$ be an ideal of a ring R. Then I is a dense in R iff I is a faithful f. g. p. ideal.

Proof. Suppose that $I \subseteq_{den} R$, then $I=\beta_1(a_1)+ \beta_2(a_2)+\ldots$, $a_i \in I$. Thus for each $a \in I$, $A= \beta_1(a_1)+ \beta_2(a_2)+\ldots$. Hence I is a f. g. and so is a projective. The faithfulness of I is a clear, because an annihilator of I equal to the annihilator of R and then equal to the zero. Conversely, if I is a faithful f. g. p. ideal, then trace(I)=⟨e⟩, and ann(trace(I))=R(1-e)=ann(I)=(0). Thus e=1, trace(I)=R. So I $\subseteq_{den}$ R.

(*) Recall that if I^{-1}={x in Q: xI$\subseteq$ R} and II^{-1}=R, then I is invertible ideal ∋I $\subseteq_{den}$ R and Q is a quotient ring.

Corollary 2.3. If I satisfies (*), then I is a dense in R.

We know that, if M is a π-R-module which has only a finite number of maximal submodules (and at least). Then M is a semisimple module. Therefore in the next lemma, we introduce the relationship between a semisimple module and a density property.

Lemma 2.4. (Theorem 8, 4.9, [4]). Every semisimple module $_RM$ is a dense in $_RM$.

We know that there are two concepts which have several relationships between a semisimple module and other algebra properties, for example a radical and a socal of the module. Rad(M) and Soc(M) are very important in the next theorem. So, Rad(M) equal to the sum of all small submodules of M and then Rad(M/Rad(M))=0. Also, Rad(M) subset of A ∋A ≤ M with Rad(M/A)=0. On the other hand, Soc(M) equal to the sum of all minimal submodules.

Theorem 2.5. Let M be an R-module. Then Soc(M) is a dense in M.

Proof. The proof is a very easy because the Soc(M) is a largest semi-simple sub-module of M. We can consider Soc(M) is a module. Thus Soc(M) $\subseteq_{den}$ M (Lemma 2.4).

In the next theorem, we study the relationship between addition complement submodule and a dense property. But before that, we need to introduce the meaning of a complement in a general, see the following definition.

Definition 2.6. Let M be an R-module and N ≤M. Then N is called addition complement in M if $N+N^*$ =M and N is a minimal in $N+N^*$ =M.

Lemma 2.7. "Let M be an R-module and N, K are two submodules of M such that M=N+K. If N∩K≪K, then K has addition complement" [4].

Theorem 2.8. If any submodule N of a module M has addition complement with Rad(M)=0, then M $\subseteq_{den}$M.

Proof. Suppose N ≤ M and it has addition complement in M. Then $N+N^*$=M and $N\cap N^*$ ≤ Rad(M)=0. Now, from Lemma 2.7, $M=N+N^*$. So M is a semisimple module. Thus M $\subseteq_{den}$M.

Recall that M is called an Artinian module if it is a satisfying the (dcc).

Corollary 2.9. Let M be an Artinian R-module with Rad(M)=0. Then M $\subseteq_{den}$M.

Proof. Since M is an Artinian module, then every N ≤ M has addition complement in M. From Lemma 2.4, M is a semisimple module and hence is a dense in M.

Example 2.10. Let M=(Q_p / Z),∋p is a prime number (Q_p=(a / p^i): a ∈Z; i ∈ N). Then (Q_p / Z) $\subset_{den}$(Q_p/Z), because (Q_p/Z) is an Artinian module.

Corollary 2.11. Let M be an Artinian R-module. Then (M/Rad(M)) $\subset_{den}$ (M/Rad(M)).

Proof. Since M is an Artinian module, then (M/Rad(M)) is a semisimple module. So (M/Rad(M)) $\subset_{den}$(M/Rad(M)).

Corollary 2.12. Let R be a semisimple ring and M be an R-module. Then M is a dense in M.

Proof. Let M=m, such that m ∈M. Then mR is also a semisimple. So M=∑mR; m ∈M as a direct sum of a semisimple module. Thus M $\subset_{den}$M.

In the next Proposition we introduce good results which connect between a free module and a dense property.

Proposition2.13. Let M be an R-module. Then every free module is a dense in M.

Proof. Let F be a free R-module on a set S. Let A and B be any two R-modules over the ring R. We consider f:A→B is a homomorphism. For all x in S, we choose a_x in A such that

$j(x)= a_x$...... (1).

Also, for all x in F, g(x) in B and f:A→B is onto, then ∃a_x in A ∋

$f(a_x)=g(x)$...... (2).

Since F is a free R-module on S, then ∃ a unique homomorphism h:F→A ∋

$$h\circ i=j \text{.......... (3).}$$

To prove $f\circ h=g$?

Let x in F. Then $x=\sum r_k x_k$, x_k in S and r_k in R, where k=1, 2, ...,n (because F is a generated by S, then F=⟨S⟩). Now

$$(f\circ h)(x)= (f\circ h)(\sum r_k x_k)$$

$$=f(h(\sum r_k x_k))$$

$$=f(\sum r_k h(x_k)); \text{ because h is a homomorphism.}$$

$$=f(\sum r_k(h(i(x_k)))).$$

Now

$$(f\circ h)= f(\sum r_k((h\circ i)(x_k)))$$

$$=f(\sum r_k(j(x))) \quad \text{(by 3)}$$

$$=f(\sum r_k a_{xk}) \quad \text{(by 1)}$$

$$= \sum r_k f(a_{xk}); \text{ because f is a homomorphism.}$$

$$= \sum r_k g(x_k) \quad \text{(by 2)}$$

$$=g\sum r_k (xk), \text{ because g is a homomorphism.}$$

So $f\circ h=g(x)$. Thus M is a projective and hence is injective module over R. Then R is a semisimple ring. Thus M is a semisimple module and hence $M \subseteq_{den} M$.

Theorem 2.14. If any field F is a fractions of integral domain R, then F is a dense in F.

Proof. Let f: I → F be a homomorphism of R-modules ∋ I is an ideal of R. For 0 ≠ r and s∈I, we have rf(s)=f(rs)=sf(r). As consequence in F, we have f(r)/r =f(s)/s for any 0 ≠r and s∈I. Denote this element by a. Define f:R→F, $f^*(r)$:=ra. Check: f^* is a homo. of modules and $f^*|_I$=fm. So F is injective. So R is a semisimple (F is a semisimple). Then $F \subseteq_{den} F$.

Example 2.15. Q is a dense in Q, because Q is an injective Z-module.

"Recall that if R is an integral domain, then an R-module M is called a divisible if for every r ∈ R − {0} and for every m ∈ M there is n ∈ M such that rn = m" [8].

Example 2.16. Z is a dense in Z, because if Z is a P.I.D, then it is an injective Z-modules and hence it is a divisible Z-module.

Example 2.17. If G is commutative group $G \cong Q \oplus Z(p^\infty)$ for all primes p , then G is a dense in G.

Recall that if M is a π-module which has only a finite number of maximal submodules (at least one), M is a semisimple module.

Theorem 2.18. Let M be a quasi-injective prime module. If M has only finite number of maximal submodules (and at least one), then M is a dense in M.

Proof. We use the characterization of the π-module. So let $0\neq K \leq M$. From the definition of stability, K satisfy invariant property. By assumption M is a prime module and put $M^{*}=M$ has no non-trivial invariant submodules. Thus M subset of K (M is the π-module), M* indicates to the quasi-injective. But M has only a finite number of maximal submodules, M is a semisimple module. Thus M is a dense in M.

Corollary 2.19. Every maximal right ideal is a direct summand of R_R., then any module M is a dense in M.

Proof. Assume that $S(R_R) \subsetneq R_R$. $\exists$ a maximal ideal I_R of R $\ni S(R_R) \subseteq IR$. So, $\exists$ $R_R = I \oplus X$. Then X is a simple submodule of R_R and hence $X \subseteq S(R_R) \subseteq I$, a contradiction. Hence $R = S(R_R)$. Then R is a semisimple ring. Thus M is a dense in M.

Corollary 2.20. Every divisible module M over the Euclidean domain R is a semisimple and hence it is a dense in M and every divisible module M over P.I.D is dense in M.

Corollary 2.21. Ever divisible module M over the field K is a dense in M.

3. Dense property And Essential Submodule

In this section, we expect to obtain useful and powerful results that clarify the relationship between the essential submodules and the density property in the module theory.

From [5], if we have a dense sub module in a non-singular module M, it is unequivocally essential. So we will rely on this fact, to prove the converse of this phrase in order to be the main entrance to demystify the relationship of the density property with the essential submodules.

Definition 3.1. "A non-zero sub module K of an R-module M is called an essential if $K\cap L\neq 0$ for each non-zero submodule L of M. equivalently $K \leq_{ess} M$ if whenever $K \cap L = 0$, $L \leq M$, then $L = 0$ "[5].

Now we introduce the converse of this fact. See the following Lemma.

Lemma 3.2. If N is an essential sub module of a non-singular R-module M, then N is dense in M.

Proof. Let x, y two elements in M, $x\neq 0$. Consider the R-homo. $f: R \to M$ define by $fr=yr$, $(r\in R)$. Since $N \leq_{ess} M$, then $f^{-1}(N)=\{r\in R: yr \in N\}=y^{-1}N$. Therefore $xy^{-1} \neq 0$. For otherwise $ann(x)\supseteq y^{-1}N$ which is a contradiction because M is a non-singular module. Hence N is a dense sub module of M.

Lemma 3.3. (Theorem (1.3) [6]): "Let M be an R-module and N be a submodule of M, then $N \leq_{ess} M$ if and only if every non-zero element of M has a non-zero multiplication in N".

Lemma 3.4. If N and K are multiplication R-modules such that N is a f. g.. Then N is dense in K iff EK=K where E=[ann(K) : ann(N)].

Theorem 3.5. Let M be a non-singular module. If every non-zero element of M has a non-zero multiplication in a submodule N, then N is dense in M.

Proof. From above two lemmas.

Corollary 3.6. Let N and K be multiplication R-modules with N finitely generated. If $ann(N)\subseteq ann(K)$, then N is dense in K.

Proof. $ann(N)\subseteq ann(K)$ implies that E=[ann(K) : ann(N)]=R and the result follow from Lemma 3.4.

Proposition 3.7. Let Z(M) ={m ∈ M such that mI = 0 for some essential ideal I of R} and $N\leq_{H\text{-}ess}M$. Then kM $\leq_{ess}M$ and it is dense in M for each $0\neq k$ in $ann_R(N)$.

Proof. Let Z(M) =0. So M is a non-singular module. Let a non-zero element k in $ann_R(N)$. Define $f: M/N\to M$ by $f(a+N)=ka$ for each (a+N). Since $N\leq_{H\text{-}ess} M$, then $\forall 0\neq f \in Hom_R(M/N, M)$; $f(M/N) \leq_{ess}M$ $\ni$N is a proper in M, so $f(M/N) = kM \leq_{ess}M$. But M is a non-singular module. Then $kM\subset_{den}M$. (see Lemma 3.2).

Theorem 3.8. Let M be a module over a Noetherian ring R and Z(M)=0. If N $N\leq_{H\text{-}ess}M$, then $ann_R(N)M$ is a dense in M.

Proof. If R is a Noetherian ring, then $ann_R(N)$ is a f.g ideal of R, $ann_R(N) = (r_1, r_2,..., r_n)$ for some r_i in $ann_R(N)$. This implies that $ann_R(N)U = \sum M$, I=1,...,n . By above Proposition 3.7, $r_iM \leq_{ess} M$ for each i=1,2, . . ., n, and so $(ann_R(N))M \leq_{ess} M$. But Z(M)=0. Thus $ann_R(N)M$ is a dense in M.

If $0 \neq N \leq M$ and N $N \leq_{H\text{-}ess} M$, then $f(M) \leq_{ess} M$ $\ni$f in $End_R(M)$, N≤ ker f. Therefore, we can introduce the following result.

Corollary 3.9. Let M be a non-singular R-module. If $N \leq_{H\text{-}ess} M$, then f(M) is a dense in M.

Corollary 3.10. If N is a non-zero rational submodule of a non-singular R-module M, then f(M) is a dense in M.

Proof. Since N is a rational submodule of M, then $Hom_R(M/N, M) \leq Hom_R(M/N, E(M))$ and so $Hom_R(M/N, E(M))=0$. Hence $Hom_R(M/N, M)=0$, Thus $N \leq_{H\text{-}ess} M$, f(M) is a dense in M (Corollary 3.9).

Example 3.11. Consider the non-singular Z-module Q, where Q is the set of all rational numbers. Since Z is a rational submodule of Q, hence $Z \leq_{H\text{-}ess} Z$. So f(Z) is a dense in Z.

Theorem 3.12. Let M be torsion free Z-module. If $N \leq_{ess} M$, then N is a dense in M.

Proof. From the definition of torsion free module, we obtain T(M)=0. We have N is an essential submodule of M. To prove that M is a non-singular module (Z(M)=0).

Let $x \in Z(M)$. So $\exists$ $I \leq_{ess} Z$ such that xI=0. But I=nz, because Z is a P.I.D. Now nxZ=0, it follows that nx=0 so $x \in T(M)=0$ (x=0). Thus Z(M)=0. Therefore M is a non-singular module with N essential submodule, and then N is a dense in M.

Theorem 3.13. Let Hom(N, M)=0. If N is a singular essential R-module, then $N \subset_{den} M$.

Proof. We need to prove that M is a non-singular. Assume that Hom(N, M)=0 for each singular M R-module. To prove that Z(M)=0. Since Z(M) is a singular R-module, then Hom(Z(M), M)=0. So i: Z(M)→M, the inclusion mapping. Therefore $i \in Hom(Z(M), M)$. So i=0. Hence Z(M)=0. But N is a singular essential R-module. Thus $N \subset_{den} M$.

From above theorem, we obtain some submodules as an example about density.

Example 3.14. Let $M=Z_{12} = \{0 , 1 , 2 , 3 , 4 , 5 , 6 , 7 , 8 , 9 , 10 , 11\} = L_6$.

The submodules of Z_{12} are

$< 0 > = L_1$

$L_2 =< 2 > = \{0, 2, 4, 6, 8, 10\}$

$< 2 > = \{0, 2, 4, 6, 8, 10\} = L_2$

$< 3 > = \{0, 3, 6, 9\} = L_3$

$< 4 > = \{0, 4, 8\} = L_4$

$< 6 > = \{0, 6\} = L_5$

But $ann(L_2)=r< 2 >=0$.

So

$L_2 \cap L_2 = L_2 \neq 0$

$L_2 \cap L_3 \neq 0 = \{0 , 6\}$

$L_2 \cap L_4 \neq 0 = \{4 , 8\}$

$L_2 \cap L_5 \neq 0 = \{0 , 6\}$

$L_2 \cap L_6 \neq 0 = < 2 >$, then $L_2 \leq_{ess} Z_{12}$.

But $Z(M)=Z(Z_{12})=\{ann(m)\cap K\neq 0;\ m\in M,\ 0\neq K\in M\}$, then $Z(Z_{12})=\{ann(m)\leq_{ess} M\}$.

But from the definition of the annihilator we have :

ann(< 2 >)=r< 2 >=0 =r{0, 2, 4, 6, 8, 10}=0.

So, ann{0, 2, 4, 6, 8, 10}=ann(< 2 >)∩ L_3=0 and ann(< 2 >)∩ L_4=0, ann(< 2 >)∩ L_5=0 and ann(< 2 >)∩ L_6=0. Thus $Z(M)=Z(Z_{12})=0$ and this means Z_{12} is a non-singular module. Now $L_2 \leq_{ess} Z_{12}$ with Z_{12} is a non-singular module imply L_2 is a dense in Z_{12}.

Also,

$L_3 \cap L_1 = 0$

$L_3 \cap L_2 = \{0 , 6\}$

$L_3\cap L_4 = L_1$ But $L_4 \neq 0$

$L_3 \leq_{ess} Z_{12}$ with Z_{12} is a non-singular module implies L_3 is a dense in Z_{12}.

But the following submodule < 4 > = {0, 4, 8} = L_4

$L_4 \cap L_2 \neq 0 = L_4$

$L_4 \cap L_3 = 0$ but $L_3 \neq 0$

$L_4 \nleq_{ess} Z_{12}$. Therefore L_4 is not a dense in Z_{12}

Also,

$L_5 \cap L_2 \neq 0 = L_5$

$L_5 \cap L_3 \neq 0 = L_5$

$L_5 \cap L_4 = 0$ but $L_4 \neq 0$

$L_5 \nleq_{ess} Z_{12}$. Therefore L_5 is not dense in Z_{12}.

Example 3.15. In Z_{24}, we find L_5 and L_3 are not a dense in Z_{24} because:

$L_5 \cap L_1 \neq 0 = L_4$

$L_5 \cap L_2 \neq 0 = L_4$

$L_5 \cap L_3 \neq 0 = L_6$

$L_5 \cap L_4 \neq 0 = L_4$

$L_5 \cap L_5 = 0$ but $L_5 \neq 0$, then $L_5 \nleq_{ess} Z_{24}$. So L_5 is not a dense in Z_{24}.

And

$L_3\cap L_1 \neq 0 = L_4$

$L_3\cap L_2 \neq 0 = L_2$

$L_3\cap L_3 \neq 0 = L_6$

$L_3\cap L_4 \neq 0 = L_4$

$L_3\cap L_5 = 0$ but $L_5 \neq 0$, then $L_3 \nleq_{ess} Z_{24}$, therefore L_3 is not a dense in Z_{24}.

Also, L_2 and L_4 are essentials in Z_{24} and hence are dense in Z_{24} because

$L_2\cap L_1 \neq 0 = L_1$

$L_2\cap L_2 \neq 0 = L_4$

$L_2\cap L_3 \neq 0 = L_3$

$L_2 \cap L_4 \neq 0 = L_4$

$L_2 \cap L_5 \neq 0 = L_5$

$L_2 \cap L_6 \neq 0 = L_6$

$L_2 \leq_{ess} Z_{24}$ with Z_{24} is a non-singular module implies L_2 is a dense in Z_{24}.

Also,

$L_4 \cap L_1 \neq 0 = L_3$

$L_4 \cap L_2 \neq 0 = L_6$

$L_4 \cap L_3 \neq 0 = L_3$

$L_4 \cap L_4 \neq 0 = L_6$

$L_4 \cap L_5 \neq 0 = L_5$

$L_4 \cap L_6 \neq 0 = L_6$, then $L_4 \leq_{ess} Z_{24}$ with Z_{24} is a non-singular module implies L_4 is a dense in Z_{24}.

Corollary3.16. If $N \leq_{ess} M$, such that it is a non-singular, then $N \subset_{den} M$.

Proof. Since $Z(N)=N\cap Z(M)$ and $Z(N)=0$, then $N\cap Z(M)=0$. So $Z(M)=0$. $N\leq_{ess} M$. Hence M is a non-singular module. Thus $N \subset_{den} M$.

Corollary3.17. Let M be a projective R-module. If $N \leq_{ess} M$ and $Z(R)=0$, then $N \subset_{den} M$.

4- Conclusion

In this paper, we introduce a new concept which is called dense in module theory. Also, we find new conditions in order to obtain that a submodule N of M is dense in M. In the new work, we prove if N and K be multiplication R-modules with N finitely generated and $ann(N) \subseteq ann(K)$, then N is dense in K. These conditions and results are new in comparison with those of the results of other conditions . These results can be extended to other properties in module theory.

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