On the existence of continuous solutions of a nonlinear quadratic fractional integral equation

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Abstract

We prove an existence theorem for a nonlinear quadratic integral equation of fractional order, in the Banach space of real functions defined and continuous on closed interval. This equation contains as a special cases numerous integral equation studied by other authors. Finally, we give an example for indicating the natural realizations of our abstract result presented in this paper.

Keywords: Quadratic integral equation; measure of noncompactnes; Darbo's fixed point theorem; fractional calculus.

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1   Introduction

Several problems in many branches of science can be modeled as an integral equation such as in physics, engineering, biology,...ect see for example([4], [6], [7], [14]). Quadratic integral equations have many useful applications in describing problems in the real world. For example, quadratic integral equations are applicable in the kinetic theory of gases, in the theory of neutron transport and in the theory of radiative transfer. Especially the so called quadratic integral equation of Chandrsekher type can be seen in many application(cf.([9],[10] [26], [12],[25]). On the other hand the fractional calculus topic is enjoying growing interest of engineers and scientists, see ([16], [17],[19],[20]) . The most used techniques to study the existence for solutions of an integral equation are based on fixed point argument for example, Banach fixed point theorem was used by many authors in order to establish existence results for different kind of integral equations (see, for example[[8], [28]]). But in the absence of compactness and the Lipschitz condition, the above mentioned theorem cannot be applied and in this case, the measure of non-compactness can be useful. Many existence results for integral equations were established using fixed point theorems involving measure of non-compactness. For most details, we refer the reader to the references therein and to([1], [2], [3], [11], [16] ,[21], [27]). In this paper we deal with quadratic integral equation of fractional order with respect to another function. Using fixed point theorem via measure of non-compactness Due to Darbo are the main tool in carrying out our proof.

2   Preliminaries

The main purpose of this paper to study the following integral equation

$$x(t) = f(t) + g(t,x(t)) + \frac{(Ax)(t)}{\Gamma(\alpha)} \int_a^t \frac{m'(s)u(t,s,x(s),x(\lambda s))}{(m(t)-m(s))^{1-\alpha}} \, ds, t \in I = [a,T_1], \qquad (1)$$

where $\alpha \in (0,1)$, $0 \le a \le T_1$, $f : I \to R$, $g : I \times R \to R$,

$u : I \times I \times R \times R \to R$, and $m : I \to R$, from the view of the theory of existence of solutions. Eq(1) can be written in the form

$$x(t) = f(t) + g(t,x(t)) + (Tx)(t)I_{a,m}^\alpha u(t,.,x(.),x(.)),$$

where $I_{a,m}^\alpha$ is the fractional integral of orderα with respect to the function m defined by

$$\frac{1}{\Gamma(\alpha)} \int_a^t \frac{m'(s)u(t,s,x(s),x(\lambda s))}{(m(t)-m(s))^{1-\alpha}} ds, \qquad ' \qquad t \in I = [a,T_1],$$

which in particular cases we have:

   (i) If $m(s) = \ln s$ and $a > 0$ then $I_{a,m}^\alpha$ is the Hadamard fractional integral,

   (ii) If $m(s) = s^\beta (\beta > 0)$ then $I_{a,m}^\alpha$ is the Erdélyi-Kober fractional integral,

   (iii) If $m(s) = s$, $(\beta = 1)$and $a > 0$ in such case $I_{a,m}^\alpha$ is the Riemann-Liouville fractional integral.

Using a measure of non-compactness in the set $C([a,T];R)$ of the real continuous functions in $[a,T]$, we need to recall some basic concepts that will need later. let E be an infinite dimensional Banach space with norm $\|.\|$ and zero element θ. The symbols $\bar{X}$ and ConvX stand for the closure and closed convex hull of a subset X of E, respectively. Now denote $\mathcal{M}_E$ the family of all nonempty and bounded subsets of E, and $\mathcal{N}_E$ and $\mathcal{N}_E^W$ denotes the family of all nonempty relatively compact and weakly relatively compact sets, respectively. The symbol $\bar{X}^W$ stands for the weak closure of a set X while $\bar{X}$ denotes its closure.

Definition 2.1 *(Measure of noncompactness)[11]*

A mapping $\mu : \mathcal{M}_E \to [0, \infty)$ is said to be a measure of noncompactness in E if it satisfies the following conditions:

(1) *the family $\ker\mu = \{X \in \mathcal{M}_E : \mu(X) = 0\}$ is nonempty and $\ker\mu \subset \mathcal{N}_E$, where $\ker\mu$ is called the kernel of the measure μ.*

(2) *$X \subset Y \Rightarrow \mu(X) \leq \mu(Y)$.*

(3) *$\mu(ConvX) = \mu(X) = \mu(\overline{X})$.*

(4) *$\mu[\lambda X + (1-\lambda)Y] \leq \lambda\mu(X) + (1-\lambda)\mu(Y), \ \lambda \in [0, 1]$.*

(5) *If $X_n \in \mathcal{M}_E, \ X_n = \bar{X}_n$ and $X_{n+1} \subset X_n$ for $n = 1, 2, \ldots$ and if*

$$\lim_{n\to\infty} \mu(X_n) = 0, \ then \ X_\infty = \bigcap_{n=1}^{\infty} X_n \neq \varphi \quad (nonempty \quad and \quad compact).$$

A function $\phi : [0, \infty) \to [0, \infty)$ is said to be a comparison function if it satisfies the following properties:

(i) ϕ is non-decreasing;

(ii) $\lim_{n\to\infty} \phi^{(n)}(t) = 0$ for all $t > 0$, where $\phi^{(n)}$ denotes the $n-$ iteration of ϕ.

Examples of comparison functions are

$$\phi(t) = \lambda t^3, \quad \lambda \in [0, 1); \quad \phi(t) = \arctan(t).$$

We denote the set of comparison functions by $\mathbf{\Phi}$ as in [15]

Lemma 2.1 *[1]: Let $\phi : [0, \infty) \to [0, \infty)$ be a nondecreasing and upper semi-continuous function. Then the following conditions are equivalent:*

(i) $\lim_{n\to\infty} \phi^{(n)}(t) = 0$ *for all $t > 0$.*

(ii) $\phi(t) < t$ *for all $t > 0$.*

Lemma 2.2 *([1],[15]): Let C be a nonempty, bounded, closed and convex subset of Banach space E. Let $U : C \to C$ be a continuous transformation such that*

$$\mu(UW) \leq \phi(\mu(W)), \quad W \subseteq C,$$

Where $\phi \in \mathbf{\Phi}$ and μ is a measure of non-compactness in E. Then U has at least one fixed point.

For given functions $\phi, \psi : [0, 1) \to \mathbb{R}$ and a real number λ, define the operator $T_{\max}$ and T by

$$T_{\max}(\phi, \psi)(t) = \max\{\phi(t), \psi(t)\}, \quad t \geq 0$$

and

$$T(\lambda, \phi)(t) = \lambda\phi(t), \quad t \geq 0.$$

Denote $C([0, \infty); \mathbb{R})$ the set of all real continuous functions in $[0, \infty)$ and $\tilde{\mathbf{\Phi}}$ by the set $\mathbf{\Phi} \cap C([0, \infty); \mathbb{R})$. In([15]), Sadarangain and Samet proved the following useful result.

Lemma 2.3 *The following properties hold:*

(i) $T_{\max}(\boldsymbol{\Phi} \times \boldsymbol{\Phi}) \subseteq \boldsymbol{\Phi}$.

(ii) $T([0,1] \times \boldsymbol{\Phi}) \subseteq \boldsymbol{\Phi}$.

Let $E = C([a, T_1]; \mathbb{R})$ be a Banach space defined on the set C of real continuous functions in $[a, T_1]$ with norm

$$\|v\| = \max\{|v(t)| : t \in [a, T_1]\}, \qquad v \in C([a, T_1]; \mathbb{R}).$$

We will use through this paper the following notations. Let $B \in \mathcal{M}_E$. For $v \in B$ and $\epsilon \geq 0$, set

$$\omega(v, \epsilon) = \sup\{|\, v(t) - v(s)\,| : t, s \in [a, T_1], \quad |\, t - s\,| \leq \epsilon\}.$$

We define the mapping $\Omega : \mathcal{M}_E \times [0, \infty) \to [0, \infty)$ by

$$\Omega(B, \epsilon) = \sup\{\omega(v, \epsilon) : v \in B\}, \quad (B, \epsilon) \in \mathcal{M}_E \times [0, \infty) \to [0, \infty).$$

It was proved in[11] that the mapping $\mu : \mathcal{M}_E \to [0, \infty)$ defined by

$$\mu(B) = \lim_{\epsilon \to 0^+} \Omega(B, \epsilon), \qquad B \in \mathcal{M}_E$$

is a measure of non-compactness in the Banach space E. In this section, we will investigate the existence theorem of our integral equation(1) under suitable assumptions:

3   Main result

Consider the following assumptions:

(a_1) $f : I \to \mathbb{R}$; is continuous nondecreasing and nonnegative function on I,

(a_2) the function $g : I \times \mathbb{R} \to \mathbb{R}$ is continuous .

(a_3) There exists a function $\phi_1 : [0, \infty) \to [0, \infty)$ such that

(i) $\phi_1(0) = 0$.

(ii) ϕ_1 is nondecreasing and continuous.

(iii) there exist $\delta \geq 0$ and $\theta \in \tilde{\boldsymbol{\Phi}}$ such that

$$\phi_1(t) \leq \delta\theta(t), \quad t \geq 0$$

(iv) for all $(t, u, v) \in I \times \mathbb{R} \times \mathbb{R}$ we have

$$|g(t, u) - g(t, v)| \leq \phi_1(|u - v|),$$

(a_4) The operator $A : C(I) \to C(I)$ is continuous and satisfies the Darbo condition for measure of non compactness μ with the constant q, A is positive operator($Ax \geq 0$ if $x \geq 0$),

(a_5) There exist nonnegative constants a and b such that

$$|(Ax)(t)| \leq (a + b\|x\|)$$

for each $x \in C(I)$ and $t \in I$,

(a_6) $u : I \times I \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is continuous function such that $u : I \times I \times \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+$ $u(t, s, x, y)$ is nondecreasing with respect to each variable separately,

(a_7) There exists a nondecreasing function $\phi : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+$ s.t

$$|u(t, s, x, y)| \leq \phi(|x|, |y|) \ \ \forall \ \ t, s \in I \text{and} \ \ \ \ x, y \in \mathbb{R},$$

(a_8) $m \in C([a, T_1], R) \cap C^1((a, T_1], R)$ with $m'(t) > 0$ for all $t \in (a, T_1]$,

(a_9) There exists a positive constant r_0 such that

$$\| f \| + \phi_1(r_0) + g^* + \frac{(a + br_0)}{\Gamma(\alpha + 1)} . \phi(r_0, r_0)(m(T_1) - m(a))^\alpha \leq r_0$$

where

$$g^* = \max\{| g(t, 0) |: \ \ \ t \in [a, T_1]\}.$$

Theorem 3.1 *Under the assumptions* $(a_1) - (a_9)$*, the Eq(1) has at least one solution* $x^* \in C([a, T_1]; \mathbb{R})$*. Moreover* $\| x^* \| \leq r_0$.

Proof. Let us consider the operator F defined on $E = C([a, T_1], R)$ by

$$(Fx)(t) = f(t) + g(t, x(t)) + \frac{(Ax)(t)}{\Gamma(\alpha)} \int_a^t \frac{m'(s)u(t, s, x(s), x(\lambda s))}{(m(t) - m(s))^{1-\alpha}} \ ds, (x, t) \in E \times [a, T_1]. \tag{2}$$

At first,we show that the operator $F : C([a, T_1], \mathbb{R}) \to C([a, T_1], \mathbb{R})$, set

$$(Ux)(t) = \frac{1}{\Gamma(\alpha)} \int_a^t \frac{m'(s)u(t, s, x(s), x(\lambda s))}{(m(t) - m(s))^{1-\alpha}} \ ds, \ \ (x, t) \in E \times [a, T_1]. \tag{3}$$

From assumption $(a_1)(a_2)(a_4)$ We have just to prove U maps E into itself, that is $Ux : [a, T_1] \to \mathbb{R}$ is continuous for every $x \in E$. Observe that Ux is well defined for all $t \in [a, T_1]$.

$$
\begin{aligned}
|(Ux)(t)| &\leq \frac{1}{\Gamma(\alpha)} \int_a^t \frac{| m'(s) || u(t, s, x(s), x(\lambda s)) |}{(m(t) - m(s))^{1-\alpha}} ds \\
&\leq \frac{1}{\Gamma(\alpha)} . \phi(\|x\|, |\lambda| . \|x\|) \int_a^t (m(t) - m(s))^{\alpha-1} ds \\
&\leq \frac{1}{\alpha\Gamma(\alpha)} . \phi(\|x\|, \|x\|)(m(t) - m(a))^\alpha \\
&\leq \frac{1}{\Gamma(\alpha + 1)} . \phi(\|x\|, \|x\|)(m(T_1) - m(a))^\alpha < \infty.
\end{aligned}
$$

$$
\begin{aligned}
| \ (Fx)(t) | &= \left| (f(t) + g(t, x(t)) + \frac{(Ax)(t)}{\Gamma(\alpha)} \int_a^t \frac{m'(s)u(t, s, x(s), x(\lambda s))}{(m(t) - m(s))^{1-\alpha}} \ ds) \right| \\
&\leq \ | f(t) | + | g(t, x(t)) - g(t, 0) | + \frac{| (Ax)(t) |}{\Gamma(\alpha)} \int_a^t | \frac{m'(s)u(t, s, x(s), x(\lambda s))}{(m(t) - m(s))^{1-\alpha}} | \ ds
\end{aligned}
$$

$$
\begin{aligned}
&\leq \quad |f(t)| + \phi_1(|x(t)|) + |g(t,0)| + \frac{|(Ax)(t)|}{\Gamma(\alpha)} \int_a^t \left| \frac{m'(s)u(t,s,x(s),x(\lambda s))}{(m(t)-m(s))^{1-\alpha}} \right| \; ds \\
&\leq \quad \|f\| + \phi_1(\|x\|) + g^* + \frac{(a+b\|x\|)}{\Gamma(\alpha)} \int_a^t \frac{|m'(s)||u(t,s,x(s),x(\lambda s))|}{(m(t)-m(s))^{1-\alpha}} ds \\
&\leq \quad \|f\| + \phi_1(\|x\|) + g^* + \frac{(a+b\|x\|)}{\Gamma(\alpha+1)} . \phi(\|x\|,\|x\|)(m(T)-m(a))^\alpha \\
&< \quad \infty.
\end{aligned}
$$

To show the continuity of Ux at the point a. Let $\{t_n\}$ be a sequence in $[a,T_1]$ such that $t_n \to a^+$ as $n \to \infty$ using the above result, for all $n \in N$, we have

$$
\begin{aligned}
|(Ux)(t_n)| &\leq \frac{1}{\Gamma(\alpha)} \phi(\|x\|,\|x\|) \int_a^t (m(t_n)-m(s))^{\alpha-1} ds \\
&\leq \frac{1}{\Gamma(\alpha+1)} . \phi(\|x\|,\|x\|)(m(t_n)-m(a))^\alpha,
\end{aligned}
$$

letting $n \to \infty$ in the above inequality and using the continuity of m, we get

$$
\lim_{n\to\infty}(Ux)(t_n) = (Ux)(a) = 0,
$$

which prove the continuity of Ux at point a. Now let $t \in (a,T_1]$ be fixed and $\{t_n\}$ be a sequence in $(a,T_1]$ such that $t_n \to t$ as $n \to \infty$. With out restrictions we suppose that $t_n \geq t$ for n large enough. We have

$$
\begin{aligned}
&|\quad (Ux)(t_n)-(Ux)(t)| \\
&= \frac{1}{\Gamma(\alpha)} \left| \int_a^{t_n} \frac{m'(s)u(t_n,s,x(s),x(\lambda s))}{(m(t_n)-m(s))^{1-\alpha}} \; ds - \int_a^t \frac{m'(s)u(t,s,x(s),x(\lambda s))}{(m(t)-m(s))^{1-\alpha}} ds \right| \\
&\leq \frac{1}{\Gamma(\alpha)} \left| \int_a^t \left(\frac{m'(s)u(t_n,s,x(s),x(\lambda s))}{(m(t_n)-m(s))^{1-\alpha}} - \frac{m'(s)u(t,s,x(s),x(\lambda s))}{(m(t)-m(s))^{1-\alpha}} \right) ds \right| \\
&+ \frac{1}{\Gamma(\alpha)} \left| \int_t^{t_n} \frac{m'(s)u(t_n,s,x(s),x(\lambda s))}{(m(t_n)-m(s))^{1-\alpha}} \; ds \right| \\
&\leq \frac{1}{\Gamma(\alpha)} \int_a^t \left(\frac{m'(s)\phi(|x|,|\lambda x|)}{(m(t_n)-m(s))^{1-\alpha}} - \frac{m'(s)\phi(|x|,|\lambda x|)}{(m(t)-m(s))^{1-\alpha}} \right) \; ds \\
&+ \frac{1}{\Gamma(\alpha)} \int_t^{t_n} \frac{m'(s)\phi(|x|,|\lambda x|)}{(m(t_n)-m(s))^{1-\alpha}} ds \\
&\leq \frac{1}{\Gamma(\alpha)} \phi(\|x\|,\|x\|) \int_a^t \left(\frac{m'(s)}{(m(t)-m(s))^{1-\alpha}} - \frac{m'(s)}{(m(t_n)-m(s))^{1-\alpha}} \right) \; ds \\
&+ \frac{1}{\Gamma(\alpha)} \phi(\|x\|,\|x\|) \left| \int_t^{t_n} \frac{m'(s)}{(m(t_n)-m(s))^{1-\alpha}} ds \right| \\
&\leq \frac{1}{\alpha\Gamma(\alpha)} \phi(\|x\|,\|x\|)[(m(t)-m(a))^\alpha + (m(t_n)-m(t))^\alpha - (m(t_n)-m(a))^\alpha] \\
&+ \frac{1}{\alpha\Gamma(\alpha)} \phi(\|x\|,|\lambda|\|x\|)[(m(t_n)-m(t))^\alpha
\end{aligned}
$$

since m is continuous function in $[a,T_1]$, then we hae $\lim_{n\to\infty} |(Ux)(t_n)-(Ux)(t)| = 0$. Then Ux is continuous at t, then as consequence, $Ux \in E$ for all $x \in E$ and $F : E \to E$ is well defined. For arbitrary fixed $x \in E$ and $t \in [a,T_1]$ we have

$$
\begin{aligned}
|(Fx)(t)| &\leq |(f(t)| + |g(t,x(t))| + \frac{|(Tx)(t)|}{\Gamma(\alpha)} \int_a^t \frac{m'(s)|u(t,s,x(s),x(\lambda x))|}{(m(t)-m(s))^{1-\alpha}} \; ds \\
&\leq |f(t)| + \phi_1(|x(t)|) + g^* + \frac{(a+b\|x\|)}{\alpha\Gamma(\alpha)} \phi(\|x\|,|\lambda|.\|x\|)(m(T_1)-m(a))^\alpha.
\end{aligned}
$$

Then

$$
\|Fx\| \leq \|f\| + \phi_1(\|x\|) + g^* + \frac{(a+b\|x\|)}{\Gamma(\alpha+1)} \phi(\|x\|,\|x\|)(m(T_1)-m(a))^\alpha,
$$

From assumption (a_9), the fact that the functions ϕ_1, ϕ are nondecreasing, and the above inequality, we infer that the operator F maps $\overline{B(0, r_0)}$ into itself, where

$$\overline{B(0, r_0)} = \{x \in E : \parallel x \parallel \le r_0\}.$$

Now we will prove that the operator $F : \overline{B(0, r_0)} \to \overline{B(0, r_0)}$ is continuous, in order to prove this, it is sufficient to show that f, g, T, U are continuous on $\overline{B(0, r_0)}$. From assumption (a_1) the function f is continuous. Firstly we show that g is continuous on $\overline{B(0, r_0)}$. To do this, let $\{x_n\} \subset \overline{B(0, r_0)}$ and $x \in \overline{B(0, r_0)}$ such that $\parallel x_n - x \parallel \to 0$ as $n \to \infty$, and we want to prove that $\parallel gx_n - gx \parallel \to 0$ as $n \to \infty$. In fact, for all $t \in [a, T]$, we have

$$
\begin{aligned}
\mid g(t, x_n(t)) - g(t, x(t)) \mid \quad &\le \quad \phi_1(\mid x_n - x \mid) \\
&\le \quad \phi_1(\parallel x_n - x \parallel).
\end{aligned}
$$

Then

$$\parallel g(x_n) - g(x) \parallel \le \phi_1(\parallel x_n - x \parallel), \quad n \in \mathbb{N}.$$

Letting $n \to \infty$ in the above inequality, using the continuity of the function ϕ_1, and the fact that $\phi_1(0) = 0$, we get

$$\lim_{n \to \infty} \parallel g(x_n) - g(x) \parallel \le \phi_1(\lim_{n \to \infty} \parallel x_n - x \parallel) = \phi_1(0) = 0.$$

Next, for the continuity of U on $\overline{B(0, r_0)}$. To do this, we fix a real number $\rho > 0$ and take arbitrary functions $x, y \in \overline{B(0, r_0)}$ such that $\parallel x - y \parallel \le \rho$. For all $t \in [a, T_1]$, we have

$$
\begin{aligned}
\mid (Ux)(t) - (Uy)(t) \mid \quad &\le \quad \frac{1}{\Gamma(\alpha)} \int_a^t \frac{m'(s) \mid u(t, s, x(s), x(\lambda s)) - u(t, s, y(s), y(\lambda s)) \mid}{(m(t) - m(s))^{1-\alpha}} \, ds \\
&\le \quad \frac{\beta_u(\rho)(m(t) - m(a))^\alpha}{\alpha \Gamma(\alpha)} \\
&\le \quad \frac{\beta_u(\rho)(m(T_1) - m(a))^\alpha}{\Gamma(\alpha + 1)},
\end{aligned}
$$

where

$$\beta_u(\rho) = \sup\{\mid u(t, s, x_2, y_2) - u(t, s, x_1, y_1) \mid : t, s \in I, x_1, x_2, y_1, y_2 \in [0, r_0]\},$$

$$\mid x_2 - x_1 \mid \le \rho, \mid y_2 - y_1 \mid \le \rho.$$

Thus,

$$\parallel Ux - Uy \parallel \le \frac{\beta_u(\rho)(m(T_1) - m(a))^\alpha}{\Gamma(\alpha + 1)}.$$

By the uniform continuity of the function u on $I \times I \times [0, r_0] \times [0, r_0]$ we have that $\beta_u(\rho) \to 0$ as $\rho^+ \to 0$ and, therefore,

$$\lim_{\rho^+ \to 0} \parallel Ux - Uy \parallel = 0.$$

Thus U is continuous on $\overline{B(0, r_0)}$, and consequently, F is continuous. Now let us take a nonempty set $W \subset \overline{B(0, r_0)}$. Let $\epsilon > 0$ be fixed, choose $x \in W$ and $t_1, t_2 \in I$ such that $\mid t_2 - t_1 \mid \le \epsilon$. without loss of generality, we assume $t_2 \ge t_1$. Then, we obtain

$$
\begin{aligned}
&\mid \quad (Ux)(t_2) - (Ux)(t_1) \mid \\
&= \quad \frac{1}{\Gamma(\alpha)} \mid \int_a^{t_2} \frac{m'(s) u(t_2, s, x(s), x(\lambda s))}{(m(t_2) - m(s))^{1-\alpha}} \, ds - \int_a^{t_1} \frac{m'(s) u(t_1, s, x(s), x(\lambda s))}{(m(t_1) - m(s))^{1-\alpha}} ds \mid
\end{aligned}
$$

$$\leq \quad \frac{1}{\Gamma(\alpha)} \mid \int_a^{t_2} \frac{m'(s)u(t_2,s,x(s),x(\lambda s))}{(m(t_2)-m(s))^{1-\alpha}} \ ds - \int_a^{t_2} \frac{m'(s)u(t_1,s,x(s),x(\lambda s))}{(m(t_2)-m(s))^{1-\alpha}} \ ds \mid$$

$$+ \quad \frac{1}{\Gamma(\alpha)} \mid \int_a^{t_2} \frac{m'(s)u(t_1,s,x(s),x(\lambda s))}{(m(t_2)-m(s))^{1-\alpha}} \ ds - \int_a^{t_1} \frac{m'(s)u(t_1,s,x(s),x(\lambda s))}{(m(t_2)-m(s))^{1-\alpha}} \ ds \mid$$

$$+ \quad \frac{1}{\Gamma(\alpha)} \mid \int_a^{t_1} \frac{m'(s)u(t_1,s,x(s),x(\lambda s))}{(m(t_2)-m(s))^{1-\alpha}} \ ds - \int_a^{t_1} \frac{m'(s)u(t_1,s,x(s),x(\lambda s))}{(m(t_1)-m(s))^{\alpha-1}} \ ds \mid$$

$$\leq \quad \frac{1}{\Gamma(\alpha)} \int_a^{t_2} m'(s) \mid u(t_2,s,x(s),x(\lambda s)) - u(t_1,s,x(s),x(\lambda s)) \mid (m(t_2)-m(s))^{\alpha-1} \ ds$$

$$+ \quad \frac{1}{\Gamma(\alpha)} \int_{t_1}^{t_2} m'(s) \mid u(t_1,s,x(s),x(\lambda s)) \mid (m(t_2)-m(s))^{\alpha-1} \ ds$$

$$+ \quad \frac{1}{\Gamma(\alpha)} \int_a^{t_1} m'(s) \mid u(t_1,s,x(s),x(\lambda s)) \mid ((m(t_2)-m(s))^{\alpha-1} - (m(t_1)-m(s))^{\alpha-1})ds$$

$$\leq \quad \frac{1}{\Gamma(\alpha)}\omega_u(\epsilon) \int_a^{t_2} m'(s)(m(t_2)-m(s))^{\alpha-1}ds$$

$$+ \quad \frac{1}{\Gamma(\alpha)}\phi(\parallel x \parallel, \parallel x \parallel) \int_{t_1}^{t_2} m'(s)(m(t_2)-m(s))^{\alpha-1}ds$$

$$+ \quad \frac{1}{\Gamma(\alpha)}\phi(\parallel x \parallel, \parallel x \parallel) \int_a^{t_1} ((m(t_2)-m(s))^{\alpha-1} - (m(t_1)-m(s))^{\alpha-1})ds$$

$$\leq \quad \frac{\omega_u(\epsilon)(m(t_2)-m(a))^\alpha}{\alpha\Gamma(\alpha)} + \frac{\phi(\parallel x \parallel, \parallel x \parallel)(m(t_2)-m(t_1))^\alpha}{\alpha\Gamma(\alpha)}$$

$$+ \quad \frac{\phi(\parallel x \parallel, \parallel x \parallel)[-(m(t_2)-m(t_1))^\alpha + (m(t_2)-m(a))^\alpha - (m(t_1)-m(a))^\alpha}{\alpha\Gamma(\alpha)}$$

$$\leq \quad \frac{\omega_u(\epsilon)(m(t_2)-m(a))^\alpha}{\Gamma(\alpha+1)} + \frac{\phi(\parallel x \parallel, \parallel x \parallel)[(m(t_2)-m(a))^\alpha - (m(t_1)-m(a))^\alpha]}{\Gamma(\alpha+1)}$$

$$\leq \quad \frac{\omega_u(\epsilon)(m(t_2)-m(a))^\alpha}{\Gamma(\alpha+1)}.$$

Then in view of our assumptions we have

$$\mid \quad (Fx)(t_2) - (Fx)(t_1) \mid \leq \mid f(t_2) - f(t_1) \mid + \mid g(t_2,x(t_2)) - g(t_1,x(t_1)) \mid$$

$$+ \quad \frac{\mid (Ax)(t_2)(Ux)(t_2) - (Ax)(t_2)(Ux)(t_1) \mid}{\Gamma(\alpha)} + \frac{\mid (Ax)(t_2)(Ux)(t_1) - (Ax)(t_1)(Ux)(t_1) \mid}{\Gamma(\alpha)}$$

$$\leq \quad \mid f(t_2) - f(t_1) \mid + \mid g(t_2,x(t_2)) - g(t_1,x(t_1)) \mid$$

$$+ \quad \frac{\mid (Ax(t_2) \mid \mid (Ux)(t_2) - (Ux)(t_1) \mid}{\Gamma(\alpha)}$$

$$+ \quad \frac{\mid (Ax)(t_2) - (Ax)(t_1) \mid \mid (Ux)(t_1) \mid}{\Gamma(\alpha)}$$

$$\leq \quad \mid f(t_2) - f(t_1) \mid + \mid g(t_2,x(t_2)) - g(t_2,x(t_1)) \mid$$

$$+ \quad \mid g(t_2,x(t_1) - g(t_1,x(t_1)) \mid + \frac{(a+b \parallel x \parallel) \mid (Ux)(t_2) - (Ux)(t_1) \mid}{\Gamma(\alpha)}$$

$$+ \quad \frac{\mid (Ax)(t_2) - (Ax)(t_1) \mid \mid (Ux)(t_1) \mid}{\Gamma(\alpha)}$$

$$\leq \quad \mid f(t_2) - f(t_1) \mid + \phi(\mid x(t_2) - x(t_1) \mid)$$

$$+ \quad \mid g(t_2,x(t_1) - g(t_1,x(t_1)) \mid$$

$$+ \quad \frac{(a+b \parallel x \parallel)[\omega_u(\epsilon)(m(t_2)-m(a))^\alpha + \phi(\parallel x \parallel, \parallel x \parallel)[2(m(t_2)-m(t_1))^\alpha]]}{\alpha\Gamma(\alpha)}$$

$$+ \quad \frac{\mid (Ax(t_2) - (Ax)(t_1) \mid \phi(\parallel x \parallel, \parallel x \parallel)(m(t_1)-m(a))^\alpha}{\alpha\Gamma(\alpha)}$$

$$
\begin{aligned}
\leq \quad & \omega(f,\epsilon) + \phi(\omega(x,\epsilon)) + \omega_g(r_0,\epsilon) \\
+ \quad & \frac{(a+br_0)[\omega_u(\epsilon)(m(t_2)-m(a))^\alpha + 2\phi(r_0,r_0)[\omega(m,\epsilon)]^\alpha]}{\Gamma(\alpha+1)} \\
+ \quad & \frac{\omega(Ax,\epsilon)\phi(r_0,r_0)(m(t_1)-m(a))^\alpha}{\Gamma(\alpha+1)} \\
\leq \quad & \omega(f,\epsilon) + \phi(\boldsymbol{\Omega}(W,\epsilon)) + \omega_g(r_0,\epsilon) \\
+ \quad & \frac{(a+br_0)[\omega_u(\epsilon)(m(t_2)-m(a))^\alpha]}{\Gamma(\alpha+1)} + \frac{\omega(Ax,\epsilon)\phi(r_0,r_0)(m(t_1)-m(a))^\alpha}{\Gamma(\alpha+1)}
\end{aligned}
$$

$$
\begin{aligned}
\leq \quad & \omega(f,\epsilon) + \phi(\boldsymbol{\Omega}(W,\epsilon)) + \omega_g(r_0,\epsilon) \\
+ \quad & \frac{(a+br_0)[\omega_u(\epsilon)(m(T_1)-m(a))^\alpha + 2\phi(r_0,r_0)[\omega(m,\epsilon)]^\alpha]}{\Gamma(\alpha+1)} \\
+ \quad & \frac{\omega(Ax,\epsilon)\phi(r_0,r_0)(m(T_1)-m(a))^\alpha}{\Gamma(\alpha+1)}
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
\omega(Fx,\epsilon) \leq \quad & \omega(f,\epsilon) + \phi(\Omega(W,\epsilon)) + \omega_g(r_0,\epsilon) \\
+ \quad & \frac{(a+br_0)[\omega_u(\epsilon)(m(T_1)-m(a))^\alpha + 2\phi(r_0,r_0)[\omega(m,\epsilon)]^\alpha]}{\Gamma(\alpha+1)} \\
+ \quad & \frac{\omega(Ax,\epsilon)\phi(r_0,r_0)(m(T_1)-m(a))^\alpha}{\Gamma(\alpha+1)}
\end{aligned}
$$

Let $\epsilon \to 0^+$, we obtain

$$
\mu(FW) \leq \phi(\mu(W)) + \frac{\mu(AW)\phi(r_0,r_0)(m(T_1)-m(a))^\alpha}{\Gamma(\alpha+1)}
$$

Taking in consideration assumption $(a_3) - (iii)$, we get

$$
\begin{aligned}
\mu(FW) \quad \leq \quad & \delta\theta(\mu(W)) + \frac{q\mu(W)\phi(r_0,r_0)(m(t_1)-m(a))^\alpha}{\Gamma(\alpha+1)} \\
\leq \quad & (\delta + \frac{q\phi(r_0,r_0)(m(T_1)-m(a))^\alpha}{\Gamma(\alpha+1)})\mathrm{T}_{\max}(\theta,I)(\mu(W)) \\
= \quad & \mathrm{T}(\lambda, \mathrm{T}_{\max}(\theta,I))(\mu(W)),
\end{aligned}
$$

where

$$
\lambda = (\delta + \frac{q\phi(r_0,r_0)(m(T_1)-m(a))^\alpha}{\Gamma(\alpha+1)}).
$$

Since $\lambda \in [0,1]$ (see(a_9)), from Lemma 2.3, $\mathrm{T}(\lambda, \mathrm{T}_{\max}(\theta,I)) \in \Phi$. Finally, by Lemma2.2, the operator F has at least one fixed point $x^* \in B(0,r_0)$, which is the solution to the Eq.(1) satisfying $\|x^*\| \leq r_0$.

4 Example

Consider the integral equation

$$
x(t) = t^3 + \frac{e^{-1}t}{8} + \frac{1}{4\Gamma(\frac{1}{2})}\int_0^t \frac{tsx(s)x(\lambda s)}{\sqrt{t-s}}ds, \quad t \in [0,1]
$$

In this example we have $a = 0$, $T_1 = 1$, $\alpha = \frac{1}{2}$, $f(t) = t^3, t \in [0,1]$ and this function satisfies assumption (a_1) with $\| f \| = 1$. Here

$$
g(t,x(t)) = \frac{e^{-1}t}{8}, \quad (t,x) \in [0,1] \times \mathbb{R}, \tag{4}
$$

satisfies assumption(a_2) with$\phi_1(t) = \theta(t) = \delta = 0$. Moreover , in this case we have $g^* = \frac{e^{-1}}{8}$ $(Ax)(t) = \frac{1}{4}$, satisfies assumption(a_5) with $a = \frac{1}{4}$, $b = 0$, $q = 0$. Also, $u(t, s, x, y) = tsxy$, satisfies assumption(a_7) with $\phi(x, y) = xy$. Moreover, $m(t) = t$, satisfies assumption(a_8)

$$\| f \| + \phi_1(r_0) + g^* + \frac{(a + br_0)}{\alpha\Gamma(\alpha)}\phi(r_0, r_0)(m(T_1) - m(a))^\alpha \leq r_0 \tag{5}$$

$$1 + \frac{e^{-1}}{8} + \frac{1}{2\Gamma(\frac{1}{2})}.r_0^2 \leq r_0$$

then

$$r_0 = \Gamma(\frac{1}{2})(\sqrt{\Gamma(\frac{1}{2})} \pm \sqrt{\Gamma(\frac{1}{2}) - 2(1 + \frac{e^{-1}}{8})}\ \ )$$

Data Availability (excluding Review articles)

Applicable.

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