On the growth of system of entire homogenous polynomials of several complex variables

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**ABSTRACT**

In this paper we study the growth of entire functions represented by homogenous polynomials of two complex variables. The characterizations of their order and type have been obtained.

**Keywords**: homogenous polynomials; order and type; lower order and lower type.
1-INTRODUCTION

If \( \nu : \mathbb{C}^2 \rightarrow \mathbb{R}^+ \), be a real – valued function such that the following conditions hold:

(i) \( \nu(z+z') \leq \nu(z) + \nu(z') \) \( \forall z,z' \in \mathbb{C}^2. \)

(ii) \( \nu(\lambda z) \leq |\lambda| \nu(z) \forall \lambda \in \mathbb{C} \).

(iii) \( \nu(z) = 0 \iff z = 0 \), Then \( \nu \) is a norm.

Let \( f(z_1,z_2) = \sum_{m,n=0}^{\infty} P_{m,n}(z_1,z_2) \) \( \ldots \) \( (*) \) be the Taylor series expansion of \( f(z_1,z_2) \) in terms of homogeneous polynomials \( P_{m,n}(z_1,z_2) \in \mathbb{C}^2 \rightarrow \mathbb{C} \) of degree \( m+n \). We have:

\[
M(r_1,r_2) = \sup_{(z_1,z_2) \in C | r_1|,r_2| | f(z_1,z_2)|, \quad t=1,2. \quad \nu = \max (\nu_1, \nu_2) \text{Is the maximum modulus of } f(z_1,z_2) \quad \forall \nu_1, \nu_2 \in \mathbb{R}^+ \text{ with respect to the norm } \nu. \]

Define:

\[
C_{m,n} = \sup_{(z_1,z_2) \in C | r_1|,r_2| } |P_{m,n}(z_1,z_2)|
\]

The order, lower order and type of entire functions are defined respectively by:

\[
\rho = \lim_{r_1, r_2 \to \infty} \sup \frac{\log \log M(r_1,r_2)}{\log r_1, r_2}
\]

\[
\lambda = \lim_{r_1, r_2 \to \infty} \inf \frac{\log M(r_1,r_2)}{\log r_1, r_2}
\]

\[
T = \lim_{r_1, r_2 \to \infty} \sup \frac{\log M(r_1,r_2)}{(r_1, r_2)}
\]

\[
t = \lim_{r_1, r_2 \to \infty} \inf \frac{\log M(r_1,r_2)}{(r_1, r_2)}
\]

In [2], D. Kumar and K.N. Arora proved the following results

\[
\rho = \lim_{m+n \to \infty} \sup \frac{\log \nu([m+n]a_{m,n})}{\log (a_{m,n})} \quad (1.1)
\]

\[
\epsilon \rho T = \lim_{m+n \to \infty} \sup \frac{\nu([m+n]a_{m,n})}{(a_{m,n})^{\epsilon \rho T}} \quad (1.2)
\]

Where

\[
\alpha_{m,n} = \left( \frac{(m+n)^{\rho T}}{m+n} \right) \quad \text{if} \quad m,n \geq 1
\]

\[
= 0 \quad \text{if} \quad m,n = 0
\]

Analogously, the lower order and lower type are defined by:

\[
\lambda = \lim_{m+n \to \infty} \inf \frac{\log \nu([m+n]a_{m,n})}{\log (a_{m,n})} \quad (1.3)
\]

\[
\epsilon \rho t = \lim_{m+n \to \infty} \inf \frac{\nu([m+n]a_{m,n})}{(a_{m,n})^{\epsilon \rho t}} \quad (1.4)
\]

In this paper we consider have system of entire functions as follows:

\[
f(z_1,z_2) = \sum_{m,n=0}^{\infty} P_{m,n}^{(i)}(z_1,z_2), \quad i=1,2,\ldots,k \quad (1.5)
\]

Then we obtain some relations between the function represented by (*) and the system of entire homogeneous polynomials (1.5) and study the relations between the coefficients in Taylor expansion of entire homogeneous polynomials and their type. Also we continue the work of H.H. Khan and R.Ali [3], Where they generalized and improve the results of R.K. Srivastava, Vinod Kumar [5] and S.S. Dalal [1].

2-Main Results.

Theorem 2.1

Let \( f_i(z_1,z_2) = \sum_{m,n=0}^{\infty} P_{m,n}^{(i)}(z_1,z_2) \) where \( i=1,2,\ldots,k \) be \( k \) entire homogeneous polynomials of finite regular growth \( p_1,p_2,\ldots,p_k \) respectively and
\[ \alpha_{m,n}^{(i)} \sim \alpha_{m,n} \] (2.1)

In order these functions have the same order that satisfy the following condition

\[ \log \left\{ \frac{|c_{m,n}^{(i)}|^{\frac{-1}{m+n}}}{|c_{m,n}^{(i-1)}|^{\frac{-1}{m+n}}} \right\} = o(\log((m+n)\alpha_{m,n})) \]

**Proof**

Since \( f_i, i=1,2,\ldots, k \) have regular growth then

\[ \lim_{m+n \to \infty} \sup \frac{\log |(m+n)a_{m,n}^{(i)}|}{\log |(m+n)a_{m,n}^{(i-1)}|} = \rho_i \]

Since the functions \( f_i (z_1,z_2) \) for \( i=1,\ldots,k \) have the same order then \( \rho = \rho_1 = \rho_2 = \cdots = \rho_k \)

Log \[ \left\{ \alpha_{m,n}^{(i)} / \alpha_{m,n}^{(i-1)} \right\} = o(\log((m+n)\alpha_{m,n})) \] as \( m+n \to \infty \).

Let us prove the converse

Let \( f_i(z_1,z_2), i=1,2,\ldots,k \) have orders \( \rho_i, i=1,2,\ldots,k \) respectively then by using condition (2.1) we have

\[ \frac{1}{\rho_i} \frac{1}{\rho_{i-1}} = \lim_{m+n \to \infty} \sup \frac{\log \left| c_{m,n}^{(i)} \right|^{\frac{-1}{m+n}}}{\log \left| (m+n)a_{m,n}^{(i)} \right|} - \lim_{m+n \to \infty} \sup \frac{\log \left| c_{m,n}^{(i-1)} \right|^{\frac{-1}{m+n}}}{\log \left| (m+n)a_{m,n}^{(i-1)} \right|} \]

\[ \lim_{m+n \to \infty} \sup \frac{\log \left| c_{m,n}^{(i)} \right|^{\frac{-1}{m+n}}}{\log \left| (m+n)a_{m,n}^{(i)} \right|} = 0, \text{Then} \quad \rho_i = \rho_{i-1} \]

**Theorem 2.2**

Let the system of homogenous polynomials \( f_i(z_1,z_2) \) for \( i=1,2,\ldots,k \), have the regular type and the same order such that

(1) \[ \lim_{m+n \to \infty} \sup \left| c_{m,n}^{(i)} \right|^{\frac{-1}{m+n}} = 0 \quad \text{for} \quad i=1,2,\ldots,k \] (2.2)

(2) \[ \alpha_{m,n}^{(i)} \sim \alpha_{m,n} \quad \text{for} \quad i=1,2,\ldots,k \]

In order these functions have the same type that satisfy the condition

\[ \log \left\{ \frac{|c_{m,n}^{(i)}|}{|c_{m,n}^{(i-1)}|} \right\} = o(\log((m+n)) \text{ as } m+n \to \infty \]

**Proof:**

The functions \( f_i(z_1,z_2), i=1,2,\ldots,k \) have the regular type therefore...
\[
\lim_{m+n \to \infty} \inf \frac{1}{e^\rho} \frac{(m+n)\alpha_m^{(i)}}{(m^{(i)})} = t_i = T_i = \lim_{m+n \to \infty} \sup \frac{1}{e^\rho} \frac{(m+n)\alpha_m^{(i)}}{(m^{(i)})}
\]

\[
\lim_{m+n \to \infty} \inf \frac{1}{e^\rho} \frac{(m+n)\alpha_m^{(i-1)}}{(m^{(i-1)})} = t_{i-1} = T_{i-1} = \lim_{m+n \to \infty} \sup \frac{1}{e^\rho} \frac{(m+n)\alpha_m^{(i-1)}}{(m^{(i-1)})}
\]

Let the functions \( f_i(z_1, z_2) \), \( i = 1, 2, \ldots, k \) have the same type that is to say

\[
\lim_{m+n \to \infty} \frac{1}{e^\rho} \frac{(m+n)\alpha_m^{(i)}}{(m^{(i)})} = \text{type } T_i = \lim_{m+n \to \infty} \sup \frac{1}{e^\rho} \frac{(m+n)\alpha_m^{(i)}}{(m^{(i)})}
\]

Hence if we take into account the condition (2) in (2.2) then we have:

\[
\lim_{m+n \to \infty} \frac{\rho}{m+n} (\log |c_{m,n}^{(i-1)}| - \log |c_{m,n}^{(i)}|) = 0
\]

At last

\[
\text{Log} \left\{ \frac{|c_{m,n}^{(i-1)}|}{|c_{m,n}^{(i)}|} \right\} = o(m+n) \quad \text{as } m+n \to \infty
\]

Let us prove the converse.

Let the functions \( f_i(z_1, z_2) \), \( i = 1, 2, \ldots, k \) have type \( T_i \) and \( T_{i-1} \) respectively.

Then \( \log T_i - \log T_{i-1} = \rho \lim_{m+n \to \infty} \frac{1}{m+n} \log \left\{ \frac{|c_{m,n}^{(i-1)}|}{|c_{m,n}^{(i)}|} \right\} = 0 \). Hence \( T_{i-1} = T_i \).

**Theorem 2.3**

Let each function of system (1.5) be an entire homogeneous polynomials of order \( \rho_i (0 < \rho_i < \infty) \) and type \( T_i (0 < T_i < \infty) \), \( t_i (0 < t_i < \infty) \), \( i = 1, \ldots, k \) respectively, if

1. \( \alpha_m^{(i)} = \alpha_{m,n} \) for \( i = 1, 2, \ldots, k \)
2. \( C_m, n \sim \prod_{i=1}^{k} \left( c_{m,n}^{(i)} \right)^{r_i} \)
3. \( \frac{(m+n)\alpha_m^{(i)}}{(m^{(i)})} \sim \prod_{i=1}^{k} \left( \frac{(m+n)\alpha_m^{(i)}}{(m^{(i)})} \right)^{r_i} \)\( \sum_{i=1}^{k} r_i = 1 \) (2.3)

Then \( f(z_1, z_2) \) is entire function and \( \prod_{i=1}^{k} (t_i)^{r_i} \leq \leq \prod_{i=1}^{k} (T_i)^{r_i} \), Where \( T, t \) are type and lower type of \( f(z_1, z_2) \) respectively.

**Proof**

It can be easily seen [4] that the necessary and sufficient condition for \( f(z_1, z_2) \) to represent an entire polynomial of two complex variables \( (z_1, z_2) \) is that:

\[
\lim_{m+n \to \infty} \frac{c_{m,n}^{(i)}}{m+n} = 0 \quad \text{for } i = 1, 2, \ldots, k
\]

Also from conditions (1, 2) in (2.3) we get

\[
\lim_{m+n \to \infty} \sup \left( \frac{c_{m,n}^{(i)}}{m+n} \right)^{1/r_i} \leq \prod_{i=1}^{k} \lim_{m+n \to \infty} \sup \left( \frac{c_{m,n}^{(i)}}{m+n} \right)^{1/r_i} = 1
\]

Hence \( f(z_1, z_2) \) is an entire polynomial.

From (1.2) we have

\[
\frac{1}{e^\rho} \left( \frac{(m+n)\alpha_m^{(i)}}{(m^{(i)})} \right)^{1/r_i} < T_i \quad \text{for } i = 1, 2, \ldots, k
\]
Hence \[ \prod_{i=1}^{k} \left( \frac{1}{\epsilon_{m,n}} \frac{(m+n) \rho_{m,n}}{m+n} \right)^{r_{i}} < \prod_{i=1}^{k} (T_{i} + \epsilon)^{r_{i}} \]

Similarly \[ \prod_{i=1}^{k} (T_{i} - \epsilon)^{r_{i}} < \prod_{i=1}^{k} \left( \frac{1}{\epsilon_{m,n}} \frac{(m+n) \rho_{m,n}}{m+n} \right)^{r_{i}} < \prod_{i=1}^{k} (T_{i} + \epsilon)^{r_{i}} \]

Taking into account the condition (3) in (2.3)

\[ \prod_{i=1}^{k} (T_{i} - \epsilon)^{r_{i}} < \prod_{i=1}^{k} \left( \frac{1}{\epsilon_{m,n}} \frac{(m+n) \rho_{m,n}}{m+n} \right)^{r_{i}} < \prod_{i=1}^{k} (T_{i} + \epsilon)^{r_{i}} \]

Passing to limits as \( m+n \rightarrow \infty \) we obtain

\[ \prod_{i=1}^{k} (T_{i})^{r_{i}} \leq T \prod_{i=1}^{k} (T_{i})^{r_{i}} \]

Thus the theorem is proved.

**Theorem 2.4**

Let \( f_{1}(z_{1}, z_{2}) = \sum_{m,n=0}^{\infty} p_{m,n}^{(1)}(z_{1}, z_{2}) \) and \( f_{2}(z_{1}, z_{2}) = \sum_{m,n=0}^{\infty} p_{m,n}^{(2)}(z_{1}, z_{2}) \) be two entire polynomials of finite non-zero orders \( \rho_{1}, \rho_{2} \) and finite non-zero types \( T_{1}, T_{2} \) respectively then the function

\[ f(z_{1}, z_{2}) = (a_{m,n} + b_{m,n})^{1/2} \]

\[ c_{m,n} = \sup_{\rho} \left| p_{m,n}(z_{1}, z_{2}) \right|, \quad a_{m,n} = \sup_{\rho} \left| p_{m,n}^{(1)}(z_{1}, z_{2}) \right| \]

\[ b_{m,n} = \sup_{\rho} \left| p_{m,n}^{(2)}(z_{1}, z_{2}) \right| \]

is an entire function such that

\( (\rho T)^{2} \leq (\rho_{1} T_{1})^{p_{1}} \) and \( (\rho_{2} T_{2})^{p_{2}} \]

Where \( \rho \) and \( T \) are order and type of \( f(z_{1}, z_{2}) \) respectively and

\[ 2/\rho = 1/\rho_{1} + 1/\rho_{2} \]

**Proof**

We can prove as the proof of theorem 2.3 that \( f(z_{1}, z_{2}) \) is an entire function where

\[ c_{m,n} \sim (a_{m,n} b_{m,n})^{1/2} \text{ and } a_{m,n}^{(1)} \sim a_{m,n} \]

Further, using (1.2) for the function \( f_{1}(z_{1}, z_{2}) \), we have

\[ \lim_{m+n \rightarrow \infty} \sup \left\{ (m+n) a_{m,n}^{(1)} \rho_{1} \right\} \leq \left( a_{m,n} \right)^{1/2} \leq \left( b_{m,n} \right)^{1/2} \]

(2.7)

\[ \leq \epsilon \rho T \]

(2.8)

From (2.7) and (2.8), we get for an arbitrary \( \epsilon > 0 \)

\[ \left( (m+n) a_{m,n}^{(1)} \right)^{1/2} \leq \left( (m+n) a_{m,n}^{(2)} \right)^{1/2} \leq \epsilon \rho T \]

Thus for \( m+n > k \) max (k1, k2) and \( 2/\rho = 1/\rho_{1} + 1/\rho_{2} \), and using the condition \( a_{m,n}^{(1)} \sim a_{m,n} \)

we have

\[ \left( (m+n) a_{m,n} \right)^{1/2} \leq \left( (m+n) a_{m,n} b_{m,n} \right)^{1/2} \leq \epsilon \rho T \]

Therefore, if \( c_{m,n} \sim (a_{m,n} b_{m,n})^{1/2} \), we have obtain

\[ \lim_{m+n \rightarrow \infty} \sup \left\{ (m+n) a_{m,n} \right\} \leq \epsilon \rho T \]

Or

5677 | Page 1

January 11 2016
\[(e \rho T)^{\frac{1}{2}} \leq (e \rho_1 T_1)^{\frac{1}{2}}, (e \rho_2 T_2)^{\frac{1}{2}},\] where \(\rho\) and \(T\) are order and type of \(f(z_1, z_2)\) respectively. Hence

\[(\rho T)^{\frac{2}{2}} \leq (\rho_1 T_1)^{\frac{2}{2}}, (\rho_2 T_2)^{\frac{2}{2}}.\]

**Corollary**

\[f(z_1, z_2) = \sum_{m,n=0}^{\infty} p_{m,n} (z_1, z_2), \; i=1,2,...,k\] be \(k\) entire polynomials of finite non-zero orders \(\rho_1, \rho_2, \ldots, \rho_k\) and finite non-zero types \(T_1, T_2, \ldots, T_k\) respectively. Then the function

\[f(z_1, z_2) = \sum_{m,n=0}^{\infty} p_{m,n} (z_1, z_2)\] with

\[e_{m,n} \sim \prod_{i=1}^{k} (a_{m,n}^{(i)})^{\frac{1}{1}}\] is an entire function such that

\[(\rho T)^{\frac{k}{2}} < \prod_{i=1}^{k} (\rho_i T_i)^{\frac{k}{2}},\] where \(\rho\) and \(T\) are order and type of \(f(z_1, z_2)\) respectively and

\[\frac{k}{\rho} = \sum_{i=1}^{k} 1/\rho_i\]

**References.**


