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Inverse scattering with non-over-determined data

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Abstract

The results of the author's theory of the inverse scattering with non-over-determined data are described.

1 Introduction

There is a large literature on inverse scattering, see [1] and references therein. We consider the potential scattering and the obstacle scattering.

The potential scattering problem consists of finding the scattering solution $u(x, \alpha, k)$:

$$[\nabla^2 + k^2 - q(x)]u = 0 \quad in \quad \mathbb{R}^3, \tag{1}$$

$$u = u_0 + v, \quad u_0 = e^{ik\alpha\cdot x} \tag{2}$$

$$v_r - ikv = O(r^{-2}), \quad r \to \infty. \tag{3}$$

Here $r := |x|$, $\alpha \in S^2$, S^2 is the unit sphere in $\mathbb{R}^3$, $q = q(x) \in L^2_{loc}(\mathbb{R}^3)$ is assumed to be compactly supported. One has

$$v(x, \alpha, k) = A(\beta, \alpha, k)\frac{e^{ikr}}{r} + O(r^{-2}), \quad r \to \infty, \ \beta = x/r. \tag{4}$$

The $A(\beta, \alpha, k)$ is called the scattering amplitude, $\beta \in S^2$ is the direction of the scattered wave.

The inverse scattering problem consists of finding $q(x)$ from the scattering amplitude A. The function A is a function of five variables. It is easy to prove that this function known for all α, $\beta \in S^2$ and $\forall k > 0$ determines q uniquely. In 1987 the author proved that a compactly supported potential q is uniquely determined by fixed-energy scattering amplitude. More precisely, the values of $A(\beta, \alpha, k_0)$ for β and α running through fixed open subsets of S^2 and $k = k_0 > 0$ fixed determine a compactly supported q uniquely, see [3], [4], [5], [1]. The author also gave stability estimates for q in terms of the scattering amplitude, see [6], [1] and references therein.

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However, the fixed-energy data is a function of four variable, while the $q(x)$ is a function of three variables. The non-over-determined data are the values of the scattering amplitude which form a three-dimensional set. For example, the values $A(-\alpha, \alpha, k)$ for all $\alpha \in S^2$ and all $k > 0$ is such a set. These are the back-scattering data at all energies. In fact, for compactly supported potentials the author proved uniqueness of the solution to the inverse scattering problem with the non-over-determined data $A(-\alpha, \alpha, k)$ known for all k in an arbitrary small open subset of $[0, \infty)$ and all α in an arbitrary small open subset of S^2. The author proved that for a compactly supported potential these data determine uniquely the values of $A(-\alpha, \alpha, k)$ for all $k > 0$ and all $\alpha \in S^2$.

The other practically interesting example of non-over-determined data for which the author proved the uniqueness of the solution to the inverse scattering problem are the values of $A(\beta, \alpha_0, k)$ known for all k in an arbitrary small open subset of $[0, \infty)$ and all β in an arbitrary small open subset of S^2, $\alpha = \alpha_0$ being fixed.

These results are first published in [13], [14], [15] and in the monograph [1].

The obstacle scattering problem consists of finding the scattering solution $u(x, \alpha, k)$. Let $D \subset \mathbb{R}^3$ be a bounded domain with a smooth connected boundary S, $D' := \mathbb{R}^3 \setminus D$. Then

$$(\nabla^2 + k^2)u = 0 \quad in \quad D', \quad u|_S = 0, \tag{5}$$

$$u = u_0 + v, \quad u_0 = e^{ik\alpha\cdot x} \tag{6}$$

$$v_r - ikv = O(r^{-2}), \quad r \to \infty. \tag{7}$$

One has

$$v(x, \alpha, k) = A(\beta, \alpha, k)\frac{e^{ikr}}{r} + O(r^{-2}), \quad r \to \infty, \ \beta = x/r. \tag{8}$$

The non-over-determined data are the values of $A(\beta, \alpha, k)$ on a two-dimensional subset of the set $S^2 \times S^2 \times [0, \infty)$. For example, such is the set $\forall\beta \in S^2$, a fixed $\alpha = \alpha_0$ and a fixed $k = k_0 > 0$.

The author proved that these non-over-determined data determine uniquely the surface S and the boundary condition on S. The boundary condition is assumed of the Dirichle, or Neumann, or impedance type. The impedance boundary condition is

$$u_N = \zeta u \quad on \quad S. \tag{9}$$

Here $\zeta = \zeta(s)$ is the boundary impedance and it is assumed that

$$Im\zeta \leq 0. \tag{10}$$

Assumption (10) guarantees the uniqueness of the solution to the obstacle scattering problem, [11].

The uniqueness theorems for inverse obstacle scattering with non-over-determined data is proved by the author in [8], [1], [16].

Let us assume that two obstacles D_1 and D_2 generate the same scattering amplitude for all $\beta \in S^2$, a fixed α and a fixed $k = k_0 > 0$, and prove that then $D_1 = D_2$ and the boundary conditions are the same. If $D_1 = D_2 := D$ then $u_1 = u_2$ in D', so $u_1 = u_2$ and $U_{1N} = u_{2N}$ on $S := \partial D$. Consequently, the boundary conditions are the same.

Let us prove that $S_1 = S_2$ if $A_1(\beta) = A_2(\beta)$ for all $\beta \in S^2$, where $A_j(\beta) := A_j(\beta, \alpha_), k_0)$, $j = 1, 2$. If $A_1(\beta) = A_2(\beta)$ then $u_1(x, \alpha_0, k_0) = u_2(x, \alpha_0, k_0)$ for all $x \in D'_{12} := D_1 \cup D_2$. This

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follows from Lemma 1.2.15 in [1], p.47. Let $D^{12} := D_1 \cap D_2$, $S_{12} := \partial D_{12}$, $S^{12} := \partial D^{12}$. One has $u_1 = u_2 := u$ in $\mathbb{R}^3 \setminus D^{12}$. By Green's formula one gets

$$u = u_0 - \int_{S_1} g(x,s) u_N ds, \quad x \in D_1' \tag{11}$$

and

$$u = u_0 - \int_{S_2} g(x,s) u_N ds, \quad x \in D_2'. \tag{12}$$

Since u and u_0 are defined in $\mathbb{R}^3 \setminus D^{12}$, so are the integrals in (11) and (12), and consequently one obtains

$$\int_{S_1} g(x,s) u_N ds = \int_{S_2} g(x,s) u_N ds \quad x \in D_{12} \setminus D^{12}. \tag{13}$$

By Green's formula one has

$$u = \int_{S_2} g(x,s) u_N ds - \int_{S_1} g(x,s) u_N ds = 0, \quad x \in D_{12} \setminus D^{12}. \tag{14}$$

Since u is analytic function of x in $\mathbb{R}^3 \setminus D^{12}$ and vanishes in $D_{12} \setminus D^{12}$ it must vanish in D_{12}'. This is a contradiction since $\lim_{|x|\to\infty} |u(x, \alpha_0, k_0)| = 1$. This contradiction proves that $D_1 = D_2$, so $S_1 = S_2$. $\square$

A study of the inverse scattering problems with non-over-determined data is of principal interest because these are the minimal data from which the unknown scatterer can be uniquely determined.

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