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An efficient scheme for accurate closed-form approximate solution of some Duffing- and Liénard-type equations

Prakash Kumar Das, Debabrata Singh, M. M. Panja*

Department of Mathematics, Visva-Bharati, Santiniketan 731 235, West Bengal, India

Abstract

In this work, authors propose some modifications Adomian decomposition method to get some accurate closed-form approximate or exact solutions of Duffing- and Liénard-type nonlinear ordinary differential equations. Results obtained by the revised scheme have been exploited subsequently to derive constraints among parameters to get the solutions to be bounded. The present scheme appears to be efficient and may be regarded as the confluence of apparently different methods for getting exact solutions for a variety of nonlinear ordinary differential equations appearing as mathematical models in several physical processes.

Keywords: Duffing-type equation, Liénard-type equation, Adomian decomposition method, Rapidly convergent approximation method, Accurate closed-form approximate solutions

1. Introduction

Nonlinear oscillation or solitary wave propagation are ubiquitous. They model numerous physical phenomena: from condensed matter physics, nonlinear optics, plasma physics, fluid dynamics, etc. to biophysics. A large class of problems in such fields have been described by autonomous Duffing- and Liénard-type ordinary differential equations (ODE). Consequently, getting exact or accurate closed-form approximate solutions for such equations has been a topic of intense research for many years [1-25]

Since its inception, Adomian decomposition method (ADM) has been used to get an approximate solution of nonlinear differential/integral equations in the diverse field of science and engineering [26-31]. However, one of its important aspects, e.g. as a tool for getting an exact solution of the equation concerned, if that is integrable, has not been yet exercised rigorously. In their attempt, authors in [32, 33] showed that appropriate modifications in the traditional ADM could provide exact solutions in compact form for a variety of nonlinear ordinary differential equations (NLODEs) involving single algebraic nonlinear interaction term. Such method has been designated as rapidly convergent approximation method (RCAM). This success encouraged us to explore whether an accurate closed-form approximate or exact solution of another family of NLODEs involving relatively difficult nonlinear interaction terms can be obtained in a compact form with the invention of further trick on the existing scheme of RCAM.

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*Corresponding author

Email addresses: prakashdas.das1@gmail.com (Prakash Kumar Das), debabratasingh01@gmail.com (Debabrata Singh), madanpanja2005@yahoo.co.in (M. M. Panja)

The objective of this work is to invent some modifications to RCAM with a view to obtain an accurate closed-form approximate solutions (reducible to exact one) of Duffing- and Liénard-type equations (DTEs/LTEs)

$$u''(x) + \mu\, u(x) - \delta\, u(x)^m - \gamma\, u(x)^{2m-1} = 0 \tag{1}$$

and

$$v''(x) + \alpha_1 \frac{v'(x)^2}{v(x)} + \mu_1\, v(x) - \delta_1\, v(x)^p - \gamma_1\, v(x)^{2p-1} = 0 \tag{2}$$

respectively. It is not straightforward to solve Eq.(2) exactly due to the presence of three nonlinear terms, the second term in particular. To avoid difficulties we introduce transformations of dependent variable and parameters involved in equation (2) given by

$$v(x) = u(x)^{\frac{1}{1+\alpha_1}},\ m = \frac{p+\alpha_1}{1+\alpha_1},\ \mu = (1+\alpha_1)\,\mu_1,\ \ \delta = (1+\alpha_1)\,\delta_1,\ \gamma = (1+\alpha_1)\,\gamma_1, \tag{3}$$

which reduce Eq.(2) to the DTE(1). Consequently, solutions of equation (1) and their properties can be used to obtain the same for solutions of LTE(2) through the straightforward use of the transformation (3).

However, the relation between two dependent variables (v and u) presented in (3) suggests that the function $v(x)$ is not smooth (due to the presence of rational exponent $\frac{1}{1+\alpha_1}$), in general. So, appropriate conditions among parameters involved in the equation are desirable, so that solutions of Eqs.(1) and (2) become smooth and bounded.

To accomplish the goal, we have first introduced some modification in the RCAM [32, 33] so that one can evaluate the sequence of higher order terms of the approximate solution involved in the scheme through recurrence relation instead of evaluation of multiple integrals. Reformulated RCAM has been employed to obtain a solution of Eq.(1) and Eq.(2) in a compact form involving the exponents and all the parameters present there. It is shown that some smooth solutions of Eq.(1) available in the literature obtained by employing different methods and multiple contexts are the particular cases of the solution derived here. These results have been used afterwards to derive constraints among parameters to get solutions of both Eqs.(1) and (2) to be bounded.

The organization of the paper is as follows. To exhibit modifications invented here on existing scheme of RCAM, its salient steps for equations with linear part involving variable coefficients have been presented in section 2. Proposed modifications of RCAM relevant to Eq.(1) and Eq.(2), their solutions involving parameters, conditions among parameters for the existence of a bounded solution, recovery of some solutions obtained by different methods have been presented in section 3 and 4 respectively. Our findings and prospects of ADM in its revised form have been summarized in section 5.

2. Salient features of rapidly convergent approximation scheme

Here we consider a model NLODE of the form

$$u''(x)\ + p(x)u'(x) + q(x)u(x) = \mathcal{N}[u](x), \quad x \in \mathbb{R}. \tag{4}$$

Here $p(x),\ q(x) \in C(\mathbb{R})$, $\mathcal{N}[u]$ is the collection of nonlinear terms. Eq.(4) can be recast into the form

$$\hat{\mathcal{O}}[u](x) = \mathcal{N}[u](x), \tag{5}$$

where the linear operator $\hat{\mathcal{O}}[\cdot]$ is given by

$$\hat{\mathcal{O}}[\cdot](x) \equiv \frac{d^2\ [\cdot]}{dx^2} + p(x)\frac{d\ [\cdot]}{dx} + q(x)\ [\cdot]$$

$$= \begin{cases} e^{-\int \lambda_1(x)dx} \frac{d}{dx}\left(e^{\int (\lambda_1(x)-\lambda_2(x))dx} \frac{d}{dx}\left(e^{\int \lambda_2(x)dx}[\cdot]\right)\right) & (\lambda_1(x), \lambda_2(x)) \neq (0,0), \\ \frac{d^2}{dx^2}\ [\cdot] & (\lambda_1(x), \lambda_2(x)) = (0,0). \end{cases} \tag{6}$$

The relation among functions $p(x),\ q(x)$ and $\lambda_1(x),\ \lambda_2(x)$ are given by

$$\lambda_1(x) = p(x) - \lambda_2(x)$$

while $\lambda_2(x)$ satisfies the Riccati equation

$$\lambda_2'(x) - q(x) + p(x)\ \lambda_2(x) - \lambda_2(x)^2 = 0. \tag{7}$$

Method for getting solution of Eq.(7) have been discussed in Refs. [34, 35, 36].

The inverse $\hat{\mathcal{O}}^{-1}$ of the operator $\hat{\mathcal{O}}$ in (6) can be regarded as a twofold integral operator

$$\hat{\mathcal{O}}^{-1}[\cdot](x) = \begin{cases} e^{-\int \lambda_2(x)dx} \int^x e^{\int (\lambda_2(s)-\lambda_1(s))ds} \int^s e^{\int \lambda_1(t)dt}[\cdot](t)\ dtds & (\lambda_1(x), \lambda_2(x)) \neq (0,0), \\ \int^x \int^s [\cdot](t)\ dtds & (\lambda_1(x), \lambda_2(x)) = (0,0). \end{cases} \tag{8}$$

This formula can be invoked to get

$$\hat{\mathcal{O}}^{-1}(u^{''}(x) + p(x)u'(x) + q(x)u(x))$$

$$= \begin{cases} u(x) - c_1\ e^{-\int \lambda_1(x)dx} - c_2\ e^{-\int \lambda_2(x)dx} \int^x e^{\int (\lambda_2(s)-\lambda_1(s))ds} ds & (\lambda_1(x), \lambda_2(x)) \neq (0,0), \\ u(x) - c_1\ - c_2\ x & (\lambda_1(x), \lambda_2(x)) = (0,0). \end{cases} \tag{9}$$

Here c_1 and c_2 are two arbitrary constants.

To obtain an accurate closed-form approximate or exact solution of Eq.(4), we write the unknown $u(x)$ and the nonlinear term $\mathcal{N}[u]$

$$u(x) = \sum_{n=0}^{\infty} \epsilon^{\frac{n}{2}}\ u_n(x), \tag{10}$$

$$\mathcal{N}[u](x) = \epsilon^p \sum_{n=0}^{\infty} \epsilon^{\frac{n}{2}} \mathcal{A}_n(u_0(x), u_1(x), \cdots, u_n(x)) \equiv \epsilon^p \sum_{n=0}^{\infty} \epsilon^{\frac{n}{2}}\ \mathcal{A}_n(x), \tag{11}$$

respectively, in a series of an artificial parameter ϵ. Here the coefficients

$$\mathcal{A}_n(x) = \frac{1}{n!}\left[\frac{d^n}{d\bar{\epsilon}^n}\mathcal{N}\left(\sum_{k=0}^{\infty} u_k \bar{\epsilon}^k\right)\right]_{\bar{\epsilon}=0},\quad n \geq 0 \tag{12}$$

are known as Adomian polynomial [26, 27].

Now using (10) and (11), one can rewrite Eq.(5) as

$$\sum_{n=0}^{\infty} \epsilon^{\frac{n}{2}}\ \hat{\mathcal{O}}\left[u_n(x)\right] = \epsilon^p \sum_{n=0}^{\infty} \epsilon^{\frac{n}{2}}\ \mathcal{A}_n(x). \tag{13}$$

It is worthy to mention here that collection of coefficients of ϵ after setting $p = \frac{1}{2}$ in (13) provide recurrence relation

$$\hat{\mathcal{O}}\left[u_0(x)\right] = 0, \quad \hat{\mathcal{O}}\left[u_n(x)\right] = \mathcal{A}_{n-1}(x), \quad n \geq 1 \tag{14}$$

involving successive corrections $u_n(x)$ and inhomogeneous terms $\mathcal{A}_{n-1}(x)$ while the choice $p = 1$ in (13) gives

$$\hat{\mathcal{O}}\left[u_0(x)\right] = 0, \quad \hat{\mathcal{O}}\left[u_1(x)\right] = 0, \; \hat{\mathcal{O}}\left[u_n(x)\right] = \mathcal{A}_{n-2}(x), \quad n \geq 2. \tag{15}$$

In case of $\lambda_1(x), \; \lambda_2(x)$ are nonzero, operating inverse operator $\hat{\mathcal{O}}^{-1}$ on both sides of the system of equations (14) and (15) followed by the use of results in (9) leads to the iterative schemes

$$u_0(x) = c_1 \; e^{-\int \lambda_1(x)dx} + c_2 \; e^{-\int \lambda_2(x)dx} \int^x e^{\int(\lambda_2(s)-\lambda_1(s))ds} ds, \; u_n(x) = \hat{\mathcal{O}}^{-1}\left[\mathcal{A}_{n-1}(x)\right], \quad n \geq 1, \tag{16}$$

and

$$u_0(x) = c_1 \; e^{-\int \lambda_1(x)dx} + c_2 \; e^{-\int \lambda_2(x)dx} \int^x e^{\int(\lambda_2(s)-\lambda_1(s))ds} ds,$$

$$u_1(x) = d_1 \; e^{-\int \lambda_1(x)dx} + d_2 \; e^{-\int \lambda_2(x)dx} \int^x e^{\int(\lambda_2(s)-\lambda_1(s))ds} ds, \; u_n(x) = \hat{\mathcal{O}}^{-1}\left[\mathcal{A}_{n-2}(x)\right], \quad n \geq 2. \tag{17}$$

Following the similar steps, the formulae for the iteration in case of $\lambda_1 = \lambda_1 = 0$ can be found as

$$u_0(x) = \bar{c}_0 \; + \bar{c}_1 \; x, \quad u_n(x) = \hat{\mathcal{O}}^{-1}\left[\mathcal{A}_{n-1}(x)\right], \quad n \geq 1, \tag{18}$$

and

$$u_0(x) = \bar{c}_0 \; + \bar{c}_1 \; x, \quad u_1(x) = \bar{d}_0 \; + \bar{d}_1 \; x, \; u_n(x) = \hat{\mathcal{O}}^{-1}\left[\mathcal{A}_{n-2}(x)\right], \quad n \geq 2. \tag{19}$$

In both cases, symbols $\bar{c}_i, \; \bar{d}_i \; (i = 0, 1)$ are arbitrary constants. It may be be observed that the recursive scheme given by the system of equations (18) is the same as used in conventional ADM. The rest three, (16), (17) and (19) seem to be new. We will now apply these schemes to obtain the accurate closed-form approximate or exact solution of DTE (1) and LTE (2).

3. The solution of Duffing-type equation (1)

The DTE (1) appear as a mathematical model in a diverse field of science [2-12]. So, knowledge about their exact solutions will be useful in the studies of relevant physical processes.

3.1. Application of recursive scheme (16)

The recursive scheme applied here has some desirable advantage over the traditional ADM. The RCAM in its revised form is able to provide a sequence $\{u_n(x), n \in \mathbb{N}\}$ of corrections without evaluating multi-fold integrations. First few elements of the sequence can provide their generating function which yields the exact solution in a straightforward way. Furthermore, the boundary conditions at $\pm\infty$ can also be accommodated more efficiently in this scheme.

3.1.1. Solution

Comparison of DTE (1) and the reference Eq.(4) gives

$$p(x) = 0,\ q(x) = \mu = -\lambda^2(\text{say}) \tag{20}$$

and

$$\mathcal{N}[u](x) = \delta\ u(x)^m + \gamma\ u(x)^{2m-1}. \tag{21}$$

Then $\lambda_1(x), \lambda_1(x)$ involved in the formulae (6), (8) and (9) are given by

$$\lambda_1(x) = -\lambda_2(x)$$

where $\lambda_2(x)$ is the solution of

$$\lambda_2'(x) = \lambda_2(x)^2 - \lambda^2.$$

It is interesting to observe that this equation admits two distinct solutions viz, $\lambda_2(x) = \pm\lambda$ and $\lambda_2(x) = \lambda\ \tanh\lambda\ x$ so that factors involved in $\hat{\mathcal{O}}$ and its inverse $\hat{\mathcal{O}}^{-1}$ may be found as $e^{\pm\ \lambda\ x}$ or $\cosh\lambda\ x$, $\mathrm{sech}\lambda\ x$. As last two functions can be expanded in series of $e^{\pm\ \lambda\ x}$, both solution may be regarded as equivalent. Since the integration of exponential function is easier to exercise, we choose

$$\lambda_1(x) = \lambda,\ \ \lambda_2(x) = -\lambda. \tag{22}$$

So the leading order approximation $u_0(x)$ in the recursive schemes (16), (17) and (19) can be found as

$$u_0(x) = c_+\ e^{\lambda\ x} + c_-\ e^{-\lambda\ x}. \tag{23}$$

To obtain a non-trivial solution of Eq.(4) satisfying boundary conditions $u(-\infty) \to\ 0\ (u(\infty) \to\ 0)$, the choice $c_+ = 0\ (c_- = 0)$ in the leading term $u_0(x)$ in the recurrence relation (16) is desirable for $\lambda > 0$. So, to obtain solutions $u(x)$ satisfying the condition $u(-\infty) = 0$, we use (22) and (21) into (16) with $n \leq 2$ to get

$$u_0(x) = c_+\ e^{\sqrt{-\mu}x},\ \ u_1(x) = -\frac{\gamma(m+1)\left(c_+e^{\sqrt{-\mu}x}\right)^{2m-1} + 4\ \delta\ c_+\ m\ e^{\sqrt{-\mu}x}\left(c_+e^{\sqrt{-\mu}x}\right)^m}{4\mu(m-1)m(m+1)},$$

$$u_2(x) = \frac{\gamma^2(m+1)\left(c_+e^{\sqrt{-\mu}x}\right)^{2m} + 8\ \gamma\ \delta\ m\ \left(c_+e^{\sqrt{-\mu}x}\right)^{m+1} + 8\delta^2c_+^2me^{2\sqrt{-\mu}x}}{32c_+^{(3-2m)}\mu^2(m-1)^2m(m+1)e^{(3-2m)\sqrt{-\mu}x}}. \tag{24}$$

It is interesting observe that $u_n, n = 0, 1, 2$ mentioned above satisfy the three terms recurrence relation

$$\begin{aligned} u_{n+2}(x) \ \ = \ \ & -\frac{(mn+m-n)\left\{\gamma(m+1)\left(c_+e^{\sqrt{-\mu}x}\right)^{2m-2} + 4\delta m\left(c_+e^{\sqrt{-\mu}x}\right)^{m-1}\right\}}{4\mu(m-1)m(m+1)(n+2)}u_{n+1}(x) \\ & -\frac{\delta^2\left\{(m-1)n+2\right\}\left(c_+e^{\sqrt{-\mu}x}\right)^{2m-2}}{4\mu^2(m-1)(m+1)^2(n+2)}u_n(x). \end{aligned} \tag{25}$$

One can exploit this algebraic relation to obtain other terms $u_n(x)(n \geq 3)$ in (10) recursively and avoid cumbersome time-consuming process of multifold integration for their evaluation. The most important aspect of this scheme is that $u_n(x),\ n \geq 0$ can be obtained as the sequence of the function generated by

$$g(x,\epsilon)\,(=\textstyle\sum_{n=0}^{\infty}\epsilon^n u_n(x))$$

$$=\left[\frac{4\mu^2 m(m+1)^2\left(c_+e^{\sqrt{-\mu}x}\right)^{m+1}}{\epsilon\{\gamma\mu(m+1)^2+\delta^2 m\epsilon\}\left(c_+e^{\sqrt{-\mu}x}\right)^{2m}+4\delta\mu m(m+1)\epsilon\left(c_+e^{\sqrt{-\mu}x}\right)^{m+1}+4\mu^2 m(m+1)^2\left(c_+e^{\sqrt{-\mu}x}\right)^2}\right]^{\frac{1}{m-1}}$$

so that the accurate closed-form approximate or exact solution $u(x)$ (given by infinite sum in (10)) of equation (1) can be found in closed form as

$$u(x)=\textstyle\sum_{n=0}^{\infty}u_n(x)=g(x,1)$$

$$=\left[\frac{4\mu^2 m(m+1)^2\left(c_+e^{\sqrt{-\mu}x}\right)^{m+1}}{\{\gamma\mu(m+1)^2+\delta^2 m\}\left(c_+e^{\sqrt{-\mu}x}\right)^{2m}+4\delta\mu m(m+1)\left(c_+e^{\sqrt{-\mu}x}\right)^{m+1}+4\mu^2 m(m+1)^2\left(c_+e^{\sqrt{-\mu}x}\right)^2}\right]^{\frac{1}{m-1}}, \tag{26}$$

where c_+ is an arbitrary constant. Similarly, the solution satisfying boundary condition $u(\infty)=0$ can be found as

$$u(x)=\textstyle\sum_{n=0}^{\infty}u_n(x)$$

$$=\left[\frac{4\mu^2 m(m+1)^2\left(c_-e^{-\sqrt{-\mu}x}\right)^{m+1}}{\{\gamma\mu(m+1)^2+\delta^2 m\}\left(c_-e^{-\sqrt{-\mu}x}\right)^{2m}+4\delta\mu m(m+1)\left(c_-e^{-\sqrt{-\mu}x}\right)^{m+1}+4\mu^2 m(m+1)^2\left(c_-e^{-\sqrt{-\mu}x}\right)^2}\right]^{\frac{1}{m-1}}, \tag{27}$$

with c_- is another arbitrary constant.

It is important to mention here that solution in (26) vanishes at $-\infty$ (∞ in case of (27)) may be infinite at the other end $\infty(-\infty)$.

3.1.2. Recovery of solutions obtained by other methods

Here we introduce notations

$$A=\frac{c^{m-1}\left(\gamma\mu(m+1)^2+\delta^2 m\right)}{4\mu^2 m(m+1)^2},\ B=\frac{\delta}{\mu(m+1)},\ C=c^{1-m}, \tag{28}$$

$$\delta^L=\frac{m+1}{m}\sqrt{-m\,\mu\,\gamma},\ \zeta^M=\frac{-2m(m+1)c^{1-m}\mu}{\{\delta m+(m+1)\sqrt{-m\gamma\mu}\}},\ \zeta^P=\frac{-2m(m+1)c^{1-m}\mu}{\{\delta m-(m+1)\sqrt{-m\gamma\mu}\}} \tag{29}$$

and assume that m, c are real constants. Then solutions in (26) and (27) can be combined into

$$u(x)=e^{-\sqrt{-\mu}\,|x|}\left[\frac{1}{Ae^{-2(m-1)\sqrt{-\mu}\,|x|}+Be^{-(m-1)\sqrt{-\mu}\,|x|}+C}\right]^{\frac{1}{m-1}},\quad x\in(-\infty,\,\infty). \tag{30}$$

One of the important aspects of the RCAM is retaining an option to keep an arbitrary constant (c here) in the solution (30) which renders it more general in the sense that solutions of Eq.(1) obtained by different methods (Exp-function, Sech-/Tanh-methods, etc.) may be treated as the special case of the solution derived here. For illustration, solutions involving cosh-functions [37, 38] may be recovered from the solution in (30) (for $\mu\,\gamma<0,\ A_\pm\neq 0$) by the substitution of $A=\mathcal{R}\,\mathrm{e}^{\Delta}$, $C=\mathcal{R}\,\mathrm{e}^{-\Delta}$ followed by some algebraic rearrangement among them. Similarly, solutions involving tanh-functions of (1) [37, 38] can also be recovered from (30) whenever $A = 0$. Over and above, solutions involving trigonometric functions [37, 38] can be recovered from (30) in case of $\mu\,\gamma > 0$. These observations establish that RCAM in its present form generalizes different methods used to get solutions involving hyperbolic, trigonometric, exponential functions, etc. of DTEs. But the solution presented above may not be bounded for an arbitrary choice of the constant c for some assigned values of parameters μ, γ, δ involved in the model. So, the availability of restrictions on parameters and arbitrary constant for which the solution is bounded may be useful during the analysis of the physical processes described by Eq.(1) as their mathematical model.

3.1.3. Condition for boundedness

For convenience, we write

$$u(x) = \bar{u}(\zeta) = \zeta^{\frac{1}{m-1}} \left[\frac{1}{A\,\zeta^2 + B\,\zeta + C}\right]^{\frac{1}{m-1}} \tag{31}$$

where $\zeta = e^{\sqrt{-\mu}\,(m-1)\,x}$. Clearly, $\zeta \in \mathbb{R}^+$ for $\mu < 0$ and $\zeta \in \mathbb{C}$ for $\mu > 0$. So, we are considering these two cases separately.

Theorem 3.1. *For $\mu < 0$, the solution (31) is bounded in $\mathbb{R}$ if the exponent m (> 1), parameters μ, γ, δ involved in the equation and the arbitrary constant c present in the solution satisfy any one of the following conditions:*

Case	*Conditions on m, c, γ, δ*			
	m	c	γ	δ
I.	$\in (1,\infty)\backslash\mathbb{Z}$	$\in \mathbb{R}^+$	$\in \mathbb{R}^+$	$< -\delta^L$
II.	*odd integer*	$\in \mathbb{R}$	$\in \mathbb{R}^+\backslash\left\{-\frac{m\delta^2}{(m+1)^2\mu}\right\}$	$< -\delta^L$
III.	*even integer*	$\in \mathbb{R}^+$	$\in \mathbb{R}^+\backslash\left\{-\frac{m\delta^2}{(m+1)^2\mu}\right\}$	$< -\delta^L$
IV.	$\in (1,\infty)$	$\in \mathbb{R}^+$	$\in \mathbb{R}^-$	$\neq \pm\,\delta^L$
V.	$\in (1,\infty)\backslash\mathbb{Z}$	$\in \mathbb{R}^+$	$-\frac{m\delta^2}{(m+1)^2\mu}$	$\in \mathbb{R}^-$
VI.	*odd integer*	$\in \mathbb{R}$	$-\frac{m\delta^2}{(m+1)^2\mu}$	$\in \mathbb{R}^-$
VII.	*even integer*	$\in \mathbb{R}^+$	$-\frac{m\delta^2}{(m+1)^2\mu}$	$\in \mathbb{R}^-$

Proof. For $\bar{u}(\zeta)$ to be bounded as $\zeta \to \infty$, the exponent $\frac{1}{m-1}$ should be positive. This gives the restriction $m > 1$.

In case of $A \neq 0$, the boundedness of $\bar{u}(\zeta)$ desires $A\,\zeta^2 + B\,\zeta + C$ to maintain same sign (positive) for $\zeta \in \mathbb{R}^+$ leading to $A > 0$ and $C > 0$. This requirement suggests that roots ζ^M and ζ^P of the equation $A\,\zeta^2 + B\,\zeta + C = 0$ are not real positive.

We first consider the roots ζ^M and ζ^P are real i.e., $\mu\,\gamma < 0$ which gives $\gamma > 0$. Then $0 < (m+1)\sqrt{-m\,\mu\,\gamma} \in \mathbb{R}^+$ leads to $\delta\,m - (m+1)\,\sqrt{-m\,\mu\,\gamma} < \delta\,m + (m+1)\,\sqrt{-m\,\mu\,\gamma}$.

Now, if m (> 1) is real other than an integer then c must be in $\mathbb{R}^+$ for the solution $u(\zeta)$ to be real. This leads to

$$\frac{-2\,m\,(m+1)\,c^{1-m}}{\delta\,m - (m+1)\,\sqrt{-m\,\mu\,\gamma}} < \frac{-2\,m\,(m+1)\,c^{1-m}}{\delta\,m + (m+1)\,\sqrt{-m\,\mu\,\gamma}}.$$

From this relation and definitions of ζ^M and ζ^P introduced in (29) it appears that $\zeta^M < \zeta^P$ (since $\mu < 0$).

So, in case of m (> 1) is real other than integer and $\gamma > 0$, the condition for $\zeta^M,\ \zeta^P \in \mathbb{R}^-$ becomes $\zeta^M < \zeta^P < 0$. The explicit parameter dependence can be recast into

$$\frac{-2\,m\,(m+1)\,c^{1-m}}{\delta\,m-(m+1)\,\sqrt{-m\;\mu\;\gamma}} > 0 \text{ and } \frac{-2\,m\,(m+1)\,c^{1-m}}{\delta\,m+(m+1)\,\sqrt{-m\;\mu\;\gamma}} > 0,$$

which are equivalent to $\delta\,m+(m+1)\,\sqrt{-m\;\mu\;\gamma} < 0$. So, the first boundedness condition becomes

$$m\ (>1) \in (1,\ \infty)\backslash\mathbb{Z},\ c \in \mathbb{R}^+,\ \gamma > 0,\ \delta < -\delta^L.$$

We next consider m (> 1) is an odd integer and $c \in \mathbb{R}$. Then $c^{1-m} \in \mathbb{R}^+$. So, using the similar arguments we can derive the second boundedness condition as

$$m\ (>1) \text{ is odd integer},\ c \in \mathbb{R},\ \gamma > 0,\ \delta < -\delta^L.$$

Similarly, in case of m (> 1) is an even integer and $c \in \mathbb{R}^+$ third condition can be found as

$$m\ (>1) \text{ is even integer},\ c \in \mathbb{R}^+,\ \gamma > 0,\ \delta < -\delta^L.$$

From the definition of $\zeta^M,\ \zeta^P$ in (29) it appears that the condition for $\zeta^M,\ \zeta^P$ to be complex is $B^2-4\,A\,C < 0$ which leads to $\gamma < 0$.

When m (> 1) is real, the condition for the solution to be bounded becomes

$$c \in \mathbb{R}^+,\ \gamma < 0,\ \gamma\ (m+1)^2+\delta^2\ m \neq 0\ (A \neq 0).$$

In case of $A = 0$, the denominator of u in (31) is linear in ζ. Consequently, conditions on $\bar{u}(\zeta)$ to be bounded lead to the root $\left(\zeta_0 = -\frac{c^{1-m}\,\mu\,(m+1)}{\delta}\right)$ of $B\,\zeta+C = 0$ is not real positive. Furthermore, the positive definiteness of the polynomial $B\ \zeta+C$ demands that $B > 0$ and $C > 0$ for $\zeta \in \mathbb{R}^+$.

If m (> 1) is real other than an integer, then $c \in \mathbb{R}^+$. The condition for solution $\bar{u}(\zeta)$ to be bounded becomes $\mu\ \delta > 0$ i.e., $\delta < 0$.

When m (> 1) is an odd integer, $c \in \mathbb{R}$ then $c^{1-m} \in \mathbb{R}^+$ and the boundedness condition can be similarly obtained as $\delta < 0$.

If m (> 1) is an even integer and $c \in \mathbb{R}^+$ we get the similar condition. □

Theorem 3.2. *For $\mu > 0$, the solution (30) is bounded in $\mathbb{R}$ if m (> 1) is real, the parameters $\mu,\ \gamma,\ \delta$ and the constant c satisfy the conditions*

$$c = \left[\frac{4\ \mu^2\ m\ (m+1)^2}{\gamma\ \mu\ (m+1)^2+\delta^2\ m}\right]^{\frac{1}{2\ (m-1)}},\ -\frac{\delta^2\ m}{\mu\ (m+1)^2} < \gamma < 0,\ \delta > 0.$$

Proof. The solution in (30) can be written in the form

$$u(x) = \left[\frac{1}{A\ e^{(m-1)\sqrt{-\mu}x}+C\ e^{-(m-1)\sqrt{-\mu}x}+B}\right]^{\frac{1}{m-1}},$$

$$x \in (-\infty, \infty). \tag{32}$$

For the solution to be real we must have $m > 1$. To prove the theorem we make the following substitutions

$$A = \frac{D}{2}\, e^{i\,\Delta}, \quad C = \frac{D}{2}\, e^{-i\,\Delta} \tag{33}$$

where the amplitude D and the phase shift Δ are both real.

Use of these substitutions reduces (32) to

$$u(x) = \left[\frac{1}{D\ cos\left[(m-1)\sqrt{\mu}\ x + \Delta\right] + B}\right]^{\frac{1}{m-1}} \tag{34}$$

where the expressions for Δ and D can be obtained as

$$\Delta = cos^{-1}\left(\frac{A+C}{D}\right), \quad D = 2\sqrt{A\ C}. \tag{35}$$

Now, Δ will be well-defined if

$$-1 \leq \frac{A+C}{D} \leq 1$$

which can be put into the form (using (35))

$$(A-C)^2 \leq 0$$

which leads to

$$A = C.$$

Use of this relation into (28) gives the arbitrary constant c as

$$c = \left[\frac{4\ \mu^2\ m\ (m+1)^2}{\gamma\ \mu\ (m+1)^2 + \delta^2\ m}\right]^{\frac{1}{2\ (m-1)}}.$$

Using this value of c in (28) one gets

$$A = C = \sqrt{\frac{\gamma\ \mu\ (m+1)^2 + \delta^2\ m}{4\ \mu^2\ m\ (m+1)^2}}. \tag{36}$$

Use of (36) in (35) leads to

$$D = \sqrt{\frac{\gamma\ \mu\ (m+1)^2 + \delta^2\ m}{\mu^2\ m\ (m+1)^2}}. \tag{37}$$

Now, for D to be real, $\gamma\ \mu\ (m+1)^2 + \delta^2\ m > 0$, which leads to determine the lower limit of γ as

$$\gamma\ >\ -\frac{\delta^2\ m}{\mu\ (m+1)^2}. \tag{38}$$

Moreover, boundedness of $u(x)$ desires $D\ cos\left[(m-1)\sqrt{\mu}\ x + \Delta\right] + B$ to maintain same sign (positive) for $x\ \in\ \mathbb{R}$. This requirement suggests that

$$B\ >\ D, \tag{39}$$

which restricts δ to $\delta > 0$. Using (28) and (37) in (39) one can obtain the restriction on γ as

$$\gamma < 0. \tag{40}$$

Combination of (38) and (40) gives

$$-\frac{\delta^2\ m}{\mu\ (m+1)^2}\ <\ \gamma\ <\ 0.$$

□

3.2. Application of recursive scheme (19)

Recursion relations in (16) are not suitable to get an exact solution of Eq.(1) bounded in $\mathbb{R}$ and satisfy $u(x) \to 0$ as $x \to \pm\infty$, when $\mu = 0$. In this case, the scheme in (19) appears to be useful.

3.2.1. Solution

We will now apply this scheme to obtain an exact bounded solution of DTE (1) for $\mu = 0$

$$u''(x) = \delta\, u(x)^m + \gamma\, u(x)^{2m-1}, \tag{41}$$

satisfying conditions $u(x) \to 0$ as $x \to \pm\infty$ and $u(0) = U_0$. Use of the condition $u(0) = U_0$ and $u(x) \to 0$ as $x \to \infty$ in the first equation of the system (19) fixes $u_0(x) = U_0 (c_0 = U_0, c_1 = 0)$. As the condition at $x = 0$ has been accommodated in the leading term $u_0(x)$, it is attributed that $u_1(0) = 0$, which determines $d_0 = 0$. For the determination of the rest unknown parameter d_1 of the recursive scheme, it is assumed that $u(x) \approxeq u_0(x) + u_1(x)$ satisfies invariant condition $\left(\{u'(x)\}^2 = \frac{2\,\delta}{m+1}\, u(x)^{m+1} + \frac{\gamma}{m}\, u(x)^{2\,m}\right)$ involving the first primitive of the Eq.(41) at $x = 0$. This claim fixes

$$d_1 = \sqrt{\frac{U_0^m\,\{\gamma(m+1)U_0^m + 2U_0\delta m\}}{m(m+1)}}. \tag{42}$$

Using the values of arbitrary constants $c_i, d_i (i = 0, 1)$ in (19), one gets for $n \leq 2$,

$$u_0(x) = U_0,\ u_1(x) = \sqrt{\frac{U_0^m\,\{\gamma(m+1)U_0^m + 2U_0\delta m\}}{m(m+1)}}\ x,\quad u_2(x) = \frac{1}{2}\left(\gamma U_0^{2m-1} + U_0^m\delta\right) x^2. \tag{43}$$

As in the previous case ($\mu \neq 0$), $u_n(x) (n = 0, 1, 2)$ satisfy the three terms recurrence relation

$$\begin{aligned} u_n(x) \quad &= \quad \frac{m(n-1) - n + 2}{n}\sqrt{\frac{(m+1)\ \gamma\ U_0^{2m-2} + 2\ m\ \delta\ U_0^{m-1}}{m(m+1)}}\ x\ u_{n-1}(x) \\ &\quad -\frac{(m-1)\{m(n-2) - n + 4\}\ \delta}{2n\ (m+1)} U_0^{m-1} x^2\ u_{n-2}(x). \end{aligned} \tag{44}$$

One can exploit this algebraic relation to obtain other terms $u_n(x) (n \geq 3)$ recursively and get the generating function for the sequence $\{u_n(x), n \geq 0\}$ as

$$g(x,\epsilon) \quad = \frac{U_0}{\left[1+(m-1)\ \left\{-\sqrt{\frac{U_0^{m-2}\ \{(m+1)\ \gamma\ U_0^m + 2\ m\ \delta\ U_0\}}{m(m+1)}}\epsilon\ x + \frac{(m-1)\ \delta\ U_0^{m-1}}{2\ (m+1)}\ \epsilon^2\ x^2\right\}\right]^{\frac{1}{m-1}}}.$$

So, the accurate closed-form approximate or exact solution $u(x)$ of Eq.(41) satisfying condition $u(0) = U_0,\ u(\infty) \to 0$ can be found in closed form as

$$u(x) = \frac{U_0}{(1 - \Lambda_1\, x + \Lambda_2\, x^2)^{\frac{1}{m-1}}}, \quad x \in \mathbb{R}^+, \tag{45}$$

where

$$\Lambda_1 = \frac{(m-1)\sqrt{U_0^m\, \{(m+1)\,\gamma\, U_0^m + 2\, m\, U_0\, \delta\}}}{U_0\, \sqrt{m(m+1)}},\ \Lambda_2 = \frac{(m-1)^2\, \delta\, U_0^{m-1}}{2\,(m+1)}. \tag{46}$$

Similarly, the solution satisfying boundary condition $u(0) = U_0$, and $u(-\infty) = 0$ can be found as

$$u(x) \quad = \frac{U_0}{(1{+}\Lambda_1\ x\ +\ \Lambda_2\ x^2)^{\frac{1}{m-1}}}, \quad x \in \mathbb{R}^-. \tag{47}$$

In resemblance with the solution given in (26) and (27) (in case of ($\mu \neq 0$)), the solution in (45) which vanishes at ∞ (at $-\infty$ in case of (47)) may be infinite at the other end $-\infty(\infty)$. One can combine these two solutions to get non-trivial accurate closed-form approximate or exact solution $u(x)$ of Eq.(41) vanishing at $\pm\infty$ and continuous in $\mathbb{R}$ as

$$u(x) = \frac{U_0}{[\Lambda_2\, x^2 - \Lambda_1\, |x| + 1]^{\frac{1}{m-1}}}, \quad x \in \mathbb{R}. \tag{48}$$

Although this solution is continuous in $\mathbb{R}$, its first and second order derivatives may not be continuous at 0. This type of solution very often appears in the studies of nonlinear waves in the name of compacton, cuspion, etc whenever it is bounded in the whole region $\mathbb{R}$. However, the solution $u(x)$ mentioned above is not always bounded there for arbitrary values of the U_0, exponent m and parameters $\delta,\ \gamma$. So, knowledge on conditions on parameters for $u(x)$ to be bounded will be instructive for their applications in the studies of physical processes.

3.2.2. Condition for boundedness

Here conditions among parameters δ, γ, m involved in the equation (41) have been derived so that the solution in (48) is real, bounded in $\mathbb{R}$.

Theorem 3.3. *The solution (48) of Eq.(1) with $\mu = 0$ is real and bounded in $\mathbb{R}$ if parameters $m,\ \gamma,\ \delta$ involved in the equation and the arbitrary constant U_0 in the solution satisfy the conditions*
$U_0\ \in\ \left(0,\ U_0^{Max} = \{\frac{2\ \delta\ m}{(m+1)\ (-\gamma)}\}^{\frac{1}{m-1}}\right),\quad \gamma < 0,\ \delta \in \mathbb{R}^+$ *for* $m \in (1,\ \infty)$.

Proof. Condition $|u(x)| \to 0$ as $|x| \to \infty$ provides $m > 1$. Now, for the solution (48) to be bounded it requires that

(a) either roots of $\Lambda_2\ x^2 - \Lambda_1\ |x| + 1 = 0$ are complex or

(b) roots of $\Lambda_2\ x^2 - \Lambda_1\ x + 1 = 0$ are negative and roots of $\Lambda_2\ x^2 + \Lambda_1\ x + 1 = 0$ are positive.

In case of (a), $\Lambda_1^2 - 4\,\Lambda_2 < 0$. Use of explicit expressions for Λ_1 and Λ_2 given in (46) into this condition provides

$$\delta > 0 \quad \text{and} \quad (m+1)\ \gamma\ U_0^{2(m-1)} < 0.$$

Since $m > 1$ and $U_0 \in \mathbb{R}$ one gets, $\gamma < 0$.

Conditions in case of (b) lead to $\Lambda_1^2 - 4\ \Lambda_2 > 0$ and $\Lambda_1 + \sqrt{\Lambda_1^2 - 4\ \Lambda_2} < 0$, which are contradictory (since $\Lambda_1 > 0$). Next, $u(x) \in \mathbb{R}$ suggests that Λ_1 and $\Lambda_2 \in \mathbb{R}$. The condition $\Lambda_1 \in \mathbb{R}$ gives

$$U_0\ \in\ \left(0,\ U_0^{Max} = \{\frac{2\ \delta\ m}{(m+1)\ (-\gamma)}\}^{\frac{1}{m-1}}\right).$$

□

4. The solution of Liénard-type equation (2)

The Liénard-type second order ODE often appears as the mathematical model in several areas of physics. There is a vast literature on properties (first integrals, solutions etc. including their applications) of LTEs. In a sequence of papers, Lakshmanan and his collaborators investigated a variety of LTEs in the Lie group theoretic framework [14-19, 21]. On the other hand, Harko and his coworkers studied a class of LTEs with their reduction to Abel-type equations [22-24] and through deformations of some "creation-" and "annihilation-" like functions [25]. In their studies, Zheng and Shang [39] showed that the amplitude part of the solution in phase-amplitude format of the nonlinear Schrödinger equation with dual-power law nonlinearities satisfies a particular form of LTE(2) and obtained solutions in various forms with the aid of technique based on the first integral of the derived equation. However, none of these studies investigated dependence of boundedness of the solutions on the parameters involved in the equation. This aspect seems to be important for the experimentalists and simulators exercising with Eq.(2) as the mathematical model of the processes under investigations.

Results derived in the previous two subsections 3.1 and 3.2 and transformations presented in (3) have been used to obtain a solution of LTE(2)and to find conditions among parameters involved in the equation and arbitrary constants present in the solution for the solution to be bounded in $\mathbb{R}$.

4.1. Case $\mu_1 \neq 0$

Using the relations in (3) and expression for the solution $u(x)$ given in (30), the solution $v(x)$ of Eq.(2) can be found as

$$v(x) = e^{\sqrt{-\frac{\mu_1}{1+\alpha_1}}\ x}\ \left[\frac{1}{A'\ e^{2(p-1)\ \sqrt{-\frac{\mu_1}{1+\alpha_1}}\ x} + B'\ e^{(p-1)\ \sqrt{-\frac{\mu_1}{1+\alpha_1}}\ x} + C'}\right]^{\frac{1}{p-1}},\quad x \in \mathbb{R} \tag{49}$$

where

$$A' = \frac{(\alpha_1+1)\left[\gamma_1\ \mu_1\ (2\alpha_1+p+1)^2 + (\alpha_1+1)\ (\alpha_1+p)\ \delta_1^2\right]}{4\ \mu_1^2\ (\alpha_1+p)\ (2\alpha_1+p+1)^2}\ c^{\frac{p-1}{\alpha_1+1}},\ B' = \frac{(\alpha_1+1)\ \delta_1}{\mu_1\ (2\alpha_1+p+1)},\quad C' = c^{\frac{1-p}{\alpha_1+1}}. \tag{50}$$

The solution $v(x)$ mentioned above may not be bounded in $\mathbb{R}$ for arbitrary choices of parameters $p,\ \mu_1,\ \alpha_1,\ \gamma_1,\ \delta_1$. As in the case of Eq.(1), two different cases have been considered for $\mu_1\ (1+\alpha_1) < 0$ and $\mu_1\ (1+\alpha_1) > 0$ separately. Here, we skip the proofs of the theorems since the principles for their derivations are same as in case of Theorem 3.1 and Theorem 3.2 of the previous section.

Theorem 4.1. *For $\mu_1\ (1+\alpha_1) < 0$, the solution (49) is bounded in $\mathbb{R}$, if the exponent p, the parameters $\mu_1, \alpha_1, \gamma_1, \delta_1$ involved in the equation and the arbitrary constant c present in the solution satisfy any one of the following conditions.*

Case	*Conditions on $p,\ c,\ \alpha_1,\ \gamma_1,\ \delta_1$*					
	$\frac{1-p}{1+\alpha_1}$	c	μ_1	α_1	γ_1	δ_1
I.	$\in \mathbb{R}^+$	$\in \mathbb{R}^+$	$\in \mathbb{R}^+$	$< -p$	$\in \mathbb{R}^+$	$\neq \delta'^{\,L}$
II.	$\in \mathbb{R}^+$	$\in \mathbb{R}^+$	$\in \mathbb{R}^+$	$\in (-p,\ -1)$	$\in \mathbb{R}^-$	$\neq \delta'^{\,L}$
III.	$\in \mathbb{R}^+$	$\in \mathbb{R}^+$	$\in \mathbb{R}^-$	$\in (-1,\ \infty)$	$\in \mathbb{R}^-$	$\neq \delta'^{\,L}$
IV.	$\in \mathbb{R}^+ \setminus \mathbb{Z}$	$\in \mathbb{R}^+$	$\in \mathbb{R}^-$	$\in (-1,\ \infty)$	$\in \mathbb{R}^+$	$< -\delta'^{\,L}$
V.	*Even integer*	$\in \mathbb{R}$	$\in \mathbb{R}^-$	$\in (-1,\ \infty)$	$\in \mathbb{R}^+$	$< -\delta'^{\,L}$
VI.	*Odd integer*	$\in \mathbb{R}^+$	$\in \mathbb{R}^-$	$\in (-1,\ \infty)$	$\in \mathbb{R}^+$	$< -\delta'^{\,L}$
VII.	$\in \mathbb{R}^+ \setminus \mathbb{Z}$	$\in \mathbb{R}^+$	$\in \mathbb{R}^+$	$\in (-\frac{p+1}{2},\ -1)$	$\in \mathbb{R}^+$	$< \delta'^{\,L}$
VIII.	*Even integer*	$\in \mathbb{R}$	$\in \mathbb{R}^+$	$\in (-\frac{p+1}{2},\ -1)$	$\in \mathbb{R}^+$	$< \delta'^{\,L}$
IX.	*Odd integer*	$\in \mathbb{R}^+$	$\in \mathbb{R}^+$	$\in (-\frac{p+1}{2},\ -1)$	$\in \mathbb{R}^+$	$< \delta'^{\,L}$
X.	$\in \mathbb{R}^+ \setminus \mathbb{Z}$	$\in \mathbb{R}^+$	$\in \mathbb{R}^+$	$\in (-p,\ -\frac{p+1}{2})$	$\in \mathbb{R}^+$	$> \delta'^{\,L}$
XI.	*Even integer*	$\in \mathbb{R}$	$\in \mathbb{R}^+$	$\in (-p,\ -\frac{p+1}{2})$	$\in \mathbb{R}^+$	$> \delta'^{\,L}$
XII.	*Odd integer*	$\in \mathbb{R}^+$	$\in \mathbb{R}^+$	$\in (-p,\ -\frac{p+1}{2})$	$\in \mathbb{R}^+$	$> \delta'^{\,L}$
XIII.	$\in \mathbb{R}^+ \setminus \mathbb{Z}$	$\in \mathbb{R}^+$	$\in \mathbb{R}^+$	$< -p$	$\in \mathbb{R}^-$	$> -\delta'^{\,L}$
XIV.	*Even integer*	$\in \mathbb{R}$	$\in \mathbb{R}^+$	$< -p$	$\in \mathbb{R}^-$	$> -\delta'^{\,L}$
XV.	*Odd integer*	$\in \mathbb{R}^+$	$\in \mathbb{R}^+$	$< -p$	$\in \mathbb{R}^-$	$> -\delta'^{\,L}$
XVI.	$\in \mathbb{R}^+ \setminus \mathbb{Z}$	$\in \mathbb{R}^+$	$\in \mathbb{R}^-$	> -1	$-\frac{(1+\alpha_1)\,(p+\alpha_1)\,\delta_1^2}{\mu_1\,(1+p+2\,\alpha_1)^2}$	$\in \mathbb{R}^-$
XVII.	*Even integer*	$\in \mathbb{R}$	$\in \mathbb{R}^-$	> -1	$-\frac{(1+\alpha_1)\,(p+\alpha_1)\,\delta_1^2}{\mu_1\,(1+p+2\,\alpha_1)^2}$	$\in \mathbb{R}^-$
XVIII.	*Odd integer*	$\in \mathbb{R}^+$	$\in \mathbb{R}^-$	> -1	$-\frac{(1+\alpha_1)\,(p+\alpha_1)\,\delta_1^2}{\mu_1\,(1+p+2\,\alpha_1)^2}$	$\in \mathbb{R}^-$
XIX.	$\in \mathbb{R}^+ \setminus \mathbb{Z}$	$\in \mathbb{R}^+$	$\in \mathbb{R}^+$	$\in (-\frac{p+1}{2},\ -1)$	$-\frac{(1+\alpha_1)\,(p+\alpha_1)\,\delta_1^2}{\mu_1\,(1+p+2\,\alpha_1)^2}$	$\in \mathbb{R}^-$
XX.	*Even integer*	$\in \mathbb{R}$	$\in \mathbb{R}^+$	$\in (-\frac{p+1}{2},\ -1)$	$-\frac{(1+\alpha_1)\,(p+\alpha_1)\,\delta_1^2}{\mu_1\,(1+p+2\,\alpha_1)^2}$	$\in \mathbb{R}^-$
XXI.	*Odd integer*	$\in \mathbb{R}^+$	$\in \mathbb{R}^+$	$\in (-\frac{p+1}{2},\ -1)$	$-\frac{(1+\alpha_1)\,(p+\alpha_1)\,\delta_1^2}{\mu_1\,(1+p+2\,\alpha_1)^2}$	$\in \mathbb{R}^-$
XXII.	$\in \mathbb{R}^+ \setminus \mathbb{Z}$	$\in \mathbb{R}^+$	$\in \mathbb{R}^+$	$< -\frac{p+1}{2}$	$-\frac{(1+\alpha_1)\,(p+\alpha_1)\,\delta_1^2}{\mu_1\,(1+p+2\,\alpha_1)^2}$	$\in \mathbb{R}^+$
XXIII.	*Even integers*	$\in \mathbb{R}$	$\in \mathbb{R}^+$	$< -\frac{p+1}{2}$	$-\frac{(1+\alpha_1)\,(p+\alpha_1)\,\delta_1^2}{\mu_1\,(1+p+2\,\alpha_1)^2}$	$\in \mathbb{R}^+$
XXIV.	*Odd integers*	$\in \mathbb{R}^+$	$\in \mathbb{R}^+$	$< -\frac{p+1}{2}$	$-\frac{(1+\alpha_1)\,(p+\alpha_1)\,\delta_1^2}{\mu_1\,(1+p+2\,\alpha_1)^2}$	$\in \mathbb{R}^+$

Here the symbol $\delta'^{\,L} = \frac{1+p+2\,\alpha_1}{(p+\alpha_1)(1+\alpha_1)}\sqrt{-(p+\alpha_1)(1+\alpha_1)\,\mu_1\,\gamma_1}\ \ (>0)$.

Theorem 4.2. *The solution (49) is bounded in* $\mathbb{R}$ *if* $p\ (>1)$ *is real, the parameters* $\alpha_1,\ \mu_1,\ \gamma_1,\ \delta_1$ *and constant* $c = c^L$ *satisfy any one of the following conditions:*

Case	Conditions on $\alpha_1,\ \mu_1,\ \gamma_1,\ \delta_1$			
	α_1	μ_1	γ_1	δ_1
I	$\in (-1,\ \infty)$	$\in \mathbb{R}^+$	$\in \left(\gamma_1^L,\ 0\right)$	$\in \mathbb{R}^+$
II	$\in (-\infty,\ -p)$	$\in \mathbb{R}^-$	$\in \left(0, \gamma_1^L\right)$	$\in \mathbb{R}^-$
III	$\in \left(-\frac{p+1}{2},\ -1\right)$	$\in \mathbb{R}^-$	$\in \left(\gamma_1^L,\ 0\right)$	$\in \mathbb{R}^+$
IV	$\in \left(-p,\ -\frac{p+1}{2}\right)$	$\in \mathbb{R}^-$	$\in \left(\gamma_1^L,\ 0\right)$	$\in \mathbb{R}^-$

where $\gamma_1^L = -\frac{(\alpha_1+1)(\alpha_1+p)\ \delta_1^2}{(2\alpha_1+p+1)^2\ \mu_1}$, *and* $c^L = \left[\frac{4\ \mu_1^2\ (\alpha_1+p)\ (2\alpha_1+p+1)^2}{(\alpha_1+1)\{\gamma_1\ \mu_1\ (2\alpha_1+p+1)^2+(\alpha_1+1)\ (\alpha_1+p)\ \delta_1^2\}}\right]^{\frac{\alpha_1+1}{2(p-1)}}$.

4.2. Case $\mu_1 = 0$

To obtain the solution $v(x)$ of Eq.(2) satisfying $v(0) = V_0,\ v(x) \to 0$ as $x \to \pm\infty$ for $\mu_1 = 0$, we use $u(x)$ given in (48) in the transformations prescribed in (3) to get

$$v(x) = \frac{V_0^{\frac{1}{1+\alpha_1}}}{\left[\Lambda_2'\ x^2 - \Lambda_1'\ |x| + 1\right]^{\frac{1}{p-1}}},\quad x \in \mathbb{R}, \tag{51}$$

where

$$\Lambda_1' = \frac{(p-1)\sqrt{V_0^{\frac{\alpha_1+p}{\alpha_1+1}}\{\gamma_1\ (2\ \alpha_1+p+1)\ V_0^{\frac{\alpha_1+p}{\alpha_1+1}} + 2\ V_0\ \delta_1\ (\alpha_1+p)\}}}{V_0\sqrt{(p+\alpha_1)\ (2\ \alpha_1+p+1)}},\quad \Lambda_2' = \frac{\delta_1\ (p-1)^2\ V_0^{\frac{p-1}{\alpha_1+1}}}{2\ (2\alpha_1+p+1)}. \tag{52}$$

Theorem 4.3. *The solution (51) of LTE (2) (for* $\mu_1 = 0$*) is bounded in* $\mathbb{R}$ *if the parameters* $p\ (>1),\ \alpha_1,\ \gamma_1,\ \delta_1$ *involved in the equation and arbitrary constant* V_0 *present in the solution satisfy one of the following conditions:*

Case	*Conditions on* $p,\ \alpha_1,\ \gamma_1,\ \delta_1,\ V_0$				
	p	α_1	γ_1	δ_1	V_0
I	$\in (1,\ \infty)$	$\in \left(-\frac{p+1}{2},\ \infty\right)$	$\in \mathbb{R}^-$	$\in \mathbb{R}^+$	$< V_0^L$
II	$\in (1,\ \infty)$	$\in (-\infty,\ -p)$	$\in \mathbb{R}^+$	$\in \mathbb{R}^-$	$> V_0^L$
III	$\in (1,\ \infty)$	$\in \left(-p,\ -\frac{p+1}{2}\right)$	$\in \mathbb{R}^-$	$\in \mathbb{R}^-$	$> V_0^L$

where $V_0^L = \left[-\frac{2\ \delta_1\ (\alpha_1+p)}{\gamma_1\ (2\alpha_1+p+1)}\right]^{\frac{\alpha_1+1}{p-1}}$.

5. Conclusion

The objective of this paper is to present an efficient scheme through some modifications on ADM to obtain an accurate closed-form approximate or exact solutions of DTE(1) and LTE(2). For this purpose, we have considered nonlinear ODE involving linear part with variable coefficients and derived four recursive schemes among which one is conventional ADM, and the rest three appear to be new. One of the advantages of the new scheme is that one can speculate the recurrence relations among the elements of a sequence of functions $\{u_n(x), n = 0, 1, \cdots\}$ from first few elements ($n \leq 3$). Availability of such relation helps one

(i) to obtain other terms ($n > 3$) of the sequence algebraically (leads to avoid the multi-fold integration), and

(ii) to find generating function ($g(x, \epsilon)$ in sections 3.1.1 and 3.2.1) of a sequence $\{u_n(x), n = 0, 1, \cdots\}$ derived in step (i) which provides the accurate closed-form approximate or exact solution, $u(x)(\equiv g(x, 1))$, in a straightforward way.

Two new recursive schemes have been applied here to obtain two accurate closed-form approximate or exact solutions for Eq.(1) and Eq.(2) each. Two of them (in (30) and (49)) are more general. This is substantiated by the recovery of several solutions of these equations by different methods. The rest two (in (48) and (51)) seems to be new. Finally, the conditions among parameters involved in Eq.(1) and Eq.(2) have been derived separately for getting their solutions derived here to be bounded. These information may be useful for the experimentalists investigating physical processes which are mathematically described by ODEs of the form (1) or (2) or the partial differential equations reducible to these equations e.g. Benjamin-Bona-Mahony(BBM)-, nonlinear Klein-Gordon-, Zakharov-, Schrödinger equations etc. Works in these directions are in progress and will be reported elsewhere.

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