Symmetry problem 1

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Abstract

A symmetry problem is solved. A new method is used. The idea of this method is to reduce to a contradiction the PDE and the over-determined boundary data on the boundary.

The new method allows one to solve other symmetry problems.

1 Introduction

Symmetry problems for PDE were studied in many publications by many authors, see, for example, [1]. In this paper a new method is given for a study of symmetry problems for PDE. Throughout we assume that $D$ is a bounded connected $C^2$—smooth domain in $\mathbb{R}^3$, $S$ is the boundary of $D$, $N$ is the unit normal to $S$, pointing out of $D$, $u_N$ is the normal derivative of $u$ on $S$, $D' = \mathbb{R}^3 \setminus D$, $S^2$ is the unit sphere in $\mathbb{R}^3$, $J_n(r)$ is the Bessel function regular at $r = 0$, $j_0(r)$ is the spherical Bessel function, $j_0'(kr) = \frac{dj_0(kr)}{dr}$, $k > 0$ is a constant, $\beta \cdot y = (\beta, y)$ is the dot product.

In [2]–[10] the author studied various symmetry problems.

Let us formulate the symmetry problem studied in this paper. Our main result is formulated in Theorem 1.

**Theorem 1.** Assume that

$$\Delta u + k^2 u = 0 \quad \text{in} \quad D, \quad u|_S = 1, \quad u_N = 0. \quad (1)$$

Then $S$ is a sphere of radius $a$ where $a$ solves the equation $j_0'(ka) = 0$.

In [5], [7] it was shown that the Pompeiu problem is equivalent to the problem (1).

In Section 2 proofs are given.

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2 Proofs

Proof of Theorem 1. Let \( g(x, y, k) := \frac{e^{ik|x-y|}}{4\pi|x-y|} \). If problem (1) has a solution then this solution is unique by the uniqueness of the solution to the Cauchy problem for elliptic equation (1). The solution to equation (1) by Green’s formula is:

\[
  u(x) = - \int_S g_N(x, t) dt, \quad x \in D; \quad u(x) = - \int_S g_N(x, t) dt = 0, \quad x \in D'.
\]

Let \( B_R = \{x : |x| \leq R\}, \ D \subset B_R \). If \( D \) is a ball \( B_a \) of radius \( a \) and \( j_0'(ka) = 0 \), then problem (1) in \( B_a \) has a solution:

\[
  u = \frac{j_0(kr)}{j_0(ka)}, \quad r = |x|.
\]

In what follows we assume that \( D \subset \mathbb{R}^2 \) and \( S \) is a closed smooth curve. Let \( r(s) = x(s)e_1 + y(s)e_2 \) be a parametric representation of \( S \), \( s \) be the arc length along \( S \) and also the corresponding to the arc length \( s \) point on \( S \), \( \{e_1, e_2\} \) is a Cartesian basis in \( \mathbb{R}^2 \). The first boundary condition in (1) is \( u(x(s), y(s)) = 1 \). Differentiating with respect to \( s \) one gets \( u_x \dot{x} + u_y \dot{y} = 0 \) and another differentiation yields

\[
  u_{xx} \dot{x}^2 + 2u_{xy} \dot{x} \dot{y} + u_{yy} \dot{y}^2 = 0, \quad \dot{x} = \frac{dx}{ds}.
\]

Here we used the formula \( u_x \dot{x} + u_y \dot{y} = 0 \). This formula can be derived as follows: \( \nabla u \cdot \mathbf{t} = \nabla u \cdot \mathbf{N} \), where \( \kappa = \kappa(s) > 0 \) is the curvature of \( S \), \( \mathbf{N} = -N \) is the unit normal pointing into \( D \), \( \nabla u \cdot \mathbf{N} = -u_N = 0 \) on \( S \). From (1) it follows that

\[
  u_{xx} + u_{yy} = -k^2 \quad \text{on} \quad S.
\]

Let us prove that (4) and (5) are not compatible at some points, except when \( S \) is a circle of radius \( a \), where \( a \) solves the equation \( j_0'(ka) = 0 \).

Denote \( u_{xx} = p = p(s) \), \( u_{xy} = q = q(s) \). Then (5) implies \( u_{yy} = -k^2 - p \) on \( S \). Let \( A \) be a \( 2 \times 2 \) matrix with elements \( A_{11} = p \), \( A_{12} = A_{21} = q \), \( A_{22} = -k^2 - p \). The equation for finding the eigenvalues \( \lambda_{1,2} \) of \( A \) is:

\[
  \lambda^2 + k^2 \lambda - p^2 - q^2 - k^2 p = 0.
\]

The eigenvalues \( \lambda_1 \) and \( \lambda_2 \) are:

\[
  \lambda_{1,2} = -\frac{k^2}{2} \pm (\frac{k^4}{4} + p^2 + q^2 + k^2 p)^{1/2}.
\]

Clearly, \( \lambda_1 + \lambda_2 = -k^2 \), \( \lambda_1 \lambda_2 = -p^2 - q^2 - k^2 p - p^2 - q^2 + q^2 = (\frac{k^2}{2} + p)^2 + q^2 \geq 0 \). Thus, \( \lambda_2 < 0 \).
The corresponding eigenvectors (non-normalized but orthogonal) can be calculated explicitly. One has
\[ e_1 = \{1, \gamma\}, \quad \gamma := \frac{q}{k^2 + p + \lambda_1} = \frac{\lambda_1 - p}{q}. \] (8)

If \( q \neq 0 \), then
\[ e_2 = \left\{ \frac{k^2 + p + \lambda_2}{q}, 1 \right\} = \{-\gamma, 1\}. \] (9)

If \( q \neq 0 \) then one checks that \( \frac{k^2 + p + \lambda_2}{q} = \frac{q}{\lambda_2 - p} \) and \( \frac{q}{\lambda_2 - p} + \frac{\lambda_2}{k^2 + p + \lambda_1} = 0 \), so \( \gamma = -\frac{k^2 + p + \lambda_2}{q} \).

Clearly, \( e_1 \cdot e_2 = 0, \quad \|e_1\|^2 = \|e_2\|^2 = 1 + \gamma^2 \), so \( \gamma^2 \) is invariant under rotations of the Cartesian coordinate system.

Denote \( \{\dot{x}, \dot{y}\} := w \). Note that \( \dot{x}^2 + \dot{y}^2 = 1 \). Let \( c_1, c_2 \) be scalar coefficients. Then
\[ c_1 e_1 + c_2 e_2 = w, \quad w := \{\dot{x}, \dot{y}\}. \] (10)

Solving explicitly this algebraic system for \( c_1 \) and \( c_2 \) one gets:
\[ c_1 = (\dot{x} + \gamma \dot{y}) \Delta^{-1}, \quad \Delta = 1 + \gamma^2, \] (11)
and
\[ c_2 = (\dot{y} - \gamma \dot{x}) \Delta^{-1}. \] (12)

Equation (4) can be written as \((Aw, w) = 0\). Substitute \( w \) from (10) into the equation \((Aw, w) = 0\) and use the orthogonality of \( e_1 \) and \( e_2 \) to get
\[ (\dot{y} - \gamma \dot{x})^2 \lambda_2 + (\dot{x} + \gamma \dot{y})^2 \lambda_1 = 0. \] (13)

We now prove that (13) leads to a contradiction unless \( S \) is a circle of radius \( a \) where \( a \) solves the equation \( J'_0(ka) = 0 \) if \( D \subset \mathbb{R}^2 \) and \( a \) solves the equation \( j'_0(ka) = 0 \) if \( D \subset \mathbb{R}^3 \).

Choose Cartesian coordinates in which \( \dot{x}(s) = -\gamma \dot{y} \). Such coordinate system does exist because the only restriction on \( \dot{x} \) and \( \dot{y} \) is \( \dot{x}^2 + \dot{y}^2 = 1 \) at all \( s \in S \). Then, since \( \lambda_2 < 0 \) equation (13) with \( \dot{x}(s) = -\gamma \dot{y} \) implies \( \dot{y}(1 + \gamma^2) = 0 \). Thus, \( \dot{y} = 0 \). Therefore, \( \dot{x} = \dot{y} = 0 \). This contradicts the relation \( \dot{x}^2 + \dot{y}^2 = 1 \). This contradiction holds for any smooth \( S \) except for a circle of a special radius, see (3).

Theorem 1 is proved. \( \square \)
References


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