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Numerical and analytic method for solvingproposal New Type for fuzzy nonlinear volterra integral equation

Sameer Qasim Hasan
Department of Mathematics, College of Education, University of Al-Mustansiriyah, Baghdad Iraq
dr.sameer_Kkasim@yahoo.com
Alan jalal Abdulqader
Department of Mathematics, College of Education, University of Al-Mustansiriyah, Baghdad Iraq
alanjalal515@yahoo.com

ABSTRACT

In this paper, we proved the existence and uniqueness and convergence of the solution of new type for nonlinear fuzzy volterra integral equation . The homotopy analysis method are proposed to solve the new type fuzzy nonlinear Volterra integral equation . We convert a fuzzy volterra integral equation for new type of kernel for integral equation, to a system of crisp function nonlinear volterra integral equation . We use the homotopy analysis method to find the approximate solution of the system and hence obtain an approximation for fuzzy solution of the nonlinear fuzzy volterra integral equation . Some numerical examples is given and results reveal that homotopy analysis method is very effective and compared with the exact solution and calculate the absolute error between the exact and AHM .Finally using the MAPLE program to solve our problem .

Keywords: Fuzzy Number; Volterra nonlinear Integral equation; fuzzy integral; Homotopy analysis method

1. Introduction

The solutions of integral equations have a major role in the field of science and engineering. Since few of these equations can be solved explicitly, it is often necessary to resort to numerical techniques which are appropriate combinations of numerical integration and interpolation [7, 12]. There are several numerical methods for solving linear Volterra integral equation [11, 23] and system of nonlinear Volterra integral equations [3]. Borzabadi and Fard in [5] obtained a numerical solution of nonlinear Fredholm integral equations of the second kind.The concept of fuzzy numbers and fuzzy arithmetic operations were first introduced by Zadeh [14],Dubois and Prade [21]. We refer the reader to [10] for more information on fuzzy numbers and fuzzy arithmetic.The topics of fuzzy integral [20] and fuzzy integral equations (FIE) which growing interest for some time, inparticular in relation to fuzzy control, have been rapidly developed in recent years. The fuzzy mapping functionwas introduced by Chang and Zadeh [14]. Later, Dubois and Prade [8] presented an elementary fuzzy calculusbased on the exten- sion principle also the concept of integration of fuzzy functions was first introduced byDubois and Prade [21]. Babolian et al. and Abbasbandy et al. in [10,11] obtained a numerical solution of linearFredholm fuzzy integral equations of the second kind, while finding an approximate solution for the fuzzynonlinear kinds.is more difficult and a numerical method in this case can be found in [4]In this paper, we present a novel and very simply numerical method (Homotopy Analysis method) for solving fuzzy nonlinear volterra integral equation .

.2.Basic concepts

Basic definitions of fuzzy number are given in [1,2,10,15,17,20] as follows:

Definition 2.1. Fuzzy number. A fuzzy number is a map $\mathbf{u}: R \to [a, b]$, which satisfying

(1) u is upper semi- continuous function,

(2) u(x) = 0 outside some interval $[a, d]$

(3) There are real numbers b,c such $a \le b \le c \le d$

i) u(x)is a monotonic increasing function on $[a, b]$

ii)u(x) is a monotonic decreasing function on $[c, d]$

iii) u(x) = 1 for all $x \in [b, c]$

The set of all fuzzy numbers (as given by Definition 13) is denoted by E^1 and is a convex cone. An alternative definition for parameter from of a fuzzy number is given by Kaleva [14].

Definition 2.2 . A fuzzy number $\breve{u}$ in parametric form is a pair $(\underline{u}, \overline{u})$of function $\underline{u}(\alpha), \overline{u}(\alpha)$, $0 \le \alpha \le 1$, which satisfies the following requiremenst:

i) $\underline{u}(\alpha)$ is a bounded left continuous non- decreasing function over [0, 1]

ii)$\overline{u}(\alpha)$ is a bounded left continuous non- increasing function over [0, 1]

iii) $\underline{u}(\alpha) \le \overline{u}(\alpha)$, $0 \le \alpha \le 1$

Definition 2.3. .For arbitrary fuzzy $u = (\underline{u}(\alpha),\overline{u}(\alpha))$,$v = (\underline{v}(\alpha),\overline{v}(\alpha))$, , $0 \le \alpha \le 1$ and scalar k, we define addition, subtraction, scalar product by k and multiplication are respectively as following:

$1 -$ addition : $\underline{(u+v)}(\alpha) = (\underline{u}(\alpha)+\underline{v}(\alpha))$, $\overline{(u+v)}(\alpha) = (\overline{u}(\alpha)+\overline{v}(\alpha))$,

$2 -$ subtraction : $\underline{(u-v)}(\alpha) = (\underline{u}(\alpha)-\underline{v}(\alpha))$, $\overline{(u-v)}(\alpha) = (\overline{u}(\alpha)-\overline{v}(\alpha))$,

$3 -$ scalar product :

$$k\breve{u}=\begin{cases}\left(k\underline{u}(\alpha),k\overline{u}(\alpha)\right), & k\ge 0\\ \left(k\underline{u}(\alpha),k\overline{u}(\alpha)\right), & k<0\end{cases} \quad (1)$$

Defined 2.4. For arbitrary Fuzzy numbers $\tilde{u},\tilde{v} \in E^1$

$$D(\tilde{u},\tilde{v}) = \max\{\sup_{\alpha\in[0,1]}\left|\underline{u}(\alpha)-\underline{v}(\alpha)\right|, \sup_{\alpha\in[0,1]}|\overline{u}(\alpha)-\overline{v}(\alpha)|\}, \quad (2)$$

In the distance between The $\tilde{u}$ and $\tilde{v}$, it is prove (E^1, D) is a complete metric space .

Definition 2.5. The integral of a fuzzy function was define in [14] by using the Riemann integral concept . Let $f:[a,b]\to E^1$. For Fuzzy function, for each partition $p=\{t_0, \dots, t_n\}$ of $[a,b]$ and for arbitrary $\xi_i \in [t_{i-1},t_i]$, $1\le i\le n$, suppose

$R_P=\sum_{i=1}^{n} f(\xi_i)(t_i - t_{i-1})$ (3)

$$\Delta := \max\{|t_i - t_{i-1}|, 1\le i\le n\}.$$

The define integral of $f(t)$ over $[a,b]$ is

$$\int_a^b f(t)dt = \lim_{\Delta\to 0} R_P , \quad (4)$$

If the fuzzy function $f(t)$ is continuous in metric D,its definite the integral exists and also

$(\underline{\int_a^b f(t;\alpha)dt}) = \int_a^b \underline{f}(t;\alpha)dt$, $(\overline{\int_a^b f(t;\alpha)dt}) = \int_a^b \overline{f}(t;\alpha)dt$ (5)

It should be noted that the fuzzy integral can be also defined using the Lebesgue – type approach. However, if $f(t)$ is continuous, both approaches yield the same value .More details about the properties of the fuzzy integral

Proposition 2.1. A function $F,G: I \to E^n$ be integrable and $\varphi \in R. \in$ then

1-$\int(F+G) = \int F + \int G$

2- $\int \varphi F = \varphi \int F$

3- $D(F,G)$IS integrable

$$4 - D\left(\int F, \int G\right) \le \int D(F,G)$$

Proposition 2.2. For any $p,q,r,s \in E^n$ and $\varphi \in R$, then the following hold

i-$(E^n, D)$is a complete metric space

ii-$D(\varphi p, \varphi q) = |\varphi| D(p,q)$

iii-$D(p+r, q+s) = D(p,q)$

iv-$D(p+q, r+s) \le D(p,r) + D(q,s)$

Definition2.6. A function $F: I \to E^n$ is called bounded if there exists a constant $M > 0$ such that $D(F(x),\tilde{0}) \le M$ for all $x \in$

Definition 2.7 . A function $F: I \to E^n$ is said to be continuous if for arbitrary fixed $x_0 \in I$ and $\varepsilon > 0$ there exists $\delta > 0$ such that if $|x - x_0| < \delta$, than $D(F(x), F(x_0)) < \varepsilon$ for each $x \in I$

3.Novel formula fuzzy nonlinear volterra integral equation

The fuzzy nonlinear integral equation with integral kernel which is discussed in this work is the fuzzy nonlinear Volterra integral equation of the second kind (FNVIE-2) as follows:

$$\tilde{u}(x) = \tilde{f}(x) + \lambda \int_a^x k((x,t,\tilde{F}_1(x,t,\tilde{u}(t)), G(t, \int_a^t \tilde{F}_2(t,s,\tilde{u}(s))ds))dt \quad (6)$$

where $\lambda \geq 0$,$\tilde{f}(x)$is a fuzzy function of x : a ≤ x ≤ b, $k((x,t,\tilde{F}_1(x,t,\tilde{u}(t))),\tilde{G}(t,\int_a^t \tilde{F}_2(t,s,\tilde{u}(s))ds))$ isanalytic functionson $[a,b] \times [a,b] \times [a,b] \times E^n \times E^n$and $\tilde{F}_1(x,t,\tilde{u}(t))$, $\tilde{F}_2(t,s,\tilde{u}(s))$ are nonlinear function on [a,b]. For solving in parametric form of Eq. (6), consider $(\underline{f}(x,\alpha),\bar{f}(x,\alpha))$ and $(\underline{u}(x,\alpha),\overline{u}(x,\alpha))$, 0 ≤ α ≤ 1 and t ,s ∈ [a,b] are parametric form of $\tilde{f}(x)$and$\tilde{u}(x)$, respectively. then, parametric form of Eq. (6) is as follows

$$\underline{u}(x,\alpha) = \underline{f}(x,\alpha) + \lambda \int_a^x \underline{k((x,t,F_{1\alpha}(x,t,u(t,\alpha)),G(t,\int_a^t F_{2\alpha}(t,s,u(s,\alpha))ds))}dt \quad (7)$$

$$\overline{u}(x,\alpha) = \bar{f}(x,\alpha) + \lambda \int_a^x \overline{k((x,t,F_{1\alpha}(x,t,u(t,\alpha)),G(t,\int_a^t F_{2\alpha}(t,s,u(s,\alpha))\,ds))}dt$$

let $x,t,s \in [a,b]$

$$\underline{k((x,t,F_{1\alpha}(x,t,u(t,\alpha)),G(t,\int_a^t F_{2\alpha}(t,s,u(s,\alpha))ds)} = \begin{cases} k(x,t,F_{1\alpha}\left(t,\underline{u}(t,\alpha)\right), k(x,t) \geq 0 \\ k(x,t,G(t,\int_a^t F_{2\alpha}(t,s,\overline{u}(s,\alpha))ds)) \ , k(x,t) < 0 \end{cases}$$

$$\overline{k((x,t,F_{1\alpha}(x,t,u(t,\alpha)),G(t,\int_a^t F_{2\alpha}(t,s,u(s,\alpha))} = \begin{cases} k(x,t,G\left(t,\int_a^t \overline{F}_{2\alpha}(t,s,\overline{u}(s,\alpha))ds\right)), k(x,t) \geq 0 \\ k(x,t,F_{1\alpha}\left(t,\underline{u}(t,\alpha)\right)), k(x,t) < 0 \end{cases}$$

(8)

where

$\underline{k(x,t,F_{1\alpha}(t,u(t,\alpha))} = k\left(x,t,\underline{F_{1\alpha}}(t,u(t,\alpha))\right) = k(x,t,F_{1\alpha}\left(t,\underline{u}(t,\alpha)\right))$ and

$$\overline{k(x,t,F_{1\alpha}(t,u(t,\alpha))} = k\left(x,t,\overline{F_{1\alpha}}(t,u(t,\alpha))\right) = k(x,t,F_{1\alpha}(t,\overline{u}(t,\alpha)))$$

For each 0≤α≤1anda≤x≤b. WecanseethatEq. (6)converttoasystem of nonlinear Volterra integral equations in crisp case for each 0 ≤ α ≤ 1 and a ≤ t ≤ b. Now, we explain homotopy analysis methods as approximating solution of this system of nonlinear integral equations in crisp case. then, we find approximate solutions for ũ(x), a ≤ x ≤ b $0 \leq c \leq x$

3.1 Homotopy analytic method "HAM"

Now we apply homotopy analytic method for solve the system (7) and obtain a recursion scheme for it. Prior to apply HAM for the system (7). We suppose that the kernel have four cases for kernal's .

$$N[\tilde{u}(x,\alpha)] = 0$$

$$\tilde{u}(x) = \tilde{f}(x) + \lambda \int_a^x k((x,t,\tilde{F}_1(x,t,\tilde{u}(t)),G(t,\int_a^t \tilde{F}_2(t,s,\tilde{u}(s))ds))dt$$

Where

$\tilde{u}(x,\alpha) = (\underline{u}(x,\alpha),\overline{u}(x,\alpha))$ and $\tilde{f}(x,\alpha) = (\underline{f}(x,\alpha),\bar{f}(x,\alpha))$

We see that eq(15) is convert to system of nonlinear crisp fuzzy volterra integral equations

$$\underline{u}(x,\alpha) = \underline{f}(x,\alpha) + \lambda \int_a^c k\left(x,t,\underline{F}_{1\alpha}\left(t,\underline{u}(t,\alpha)\right)\right)dt + \lambda \int_c^x k(x,t,G\left(t,\int_a^t \overline{F}_{2\alpha}(t,s,\overline{u}(s,\alpha))ds\right)dt)$$

$$\overline{u}(x,\alpha) = \bar{f}(x,\alpha) + \lambda \int_a^c k(x,t,G\left(t,\int_a^t \overline{F}_{2\alpha}(t,s,\overline{u}(s,\alpha))ds\right)dt) + \lambda \int_c^x k\left(x,t,\underline{F}_{1\alpha}\left(t,\underline{u}(t,\alpha)\right)\right)dt$$

(9)

For solving system (9) by HAM, we construct the zeroth-order deformation equations:

$$(1-p)L[\underline{\emptyset}(x,p;\alpha) - \underline{w}_0(x;\alpha)] = phH[\underline{\emptyset}(x,p;\alpha) - \underline{f}(x,\alpha)$$

$$-\lambda[\int_a^c k((x,t,\underline{F}_{1\alpha}\left(x,t,\underline{\emptyset}(t,p;\alpha)\right)) + \int_c^x k(x,t,G(t,\int_a^t \overline{F}_{2\alpha}(t,s,\overline{\emptyset}(s,p,\alpha))ds)\,dt]$$

$$(1-p)L[\overline{\emptyset}(x,p;\alpha)-\overline{w}_0(x;\alpha)] = phH[\overline{\emptyset}(x,p;\alpha)-\overline{f}(x,\alpha)$$
$$-\lambda\left[\int_a^c k(x,t,G(t,\int_a^t \overline{F}_{2\alpha}(t,s,\overline{\emptyset}(s,p,\alpha))ds\right)dt+\int_c^x k((x,t,\underline{F}_{1\alpha}\left(x,t,\underline{\emptyset}(t,p;\alpha)\right)dt]$$
(10)

Where $p \in [a,b]$ is the embedding parameter, his nonzero auxiliary parameter, L is an auxiliary linear operator $H(x)$ is an auxiliary function , $(\underline{w}_0(x;\alpha)\,,\overline{w}_0(x;\alpha))$ are initial guesses of $\underline{F}_{1\alpha}\left(x,t,\underline{\emptyset}(t,p;\alpha)\right),\overline{F}_{1\alpha}\left(t,s,\overline{\emptyset}(t,p;\alpha)\right),\overline{F}_{2\alpha}\left(x,t,\overline{\emptyset}(t,p;\alpha)\right) and\ \underline{F}_{2\alpha}\left(t,s,\underline{\emptyset}(t,p;\alpha)\right)$ respectively and $\underline{\emptyset}(t,p;\alpha) and\ \overline{\emptyset}(t,p;\alpha)$ are unknown functions. Using the above zeroth – order deformation equations ,with assumption

$L(u)=u\ and\ H(x)=1$, we have

$$(1-p)[\underline{\emptyset}(x,p;\alpha)-\underline{w}_0(x;\alpha)] = ph[\underline{\emptyset}(x,p;\alpha)-\underline{f}(x,\alpha)$$
$$-\lambda[\int_a^c k((x,t,\underline{F}_{1\alpha}\left(x,t,\underline{\emptyset}(t,p;\alpha)\right))+\int_c^x k(x,t,G(t,\int_a^t \overline{F}_{2\alpha}(t,s,\overline{\emptyset}(s,p,\alpha))ds)\,dt]$$
$$(1-p)[\overline{\emptyset}(x,p;\alpha)-\overline{w}_0(x;\alpha)] = ph[\overline{\emptyset}(x,p;\alpha)-\overline{f}(x,\alpha)$$
$$-\lambda[\int_a^c k(x,t,G(t,\int_a^t \overline{F}_{2\alpha}(t,s,\overline{\emptyset}(s,p,\alpha))ds)+\int_c^x k((x,t,\underline{F}_{1\alpha}\left(x,t,\underline{\emptyset}(t,p;\alpha)\right))dt]$$
(11)

Obviously ,when $p=0\ and\ p=1$ and since $h \neq 0$, we obtain

$$\underline{\emptyset}(x,0;\alpha)=\underline{w}_0(x;\alpha)$$

$\overline{\emptyset}(x,0;\alpha)$=$\overline{w}_0(x;\alpha)$

And

$$\underline{\emptyset}(x,1;\alpha)=\underline{f}(x,\alpha)+\lambda\int_a^c k((x,t,\underline{F}_{1\alpha}\left(x,t,\underline{\emptyset}(t,1;\alpha)\right))dt+\lambda\int_c^x k(x,t,G(t,\int_a^t \overline{F}_{2\alpha}(t,s,\overline{\emptyset}(s,1,\alpha))ds)\,dt$$

$$\overline{\emptyset}(x,1;\alpha)=\overline{f}(x,\alpha)+\lambda\int_a^c k(x,t,G(t,\int_a^t \overline{F}_{2\alpha}(t,s,\overline{\emptyset}(s,1,\alpha))ds)\,dt\ +\int_a^x k((x,t,\underline{F}_{1\alpha}\left(x,t,\underline{\emptyset}(t,1;\alpha)\right))dt \quad (12)$$

Respectively . Thus as p increases from $0\ to\ 1$, the function$\underline{\emptyset}(x,p;\alpha)$

$\overline{\emptyset}(x,p;\alpha)$ deforms from the initial guesses $\underline{w}_0(x;\alpha)\,,\overline{w}_0(x;\alpha)$ to the solution of $\underline{F}_{1\alpha}\left(x,t,\underline{\emptyset}(t,p;\alpha)\right),\overline{F}_{1\alpha}\left(t,s,\overline{\emptyset}(t,p;\alpha)\right),\overline{F}_{2\alpha}\left(x,t,\overline{\emptyset}(t,p;\alpha)\right) and\ \underline{F}_{2\alpha}\left(t,s,\underline{\emptyset}(t,p;\alpha)\right)$.

Expanding $\underline{\emptyset}(x,p;\alpha)$ in Taylors series with respect p gives

$$\underline{\emptyset}(x,p;\alpha)=\underline{w}_0(x;\alpha)\ +\sum_{m=1}^{\infty}\underline{u}_m(x,\alpha)p^m \quad (13)$$

$$\overline{\emptyset}(x,p;\alpha)=\overline{w}_0(x;\alpha)+\sum_{m=1}^{\infty}\overline{u}_m(x,\alpha)p^m$$

Where

$$\underline{u}_m(x,r)=\frac{1}{m!}\frac{\partial^m\underline{\emptyset}(t,p;r)}{\partial p^m}\bigg|_{p=0}$$

$$\overline{u}_m(x,r)=\frac{1}{m!}\frac{\partial^m\overline{\emptyset}(t,p;r)}{\partial p^m}\bigg|_{p=0} \quad , \quad m\geq 1$$

It should be noted that $\underline{\emptyset}(x,p;\alpha) = \underline{w}_0(x;\alpha)$ *and* $\overline{\emptyset}(x,p;\alpha) = \overline{w}_0(x;\alpha)$differential the zeroth–order deformation equation (10) m times with respect to the embedding parameter p and dividing them by $m!$ and$n!$ finally setting $p = 0$,we obtain the so called $m - th$ order deformation equations

$$\underline{u}_m(x;r) = h[\underline{u}_{m-1}(x;r) - (1-\mathcal{X}_m)\underline{f}(x;r) - \lambda\int_a^c \underline{\boldsymbol{R}}_{m-1}(x,t;\alpha)\,dt - \lambda\int_c^x \overline{\boldsymbol{R}}_{m-1}(x,t;\alpha)\,dt$$

$$\overline{u}_m(x;r) = h[\overline{u}_{m-1}(x;r) - (1-\mathcal{X}_m)\overline{f}(x;r) - \lambda\int_c^x \overline{\boldsymbol{R}}_{m-1}(x,t;\alpha)\,dt - \lambda\int_a^c \underline{\boldsymbol{R}}_{m-1}(x,t;\alpha)\,dt$$

(14)

Where

$$\mathcal{X}_m = \begin{cases} 0, m = 1 \\ 1, m \neq 1 \end{cases} \qquad m \geq 1$$

$$\underline{\emptyset}(x,\alpha) = \underline{w}_0(x;\alpha) \text{ , and}$$

$$\underline{\boldsymbol{R}}_{m-1}(x,t;\alpha) = \frac{1}{(m-1)!}\frac{\partial^{m-1}k\left(x,t,\underline{F}_{1\alpha}\left(x,\underline{\emptyset}(x,p;\alpha)\right)\right)}{\partial p^{m-1}}\Bigg|_{p=0}$$

and

$\overline{\emptyset}(x,\alpha)$=$\overline{w}_0(x;\alpha)$

$$\overline{\boldsymbol{R}}_{m-1}(x,t;\alpha) = \frac{1}{(m-1)!}\frac{\partial^{m-1}k(x,t,G(t,\int_a^t \overline{F}_{2\alpha}(t,s,\overline{\emptyset}(s,p,\alpha)ds)}{\partial p^{m-1}}\Bigg|_{p=0} \qquad (15)$$

From Eqs (14) and (15), we have that

$$\underline{u}_0(x,\alpha) = 0$$

$$\underline{u}_1(x;\alpha) = h\underline{u}_0(x;\alpha) - h\underline{f}(x;\alpha) - \lambda h[\int_0^c \underline{\boldsymbol{R}}_0(x,t;\alpha)dt + \int_c^x \overline{\boldsymbol{R}}_0(x,t;\alpha)\,dt]$$

$$\underline{u}_m(x;\alpha) = (1+h)\underline{u}_{m-1}(x;\alpha) - \lambda h[\int_0^c \underline{\boldsymbol{R}}_{m-1}(x,t;\alpha)dt + \int_c^x \overline{\boldsymbol{R}}_{m-1}(x,t;\alpha)dt]$$

Similary

$$\overline{u}_0(x,\alpha) = 0$$

$$\overline{u}_1(x;\alpha) = h\underline{u}_0(x;\alpha) - h\underline{f}(x;\alpha) - \lambda h\,[\int_0^c \overline{\boldsymbol{R}}_0(x,t;\alpha)dt + \int_c^x \underline{\boldsymbol{R}}_0(x,t;\alpha)dt]$$

$$\overline{u}_m(x;\alpha) = (1+h)\underline{u}_{m-1}(x;\alpha) - \lambda h\,[\int_0^c \overline{\boldsymbol{R}}_{m-1}(x,t;\alpha)dt + \int_c^x \underline{\boldsymbol{R}}_{m-1}(x,t;\alpha)dt]$$

(16)

where,

$$\underline{\boldsymbol{R}}_0(x,t;\alpha) = k(x,t,\underline{F}_{1\alpha}\left(t,\underline{u_0}(t,\alpha)\right)$$

$$\overline{\boldsymbol{R}}_0(x,t;\alpha) = k(x,t,G(t,\int_a^t \overline{F}_{2\alpha}(t,s,\overline{u}_0(s,\alpha))ds) \qquad (17)$$

Thus, if we choose a proper values of h , the series (8) is convergent at $p = 1$. Hence the solution of system (9) in series from (homotopy solution series) is obtain as

$$\underline{U}(x,\alpha)=\underline{u}_0(x;\alpha)+\sum_{m=1}^{\infty}\underline{u}_m(x;\alpha)$$

$\overline{U}(x,\alpha)=\overline{u}_0(x;\alpha)+\sum_{m=1}^{\infty}\overline{u}_m(x;\alpha)$
(18)

We denoted the $n-th$ order approximate solution with

$$\underline{U_j}(x,r)=\underline{u}_0(x;r)+\sum_{m=1}^{\infty}\underline{u}_m(x;r)$$

$\overline{U_j}(x,r)=\overline{u}_0(x;r)+\sum_{m=1}^{\infty}\overline{u}_m(x;r)$ (19)

$$j\geq 1$$

Theorem 2.1: (**. Existence and uniqueness)**

The following conditions are satisfy

1- $f:[a,b]\to E^n$ is a continuous and bounded
2- $K:[a,b]\times[a,b]\times[a,b]\times E^n\times E^n\to E^n$ is a continuous function
3- $if\ u,u^*:[a,b]\to E^n$ are continuous, then the lipschitz condition

$$D(k(x_1,t_1,\widetilde{H}_{11}(t),G(t,H_{12}(t),k(x_2,t_2,\widetilde{H}_{21}(t),G\big(t,H_{22}(t)\big)\leq L(D(x_1,x_2)+ D(t_1,t_2)+D\big(H_{11}(t),H_{21}(t)\big)+D\big(G(t_1,H_{12}(t)),G(t_2,H_{22}(t)\big))$$

4- $D(F_1\big(x_1,t_1,\tilde{u}(t)\big),F_1(x_2,t_2,\tilde{u}^*(t))\leq L_1(\,D(x_1,x_2)+D(t_1,t_2)+D\big(\tilde{u}(x),\tilde{u}^*(x)\big))$

5- $D(F_2\big(s_1,t_1,\tilde{u}(t)\big),F_2(s_2,t_2,\tilde{u}^*(t))\leq L_2((\,D(s_1,s_2)+D(t_1,t_2)+\ D(\tilde{u}(x),\tilde{u}^*(x))$

6- $D(G(t_1,\int_a^t F_2\big(s_1,t_1,\tilde{u}(s)\big)ds\,,G(t_2,\int_a^t F_2\big(s_2,t_2,\tilde{u}(s)\big)ds)\leq L_3(D(t_1,t_2)+ (b-a)L_2(D(s_1,s_2)+D(t_1,t_2)+\ D\big(\tilde{u}(x),\tilde{u}^*(x)\big))$

We take an initial guess $\overline{u}_0(x)=\overline{f}(x)$, an auxiliary operator $L\overline{u}=\overline{u}$, an nonzero auxiliary parameter h=-1 and auxiliary function H(x)=1, this is substituted in Eq(19) to give the recurrence relation .

Then there exist a unique fuzzy solution of u(x) in Eq(19) and the successive iteration

$$\omega_0(x)=f(x)$$

$$\omega_{n+1}(x)=f(x)+\sum_{i=1}^{n+1}\int_a^x k((x,t,\tilde{F}_1\big(x,t,\tilde{u}_{i-1}(t)\big),G(t,\int_a^t\tilde{F}_2\big(t,s,\tilde{u}_{i-1}(s)\big)ds))dt\quad n\geq 0$$

(20)

and u(x) is convergent uniformly on [a,b] where

$$u_0(x)=f(x)$$

$$u_n(x)=\int_a^x k((x,t,\tilde{F}_1\big(x,t,\tilde{u}_{n-1}(t)\big),G(t,\int_a^t\tilde{F}_2\big(t,s,\tilde{u}_{n-1}(s)\big)ds))dt\ ,\qquad n\geq 1$$

(21)

Lemma 2.1: Consider $f(x)$ is a bounded on [a,b] then we will prove the $u_n(x)$, is a bounded and $u_n(x)$ is countinuous , where $n \geq 0$

Proof: we will proved the part 1 from lemma 2.1 $u_0(x) = f(x)$ is a bounded. Now we proved $u_{n-1}(x)$ is a bounded . and proposition (2.1), we have

$$D(\tilde{u}_n(x), \tilde{0}) = D(\int_a^x k((x,t,\tilde{F}_1(x,t,\tilde{u}_{n-1}(t)), G(t, \int_a^t \tilde{F}_2(t,s,\tilde{u}_{n-1}(s))ds))dt , \tilde{0})$$

$$\leq \int_a^x D(k((x,t,\tilde{F}_1(x,t,\tilde{u}_{n-1}(t)), G(t, \int_a^t \tilde{F}_2(t,s,\tilde{u}_{n-1}(s))ds) , \tilde{0})\, dt$$

$$\leq (b-a)L(D(x,\tilde{0}) + D(t,\tilde{0}) + D(\tilde{F}_1(x,t,\tilde{u}_{n-1}(t)),\tilde{0}) + D(G(t, \int_a^t \tilde{F}_2(t,s,\tilde{u}_{n-1}(s))ds) , \tilde{0})$$

$$\leq (b-a)L(D(x,\tilde{0}) + D(t,\tilde{0}) + L_1[D(x,\tilde{0}) + D(t,\tilde{0}) + D\left((\tilde{u}_{n-1}(t)),\tilde{0}\right)]$$

$$+L_2[D(t,\tilde{0}) + \int_0^t L_3[D(t,\tilde{0}) + D(s,\tilde{0}) + D(\tilde{u}_{n-1}(s),\tilde{0}]$$

$$\leq L(D(x,\tilde{0}) + D(t,\tilde{0}) + L_1\left[D(x,\tilde{0}) + D(t,\tilde{0}) + D\left((\tilde{u}_{n-1}(t)),\tilde{0}\right)\right] + L_2[D(t,\tilde{0})$$

$$+(b-a)L_3[D(t,\tilde{0}) + D(s,\tilde{0}) + D(\tilde{u}_{n-1}(s),\tilde{0})]$$

$$\leq (b-a)(L+L_1)(D(x,\tilde{0}) + (b-a)[L+L_1+L_2+(b-a)L_3]D(t,\tilde{0})$$

$$+(b-a)L_3D(s,\tilde{0}) + (b-a)[(b-a)(L_3+L))D((\tilde{u}_{n-1}(t)),\tilde{0})]$$

$$\leq (b-a)(L+L_1)supD(x,0) + (b-a)[(L+L_1+L_2) + (b-a)L_3)supD(t,\tilde{0})$$

$$+(b-a)L_3supD(s,\tilde{0}) + (b-a)[(b-a)(L_3+L)supD(\tilde{u}_{n-1}(t)),\tilde{0})]$$

$\tilde{u}_n(x)$ is a bounded , $\tilde{0}$ *is the zero function*

Now we will proof part 2 from lemma 2.1 , $\tilde{u}_n(x)$ is a continuous by using propostion (2.1)and (2.2), we have , $a \leq x \leq x_1 \leq b$

$$(\tilde{u}_n(x), \tilde{u}_n(x_1)) = D(\int_a^x k((x,t,\tilde{F}_1(x,t,\tilde{u}_{n-1}(t)), G(t, \int_a^t \tilde{F}_2(t,s,\tilde{u}_{n-1}(s))ds))dt$$

$$,\int_a^{x_1} k((x_1,t,\tilde{F}_1(x_1,t,\tilde{u}_{n-1}(t)), G(t, \int_a^t \tilde{F}_2(t,s,\tilde{u}_{n-1}(s))ds))dt)$$

$$= D(\int_a^x k((x,t,\tilde{F}_1(x,t,\tilde{u}_{n-1}(t)), G(t, \int_a^t \tilde{F}_2(t,s,\tilde{u}_{n-1}(s))ds))dt \quad ,$$
$$,\int_a^x k((x_1,t,\tilde{F}_1(x_1,t,\tilde{u}_{n-1}(t)), G(t, \int_a^t \tilde{F}_2(t,s,\tilde{u}_{n-1}(s))ds))dt) \; ,$$
$$+\int_x^{x_1} k((x_1,t,\tilde{F}_1(x_1,t,\tilde{u}_{n-1}(t)), G(t, \int_a^t \tilde{F}_2(t,s,\tilde{u}_{n-1}(s))ds))dt) \;)$$

$$\leq D(\int_a^x k(((x,t,\tilde{F}_1(x,t,\tilde{u}_{n-1}(t)), G(t, \int_a^t \tilde{F}_2(t,s,\tilde{u}_{n-1}(s))ds))dt \quad ,\int_a^x k((x_1,t,\tilde{F}_1(x_1,t,\tilde{u}_{n-1}(t)), G(t, \int_a^t \tilde{F}_2(t,s,\tilde{u}_{n-1}(s))ds))dt)$$

$$+D(\int_x^{x_1} k((x_1,t,\tilde{F}_1(x_1,t,\tilde{u}_{n-1}(t)), G(t, \int_a^t \tilde{F}_2(t,s,\tilde{u}_{n-1}(s))ds))dt) , \tilde{0})$$

$$\leq \int_a^x D(\; k((x,t,\tilde{F}_1(x,t,\tilde{u}_{n-1}(t)), G(t, \int_a^t \tilde{F}_2(t,s,\tilde{u}_{n-1}(s))ds)), k((x_1,t,\tilde{F}_1(x_1,t,\tilde{u}_{n-1}(t)), G(t, \int_a^t \tilde{F}_2(t,s,\tilde{u}_{n-1}(s))ds))\, dt)$$

$$+\int_{x}^{x_1} D\left(k((x_1,t,\tilde{F}_1(x_1,t,\tilde{u}_{n-1}(t)), G\left(t,\int_a^t \tilde{F}_2(t,s,\tilde{u}_{n-1}(s))ds)\right)dt) \right),\tilde{0})dt$$

$$\leq (b-a)supD(k((x,t,\tilde{F}_1(x,t,\tilde{u}_{n-1}(t)),G(t,\int_a^t \tilde{F}_2(t,s,\tilde{u}_{n-1}(s))ds)),k((x_1,t,\tilde{F}_1(x_1,t,\tilde{u}_{n-1}(t)),G(t,\int_a^t \tilde{F}_2(t,s,\tilde{u}_{n-1}(s))ds)))$$

$$+(x_1-x)[(L+L_1)supD(x,0)+[(L+L_1+L_2)+(b-a)L_3)supD(t,\tilde{0})$$

$$+L_3supD(s,\tilde{0})+[(b-a)(L_3+L)supD(\tilde{u}_{n-1}(t)),\tilde{0})]$$

K is continuous , we have

$$D(\tilde{u}_n(x),\tilde{u}_n(x_1)) \to 0 \quad as\ x \to x_1$$

$u_n(x)$ is continuous on [a,b]

proof

we will prove that $\omega_n(x)$, $n \geq 0$ are bounded on [a,b]. Then $\omega_0(x) = f(x)$

is a bounded by the assumption that $\omega_{n-1}(x)$ is a bounded, from Eq(20) and propostion (2.1) we will get .

$$D(\omega_n(x),\tilde{0}) = D(f(x)+\sum_{i=1}^{n}\int_a^x k((x,t,\tilde{F}_1(x,t,\tilde{u}_{i-1}(t)),G(t,\int_a^t \tilde{F}_2(t,s,\tilde{u}_{i-1}(s))ds))dt\ ,\tilde{0})$$

$$= D(f(x)+\sum_{i=1}^{n-1}\int_a^x k((x,t,\tilde{F}_1(x,t,\tilde{u}_{i-1}(t)),G(t,\int_a^t \tilde{F}_2(t,s,\tilde{u}_{i-1}(s))ds))dt$$

$$+\int_a^x k((x,t,\tilde{F}_1(x,t,\tilde{u}_{n-1}(t)),G(t,\int_a^t \tilde{F}_2(t,s,\tilde{u}_{n-1}(s))ds))dt\ ,\tilde{0})$$

$$= D(\omega_{n-1}(x)+\int_a^x k((x,t,\tilde{F}_1(x,t,\tilde{u}_{n-1}(t)),G(t,\int_a^t \tilde{F}_2(t,s,\tilde{u}_{n-1}(s))ds))dt\ ,\tilde{0})$$

$$\leq D(\omega_{n-1}(x),\tilde{0}) + D(\int_a^x k((x,t,\tilde{F}_1(x,t,\tilde{u}_{n-1}(t)),G(t,\int_a^t \tilde{F}_2(t,s,\tilde{u}_{n-1}(s))ds))dt\ ,\tilde{0})$$

$$= D(\omega_{n-1}(x),\tilde{0}) + D(\omega_n(x),\tilde{0})$$

From lemma (2.1) we have that $\omega_n(x)$ is a bounded . and $\{\omega_n(x)\}_{n=0}^{\infty}$ is a sequence of bounded functions on [a,b]

We will prove $\omega_n(x)$ are continuous on [a,b] , by using Lemma (2.1) and proposition (2.2) and(2.2) for $a \leq x \leq x_1 \leq b$, we get

$$D(\omega_n(x),\omega_n(x_1)) = D((f(x)+\sum_{i=1}^{n}\int_a^x k((x,t,\tilde{F}_1(x,t,\tilde{u}_{i-1}(t)),G(t,\int_a^t \tilde{F}_2(t,s,\tilde{u}_{i-1}(s))ds))dt$$

$$,f(x_1)+\sum_{i=1}^{n}\int_a^{x_1} k((x_1,t,\tilde{F}_1(x_1,t,\tilde{u}_{i-1}(t)),G(t,\int_a^t \tilde{F}_2(t,s,\tilde{u}_{i-1}(s))ds))dt\ )$$

$$\leq D(f(x),f(x_1)) + D(\sum_{i=1}^{n}\int_a^x k((x,t,\tilde{F}_1(x,t,\tilde{u}_{i-1}(t)),G(t,\int_a^t \tilde{F}_2(t,s,\tilde{u}_{i-1}(s))ds))dt$$

$$,\sum_{i=1}^{n}\int_{a}^{x_1} k((x_1,t,\tilde{F}_1(x_1,t,\tilde{u}_{i-1}(t)),G(t,\int_a^t \tilde{F}_2(t,s,\tilde{u}_{i-1}(s))ds))dt\ )$$

$$= D(f(x),f(x_1)) + D(\sum_{i=1}^{n}\int_{a}^{x} k((x,t,\tilde{F}_1(x,t,\tilde{u}_{i-1}(t)),G(t,\int_a^t \tilde{F}_2(t,s,\tilde{u}_{i-1}(s))ds))dt\ ,$$

$$\sum_{i=1}^{n}\int_{a}^{x} k((x_1,t,\tilde{F}_1(x_1,t,\tilde{u}_{i-1}(t)),G(t,\int_a^t \tilde{F}_2(t,s,\tilde{u}_{i-1}(s))ds))dt\ ) +$$

$$\sum_{i=1}^{n}\int_{x}^{x_1} k((x_1,t,\tilde{F}_1(x_1,t,\tilde{u}_{i-1}(t)),G(t,\int_a^t \tilde{F}_2(t,s,\tilde{u}_{i-1}(s))ds))dt\ ))$$

$$\leq D(f(x),f(x_1)) + D(\sum_{i=1}^{n}\int_{a}^{x} k((x,t,\tilde{F}_1(x,t,\tilde{u}_{i-1}(t)),G(t,\int_a^t \tilde{F}_2(t,s,\tilde{u}_{i-1}(s))ds))dt\ ,$$

$$\sum_{i=1}^{n}\int_{a}^{x} k((x_1,t,\tilde{F}_1(x_1,t,\tilde{u}_{i-1}(t)),G(t,\int_a^t \tilde{F}_2(t,s,\tilde{u}_{i-1}(s))ds))dt\ ))$$
$$+D(\sum_{i=1}^{n}\int_{x}^{x_1} k((x_1,t,\tilde{F}_1(x_1,t,\tilde{u}_{i-1}(t)),G(t,\int_a^t \tilde{F}_2(t,s,\tilde{u}_{i-1}(s))ds))dt\ ),\tilde{0})$$

$$\leq D(f(x),f(x_1)) + \int_a^x D(\sum_{i=1}^{n} k\,(x,t,\tilde{F}_1(x,t,\tilde{u}_{i-1}(t)),G(t,\int_a^t \tilde{F}_2(t,s,\tilde{u}_{i-1}(s))ds)\ ,$$

$$\sum_{i=1}^{n}\int_{a}^{x} k((x_1,t,\tilde{F}_1(x_1,t,\tilde{u}_{i-1}(t)),G(t,\int_a^t \tilde{F}_2(t,s,\tilde{u}_{i-1}(s))ds)\ ))dt$$

$$+\int_x^{x_1} D(\sum_{i-1}^{n} k\,((x_1,t,\tilde{F}_1(x_1,t,\tilde{u}_{i-1}(t)),G(t,\int_a^t \tilde{F}_2(t,s,\tilde{u}_{i-1}(s))ds)),\tilde{0})dt$$

$$\leq D(f(x),f(x_1)) + (b-a)\,sup\,D[\sum_{i=1}^{n} k((x,t,\tilde{F}_1(x,t,\tilde{u}_{i-1}(t)),G(t,\int_a^t \tilde{F}_2(t,s,\tilde{u}_{i-1}(s))ds)\ ,$$

$$,\sum_{i=1}^{n} k((x_1,t,\tilde{F}_1(x_1,t,\tilde{u}_{i-1}(t)),G(t,\int_a^t \tilde{F}_2(t,s,\tilde{u}_{i-1}(s))ds)\ ]$$

$$+(x_1-x)[sup\,D\,[\sum_{i-1}^{n} k\,((x_1,t,\tilde{F}_1(x_1,t,\tilde{u}_{i-1}(t)),G(t,\int_a^t \tilde{F}_2(t,s,\tilde{u}_{i-1}(s))ds)),\tilde{0}]\,]$$

$$D(\omega_n(x),\omega_n(x_1)) \to 0\ ,\ \ x \to x_1$$

and the sequence $\{\omega_n(x)\}_{n=0}^{\infty}$ is continuous on [a,b]

in this part we will prove the sequence $\{\omega_n(x)\}_{n=0}^{\infty}$ is convergent uniformly for $n \geq 0$ we get

$$D(\omega_{n+1}(x),\omega_n(x)) = D(f(x) + \sum_{i=1}^{n+1}\int_{a}^{x} k((x,t,\tilde{F}_1(x,t,\tilde{u}_{i-1}(t)),G(t,\int_a^t \tilde{F}_2(t,s,\tilde{u}_{i-1}(s))ds))dt,\omega_n(x))$$

$$= D(f(x) + \sum_{i=1}^{n}\int_{a}^{x} k((x,t,\tilde{F}_1(x,t,\tilde{u}_{i-1}(t)),G(t,\int_a^t \tilde{F}_2(t,s,\tilde{u}_{i-1}(s))ds))dt$$
$$+\int_a^x k((x,t,\tilde{F}_1(x,t,\tilde{u}_n(t)),G(t,\int_a^t \tilde{F}_2(t,s,\tilde{u}_n(s))ds))dt,\omega_n(x))$$

$$= D(\omega_n(x) + \int_a^x k((x,t,\tilde{F}_1(x,t,\tilde{u}_n(t)),G(t,\int_a^t \tilde{F}_2(t,s,\tilde{u}_n(s))ds))dt,\omega_n(x))$$

$$= D(\int_a^x k((x,t,\tilde{F}_1\big(x,t,\tilde{u}_n(t)\big),G(t,\int_a^t \tilde{F}_2\big(t,s,\tilde{u}_n(s)\big)ds))dt,\tilde{0})$$

$$\leq \int_a^x D(k((x,t,\tilde{F}_1\big(x,t,\tilde{u}_n(t)\big),G(t,\int_a^t \tilde{F}_2\big(t,s,\tilde{u}_n(s)\big)ds)),\tilde{0})dt$$

$$\leq (b-a)[(L+L_1)supD(x,0)+(b-a)[(L+L_1+L_2)+(b-a)L_3)supD(t,\tilde{0})$$
$$+(b-a)L_3supD\big(s,\tilde{0}\big)+(b-a)[(b-a)(L_3+L)supD\big(\tilde{u}_n(t)\big),\tilde{0})]]$$

Now we obtain

$$sup\ D\big(\omega_{n+1}(x),\omega_n(x)\big) \leq (b-a)[(L+L_1)supD(x,0)+(b-a)[(L+L_1+L_2)+(b-a)L_3)supD(t,\tilde{0})$$
$$+(b-a)L_3supD\big(s,\tilde{0}\big)+(b-a)[(b-a)(L_3+L)supD\big(\tilde{u}_n(t)\big),\tilde{0})]]$$

$$D\big(u_n(x),\tilde{0}\big) = D(\int_a^x k((x,t,\tilde{F}_1\big(x,t,\tilde{u}_{n-1}(t)\big),G(t,\int_a^t \tilde{F}_2\big(t,s,\tilde{u}_{n-1}(s)\big)ds))dt,\tilde{0})$$

$$\leq \int_a^x D(k((x,t,\tilde{F}_1\big(x,t,\tilde{u}_{n-1}(t)\big),G(t,\int_a^t \tilde{F}_2\big(t,s,\tilde{u}_{n-1}(s)\big)ds)),\tilde{0})dt$$

$$\leq (b-a)[(L+L_1)supD(x,0)+(b-a)[(L+L_1+L_2)+(b-a)L_3)supD(t,\tilde{0})$$
$$+(b-a)L_3supD\big(s,\tilde{0}\big)+(b-a)[(b-a)(L_3+L)supD\big(\tilde{u}_{n-1}(t)\big),\tilde{0})]]$$

$$D(u_{n-1}(x),\tilde{0}) \leq (b-a)[(L+L_1)supD(x,0)+(b-a)[(L+L_1+L_2)+(b-a)L_3)supD(t,\tilde{0})$$
$$+(b-a)L_3supD\big(s,\tilde{0}\big)+(b-a)[(b-a)(L_3+L)supD\big(\tilde{u}_{n-2}(t)\big),\tilde{0})]]$$

we obtain

$$D\big(u_n(x),\tilde{0}\big) \leq (b-a)[(L+L_1)D(x,0)+(b-a)[(L+L_1+L_2)+(b-a)L_3)D(t,\tilde{0})$$
$$+(b-a)L_3D\big(s,\tilde{0}\big)+(b-a)[(b-a)(L_3+L)D\big(\tilde{u}_{n-1}(t)\big),\tilde{0})]]$$
$$\leq [(b-a)[(L+L_1)D(x,0)+(b-a)[(L+L_1+L_2)+(b-a)L_3)D(t,\tilde{0})$$
$$+(b-a)L_3D\big(s,\tilde{0}\big)+(b-a)[(b-a)(L_3+L)]^2D\big(\tilde{u}_{n-2}(t)\big),\tilde{0})]]$$
$$\leq [(b-a)[(L+L_1)D(x,0)+(b-a)[(L+L_1+L_2)+(b-a)L_3)D(t,\tilde{0})$$
$$+(b-a)L_3D\big(s,\tilde{0}\big)+(b-a)[(b-a)(L_3+L)]^n sup\ D\big(f(x),\tilde{0}\big)$$

$$D\big(\omega_{n+1}(x),\omega_n(x)\big) \leq supD\big(\omega_{n+1}(x),\omega_n(x)\big) \leq$$
$$[(b-a)[(L+L_1)D(x,0)+(b-a)[(L+L_1+L_2)+(b-a)L_3)D(t,\tilde{0})$$
$$+(b-a)L_3D\big(s,\tilde{0}\big)+(b-a)[(b-a)(L_3+L)]^{n+1} sup\ D\big(f(x),\tilde{0}\big)$$

$the\ series$ $(b-a)[(L+L_1)D(x,0)+(b-a)[(L+L_1+L_2)+(b-a)L_3)D(t,\tilde{0})$

$+(b-a)L_3D\big(s,\tilde{0}\big)+(b-a)[(b-a)(L_3+L)]sup\ D\big(f(x),\tilde{0}\big)\sum_{n=0}^{\infty}[\,[(b-a)[(L+L_1)D(x,0)$

$+(b-a)[(L+L_1+L_2)+(b-a)L_3)D(t,\tilde{0})\ +(b-a)L_3D\big(s,\tilde{0}\big)+(b-a)[(b-a)(L_3+L)]^n$

$the\ series$ $\sum_{n=0}^{\infty} D\big(\omega_{n+1}(x),\omega_n(x)\big)$ is uniformly convergent on [a,b], and the uniformly convergent of the sequence $\{\omega_n(x)\}_{n=0}^{\infty}$. if denoted $u(x) = lim_{n\to\infty}\,\omega_n(x)$, then u(x) is satisfy all the conditions for theorem (2.1)

We prove the uniqueness of solution > Let u(x) and $u^*(x)$ are two continuous solution of Eq(7) on [a,b]

$$D\big(\tilde{u}(x),\tilde{u}^*(x)\big) \geq 0$$

and

$$D\big(\tilde{u}(x),\tilde{u}^*(x)\big) = D(\tilde{u}(x)+\omega_n(x),\tilde{u}^*(x)+\omega_n(x))$$
$$\leq D\big(\tilde{u}(x),\omega_n(x)\big) + D(\tilde{u}^*(x),\omega_n(x))$$
$$D\big(\tilde{u}(x),\omega_n(x)\big) \to 0$$
$$D(\tilde{u}^*(x),\omega_n(x)) \to 0$$

when $n \to \infty$, then $D\big(\tilde{u}(x),\tilde{u}^*(x)\big) = 0$, then $\tilde{u}(x) = \tilde{u}^*(x)$

3.2. numerical example

In this prrt we will discuss case of new formula by using homotopy analysis method.

And comparing the approximate method with he known exact solution and calculate the absolute error between the exact and approximate. Also give some finger for all cases.

$\underline{k\left(x,t,F_{1\alpha}\big(t,u(t,\alpha)\big)\right)} \geq 0,\ \overline{k(x,t,G(t,\int_a^t F_{2\alpha}\big(t,s,u(s,\alpha)\big)ds)} \geq 0, \overline{k(x,t,G(t,\int_a^t F_{2\alpha}\big(t,s,u(s,\alpha)\big)ds)} < 0$

$\underline{k\left(x,t,F_{1\alpha}\big(t,u(t,\alpha)\big)\right)} < 0,\quad 0 \leq x \leq t \leq b\ , a \leq s \leq t$

$$\underline{u}(x,\alpha) = \underline{f}(x,\alpha) + \lambda\int_a^c k(x,t,\underline{F}_{1\alpha}\left(x,t,\underline{u}(t,\alpha)\right)dt + \lambda\int_c^x k(x,t,G(t,\int_a^t \overline{F}_{2\alpha}\big(t,s,\overline{u}(s,\alpha)\big)ds)dt$$

$$\overline{u}(x,\alpha) = \overline{f}(x,\alpha) + \lambda\int_a^c k(x,t,G(t,\int_a^t \overline{F}_{2\alpha}\big(t,s,\overline{u}(s,\alpha)\big)ds)dt + \lambda\int_c^x k(x,t,\underline{F}_{1\alpha}\left(x,t,\underline{u}(t,\alpha)\right)dt$$

Example 1:

Now we discus this example

We will discuss Now the case of formula 2 , $\underline{k\left(x,t,F_{1\alpha}\big(t,u(t,\alpha)\big)\right)} \geq 0$, $\overline{G\left(t,\int_a^t F_{2\alpha}\big(t,s,u(s,\alpha)\big)ds\right)} \geq 0$

And $\underline{k\left(x,t,F_{1\alpha}\big(t,u(t,\alpha)\big)\right)} < 0$, $\overline{G\left(t,\int_a^t F_{2\alpha}\big(t,s,U(s,\alpha)\big)ds\right)} < 0$

$$\underline{u}(x,\alpha) = \underline{f}(x,\alpha) + \lambda\int_a^c k\left(x,t,\underline{F}_{1\alpha}\left(t,\underline{u}(t,\alpha)\right)\right)dt + \lambda\int_c^x k(x,t,G\left(t,\int_a^t \overline{F}_{2\alpha}\big(t,s,\overline{u}(s,\alpha)\big)ds\right)dt$$

$$\overline{u}(x,\alpha) = \overline{f}(x,\alpha) + \lambda\int_a^c k(x,t,G\left(t,\int_a^t \overline{F}_{2\alpha}\big(t,s,\overline{u}(s,\alpha)\big)ds\right)dt + \lambda\int_c^x k\left(x,t,\underline{F}_{1\alpha}\left(t,\underline{u}(t,\alpha)\right)\right)dt$$

Where

$$k\left(x,t,\underline{F}_{1\alpha}\left(t,\underline{u}(t,\alpha)\right)\right) = xt(\underline{u}(x,\alpha))^2$$

$$k(x,t,G\left(t,\int_a^t \overline{F}_{2\alpha}\big(t,s,\overline{u}(s,\alpha)\big)ds\right) = xt\int_a^t (\overline{u}\,(s,\alpha))^{\,2}ds$$

$\underline{f}(x,\alpha)$ =x^2*r-(1/5)*x^6*r^2-(1/30)*x^6*(2-r)^2

$\overline{f}(x,\alpha)$ =x^2*(2-r)-(1/5)*x^6*r^2-(1/30)*x^6*(2-r)^2 the intimal condition is $\underline{u}_0(x,\alpha) = \overline{u}_0(x,\alpha) = 0$ $, and \ \ 0 \le \alpha \le 1$

the exact solution $\underline{u}(x,\alpha) = x^2\alpha \ and \ \overline{u}(x,\alpha) = x^2(2-\alpha)$

$$\underline{u}_0(x,\alpha) = 0$$

$\underline{u}_1(x,\alpha)$ =-h*(x^2*r-(1/5)*x^6*r^2-(1/30)*x^6*(2-r)^2)

$$\underline{u}_2(x,\alpha) = (1+h)\underline{u}_1(x,\alpha) - h\lambda[\int_0^c \overline{\boldsymbol{R}}_1(x,t;\alpha)dt + \int_c^x \underline{\boldsymbol{R}}_1(x,t;\alpha)dt]$$

=-(1+h)*h*(x^2*r-(1/5)*x^6*r^2-(1/30)*x^6*(2-r)^2)-h*(x*h^2*(0.9390024038e-5*(-(1/5)*r^2-(1/30)*(2-r)^2)^2+0.4340277778e-3*r*(-(1/5)*r^2-(1/30)*(2-r)^2)+0.6250000000e-2*r^2)+h^2*((1/182)*(-(1/5)*r^2-(1/30)*(2-r)^2)^2*x^14+(1/45*(2-r))*(-(1/5)*r^2-(1/30)*(2-r)^2)*x^10+(1/30)*x^6*(2-r)^2)-h^2*(3.353580014*10^(-7)*(-(1/5)*r^2-(1/30)*(2-r)^2)^2+(0.2170138889e-4*(2-r))*(-(1/5)*r^2-(1/30)*(2-r)^2)+0.5208333333e-3*(2-r)^2))

$\underline{u}_3(x,\alpha)$ = (1+h)*(-(1+h)*h*(x^2*r-(1/5)*x^6*r^2-(1/30)*x^6*(2-r)^2)-h*(x*h^2*(0.9390024038e-5*(-(1/5)*r^2-(1/30)*(2-r)^2)^2+0.4340277778e-3*r*(-(1/5)*r^2-(1/30)*(2-r)^2)+0.6250000000e-2*r^2)+h^2*((1/182)*(-(1/5)*r^2-(1/30)*(2-r)^2)^2*x^14+(1/45*(2-r))*(-(1/5)*r^2-(1/30)*(2-r)^2)*x^10+(1/30)*x^6*(2-r)^2)-h^2*(3.353580014*10^(-7)*(-(1/5)*r^2-(1/30)*(2-r)^2)^2+(0.2170138889e-4*(2-r))*(-(1/5)*r^2-(1/30)*(2-r)^2)+0.5208333333e-3*(2-r)^2)))-h*(2.548432250*10^(-11)*h^6*r^2*(-.2000000000*r^2-0.3333333333e-1*(2.-1.*r)^2)^2+0.5259697569e-4*h^6*r*(-.2000000000*r^2-0.3333333333e-1*(2.-1.*r)^2)^3*x^26+9.781275041*10^(-9)*h^4*(h^2*(0.9390024038e-5*(-.2000000000*r^2-0.3333333333e-1*(2.-1.*r)^2)^2+(0.4340277778e-3*(2.-1.*r))*(-.2000000000*r^2-0.3333333333e-1*(2.-1.*r)^2)+0.6250000000e-2*(2.-1.*r)^2)-1.*x*h^2*(0.9390024038e-5*(-.2000000000*r^2-0.3333333333e-1*(2.-1.*r)^2)^2+0.4340277778e-3*r*(-.2000000000*r^2-0.3333333333e-1*(2.-1.*r)^2)+0.6250000000e-2*r^2))*(-.2000000000*r^2-0.3333333333e-1*(2.-1.*r)^2)^2-(0.2170138889e-4*(1.+h))*h*(2.-1.*r)*(-(1.*(1.+h))*h*(-.2000000000*r^2-0.3333333333e-1*(2.-1.*r)^2)-.2000000000*h^3*r^2)+0.3333333333e-1*(1.+h)^2*h^2*(2.-1.*r)^2*x^6+1.917897066*10^(-9)*(1.+h)*h^4*(2.-1.*r)*(-.2000000000*r^2-0.3333333333e-1*(2.-1.*r)^2)^2-5.540591526*10^(-9)*(-(1.*(1.+h))*h*(-.2000000000*r^2-0.3333333333e-1*(2.-1.*r)^2)-.2000000000*h^3*r^2)*h^3*r*(-.2000000000*r^2-0.3333333333e-1*(2.-1.*r)^2)-7.939346623*10^(-11)*(-(1.*(1.+h))*h*(-.2000000000*r^2-0.3333333333e-1*(2.-1.*r)^2)-.2000000000*h^3*r^2)*h^3*(-.2000000000*r^2-0.3333333333e-1*(2.-1.*r)^2)^2+8.220223064*10^(-7)*h^4*(h^2*(0.9390024038e-5*(-.2000000000*r^2-0.3333333333e-1*(2.-1.*r)^2)^2+(0.4340277778e-3*(2.-1.*r))*(-.2000000000*r^2-0.3333333333e-1*(2.-1.*r)^2)+0.6250000000e-2*(2.-1.*r)^2)-1.*x*h^2*(0.9390024038e-5*(-.2000000000*r^2-0.3333333333e-1*(2.-1.*r)^2)^2+0.4340277778e-3*r*(-.2000000000*r^2-0.3333333333e-1*(2.-1.*r)^2)+0.6250000000e-2*r^2))*r*(-.2000000000*r^2-0.3333333333e-1*(2.-1.*r)^2)+(0.5494505494e-2*((.4444444444*(1.+h))*h^4*(2.-1.*r)*r*(-.2000000000*r^2-0.3333333333e-1*(2.-1.*r)^2)+(-(1.*(1.+h))*h*(-.2000000000*r^2-0.3333333333e-1*(2.-1.*r)^2)-.2000000000*h^3*r^2)^2))*x^14+0.1041666667e-

$\underline{U}(x,\alpha)$ = -h*(x^2*r-(1/5)*x^6*r^2-(1/30)*x^6*(2-r)^2)-(1+h)*h*(x^2*r-(1/5)*x^6*r^2-(1/30)*x^6*(2-r)^2)-h*(x*h^2*(0.9390024038e-5*(-(1/5)*r^2-(1/30)*(2-r)^2)^2+0.4340277778e-3*r*(-(1/5)*r^2-(1/30)*(2-r)^2)+0.6250000000e-2*r^2)+h^2*((1/182)*(-(1/5)*r^2-(1/30)*(2-r)^2)^2*x^14+(1/45*(2-r))*(-(1/5)*r^2-(1/30)*(2-r)^2)*x^10+(1/30)*x^6*(2-r)^2)-h^2*(3.353580014*10^(-7)*(-(1/5)*r^2-(1/30)*(2-r)^2)^2+(0.2170138889e-4*(2-r))*(-(1/5)*r^2-(1/30)*(2-r)^2)+0.5208333333e-3*(2-r)^2))+= (1+h)*(-(1+h)*h*(x^2*r-(1/5)*x^6*r^2-(1/30)*x^6*(2-r)^2)-h*(x*h^2*(0.9390024038e-5*(-(1/5)*r^2-(1/30)*(2-r)^2)^2+0.4340277778e-3*r*(-(1/5)*r^2-(1/30)*(2-r)^2)+0.6250000000e-2*r^2)+h^2*((1/182)*(-(1/5)*r^2-(1/30)*(2-r)^2)^2*x^14+(1/45*(2-r))*(-(1/5)*r^2-(1/30)*(2-r)^2)*x^10+(1/30)*x^6*(2-r)^2)-h^2*(3.353580014*10^(-7)*(-

(1/5)*r^2-(1/30)*(2-r)^2)^2+(0.2170138889e-4*(2-r))*(-(1/5)*r^2-(1/30)*(2-
r)^2)+0.5208333333e-3*(2-r)^2)))-h*(2.548432250*10^(-11)*h^6*r^2*(-.2000000000*r^2-
0.3333333333e-1*(2.-1.*r)^2)^2+0.5259697569e-4*h^6*r*(-.2000000000*r^2-
0.3333333333e-1*(2.-1.*r)^2)^3*x^26+9.781275041*10^(-9)*h^4*(h^2*(0.9390024038e-
5*(-.2000000000*r^2-0.3333333333e-1*(2.-1.*r)^2)^2+(0.4340277778e-3*(2.-1.*r))*(-
.2000000000*r^2-0.3333333333e-1*(2.-1.*r)^2)+0.6250000000e-2*(2.-1.*r)^2)-
1.*x*h^2*(0.9390024038e-5*(-.2000000000*r^2-0.3333333333e-1*(2.-
1.*r)^2)^2+0.4340277778e-3*r*(-.2000000000*r^2-0.3333333333e-1*(2.-
1.*r)^2)+0.6250000000e-2*r^2))*(-.2000000000*r^2-0.3333333333e-1*(2.-1.*r)^2)^2-
(0.2170138889e-4*(1.+h))*h*(2.-1.*r)*(-(1.*(1.+h))*h*(-.2000000000*r^2-0.3333333333e-
1*(2.-1.*r)^2)-.2000000000*h^3*r^2)+0.3333333333e-1*(1.+h)^2*h^2*(2.-
1.*r)^2*x^6+1.917897066*10^(-9)*(1.+h)*h^4*(2.-1.*r)*(-.2000000000*r^2-0.3333333333e-
1*(2.-1.*r)^2)^2-5.540591526*10^(-9)*(-(1.*(1.+h))*h*(-.2000000000*r^2-0.3333333333e-
1*(2.-1.*r)^2)-.2000000000*h^3*r^2)*h^3*r*(-.2000000000*r^2-0.3333333333e-1*(2.-
1.*r)^2)-7.939346623*10^(-11)*(-(1.*(1.+h))*h*(-.2000000000*r^2-0.3333333333e-1*(2.-
1.*r)^2)-.2000000000*h^3*r^2)*h^3*(-.2000000000*r^2-0.3333333333e-1*(2.-
1.*r)^2)^2+8.220223064*10^(-7)*h^4*(h^2*(0.9390024038e-5*(-.2000000000*r^2-
0.3333333333e-1*(2.-1.*r)^2)^2+(0.4340277778e-3*(2.-1.*r))*(-.2000000000*r^2-
0.3333333333e-1*(2.-1.*r)^2)+0.6250000000e-2*(2.-1.*r)^2)-1.*x*h^2*(0.9390024038e-5*(-
.2000000000*r^2-0.3333333333e-1*(2.-1.*r)^2)^2+0.4340277778e-3*r*(-.2000000000*r^2-
0.3333333333e-1*(2.-1.*r)^2)+0.6250000000e-2*r^2))*r*(-.2000000000*r^2-
0.3333333333e-1*(2.-1.*r)^2)+(0.5494505494e-2*((.4444444444*(1.+h))*h^4*(2.-1.*r)*r*(-
.2000000000*r^2-0.3333333333e-1*(2.-1.*r)^2)+(-(1.*(1.+h))*h*(-.2000000000*r^2-
0.3333333333e-1*(2.-1.*r)^2)-.2000000000*h^3*r^2)^2))*x^14+0.1041666667e-
1*h^2*(h^2*(0.9390024038e-5*(-.2000000000*r^2-0.3333333333e-1*(2.-
1.*r)^2)^2+(0.4340277778e-3*(2.-1.*r))*(-.2000000000*r^2-0.3333333333e-1*(2.-
1.*r)^2)+0.6250000000e-2*(2.-1.*r)^2)-1.*x*h^2*(0.9390024038e-5*(-.2000000000*r^2-
0.3333333333e-1*(2.-1.*r)^2)^2+0.4340277778e-3*r*(-.2000000000*r^2-0.3333333333e-
1*(2.-1.*r)^2)+0.6250000000e-2*r^2))*(1.+h)*(2.-1.*r)+(0.3267973856e-
2*((.1538461538*(1.+h))*h^4*(2.-1.*r)*(-.2000000000*r^2-0.3333333333e-1*(2.-1.*r)^2)^2-
(.4444444444*(-(1.*(1.+h))*h*(-.2000000000*r^2-0.3333333333e-1*(2.-1.*r)^2)-
.2000000000*h^3*r^2))*h^3*r*(-.2000000000*r^2-0.3333333333e-1*(2.-
1.*r)^2)))*x^18+(0.2164502165e-2*(-(.1538461538*(-(1.*(1.+h))*h*(-.2000000000*r^2-
0.3333333333e-1*(2.-1.*r)^2)-.2000000000*h^3*r^2))*h^3*(-.2000000000*r^2-
0.3333333333e-1*(2.-1.*r)^2)^2+0.4938271605e-1*h^6*r^2*(-.2000000000*r^2-
0.3333333333e-1*(2.-1.*r)^2)^2))*x^22+7.837560129*10^(-13)*h^6*r*(-.2000000000*r^2-
0.3333333333e-1*(2.-1.*r)^2)^3+x*(1.939051537*10^(-15)*h^6*(-.2000000000*r^2-
0.3333333333e-1*(2.-1.*r)^2)^4+2.911093762*10^(-13)*h^6*(2.-1.*r)*(-.2000000000*r^2-
0.3333333333e-1*(2.-1.*r)^2)^3-2.495223225*10^(-10)*(-(1.*(1.+h))*h*(-.2000000000*r^2-

$\overline{u}_1(x,\alpha)$ = -h*(x^2*(2-r)-(1/5)*x^6*r^2-(1/30)*x^6*(2-r)^2)

$\overline{u}_2(x,\alpha)$ = -(1+h)*h*(x^2*(2-r)-(1/5)*x^6*r^2-(1/30)*x^6*(2-r)^2)-h*(h^2*(0.9390024038e-5*(-(1/5)*r^2-
(1/30)*(2-r)^2)^2+(0.4340277778e-3*(2-r))*(-(1/5)*r^2-(1/30)*(2-r)^2)+0.6250000000e-2*(2-
r)^2)+x*h^2*((1/13)*(-(1/5)*r^2-(1/30)*(2-r)^2)^2*x^13+(2/9)*r*(-(1/5)*r^2-(1/30)*(2-
r)^2)*x^9+(1/5)*x^5*r^2)-x*h^2*(0.9390024038e-5*(-(1/5)*r^2-(1/30)*(2-r)^2)^2+0.4340277778e-3*r*(-
(1/5)*r^2-(1/30)*(2-r)^2)+0.6250000000e-2*r^2))

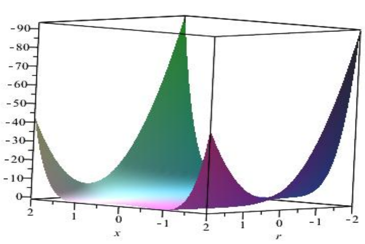
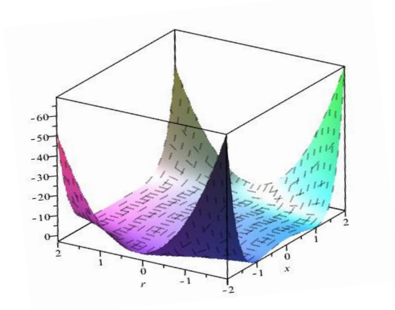
$\overline{u}_3(x,\alpha)$ = (1+h)*(-(1+h)*h*(x^2*(2-r)-(1/5)*x^6*r^2-(1/30)*x^6*(2-r)^2)-
h*(h^2*(0.9390024038e-5*(-(1/5)*r^2-(1/30)*(2-r)^2)^2+(0.4340277778e-3*(2-r))*(-
(1/5)*r^2-(1/30)*(2-r)^2)+0.6250000000e-2*(2-r)^2)+x*h^2*((1/13)*(-(1/5)*r^2-
(1/30)*(2-r)^2)^2*x^13+(2/9)*r*(-(1/5)*r^2-(1/30)*(2-r)^2)*x^9+(1/5)*x^5*r^2)-

x*h^2*(0.9390024038e-5*(-(1/5)*r^2-(1/30)*(2-r)^2)^2+0.4340277778e-3*r*(-(1/5)*r^2-(1/30)*(2-r)^2)+0.6250000000e-2*r^2)))-h*(0.6801333060e-5*h^6*(-.2000000000*r^2-0.3333333333e-1*(2.-1.*r)^2)^4*t^30+0.5259697569e-4*h^6*r*(-.2000000000*r^2-0.3333333333e-1*(2.-1.*r)^2)^3*t^26+(0.2164502165e-2*(-(.1538461538*(-(1.*(1.+h))*h*(-.2000000000*r^2-0.3333333333e-1*(2.-1.*r)^2)-.2000000000*h^3*r^2))*h^3*(-.2000000000*r^2-0.3333333333e-1*(2.-1.*r)^2)^2+0.4938271605e-1*h^6*r^2*(-.2000000000*r^2-0.3333333333e-1*(2.-1.*r)^2)^2))*t^22+(0.3267973856e-2*((.1538461538*(1.+h))*h^4*(2.-1.*r)*(-.2000000000*r^2-0.3333333333e-1*(2.-1.*r)^2)^2-(.4444444444*(-(1.*(1.+h))*h*(-.2000000000*r^2-0.3333333333e-1*(2.-1.*r)^2)-.2000000000*h^3*r^2))*h^3*r*(-.2000000000*r^2-0.3333333333e-1*(2.-1.*r)^2)))*t^18+0.6410256412e-3*h^4*(h^2*(0.9390024038e-5*(-.2000000000*r^2-0.3333333333e-1*(2.-1.*r)^2)^2+(0.4340277778e-3*(2.-1.*r))*(-.2000000000*r^2-0.3333333333e-1*(2.-1.*r)^2)+0.6250000000e-2*(2.-1.*r)^2)-1.*x*h^2*(0.9390024038e-5*(-.2000000000*r^2-0.3333333333e-1*(2.-1.*r)^2)^2+0.4340277778e-3*r*(-.2000000000*r^2-0.3333333333e-1*(2.-1.*r)^2)+0.6250000000e-2*r^2))*(-.2000000000*r^2-0.3333333333e-1*(2.-1.*r)^2)^2*t^16+(0.5494505494e-2*((.4444444444*(1.+h))*h^4*(2.-1.*r)*r*(-.2000000000*r^2-0.3333333333e-1*(2.-1.*r)^2)+(-(1.*(1.+h))*h*(-.2000000000*r^2-0.3333333333e-1*(2.-1.*r)^2)-.2000000000*h^3*r^2)^2))*t^14+0.3367003367e-2*h^4*(h^2*(0.9390024038e-5*(-.2000000000*r^2-0.3333333333e-1*(2.-1.*r)^2)^2+(0.4340277778e-3*(2.-1.*r))*(-.2000000000*r^2-0.3333333333e-1*(2.-1.*r)^2)+0.6250000000e-2*(2.-1.*r)^2)-1.*x*h^2*(0.9390024038e-5*(-.2000000000*r^2-0.3333333333e-1*(2.-1.*r)^2)^2+0.4340277778e-3*r*(-.2000000000*r^2-0.3333333333e-1*(2.-1.*r)^2)+0.6250000000e-2*r^2))*r*(-.2000000000*r^2-0.3333333333e-1*(2.-1.*r)^2)*t^12-(0.2222222222e-1*(1.+h))*h*(2.-1.*r)*(-(1.*(1.+h))*h*(-.2000000000*r^2-0.3333333333e-1*(2.-1.*r)^2)-.2000000000*h^3*r^2)*t^10-0.3571428571e-1*h*(h^2*(0.9390024038e-5*(-.2000000000*r^2-0.3333333333e-1*(2.-1.*r)^2)^2+(0.4340277778e-3*(2.-1.*r))*(-.2000000000*r^2-0.3333333333e-1*(2.-1.*r)^2)+0.6250000000e-2*(2.-1.*r)^2)-1.*x*h^2*(0.9390024038e-5*(-.2000000000*r^2-0.3333333333e-1*(2.-1.*r)^2)^2+0.4340277778e-3*r*(-.2000000000*r^2-0.3333333333e-1*(2.-1.*r)^2)+0.6250000000e-2*r^2))*(-(1.*(1.+h))*h*(-.2000000000*r^2-0.3333333333e-1*(2.-1.*r)^2)-.2000000000*h^3*r^2)*t^8+0.3333333333e-1*(1.+h)^2*h^2*(2.-1.*r)^2*t^6+.1666666667*h^2*(h^2*(0.9390024038e-5*(-.2000000000*r^2-0.3333333333e-1*(2.-1.*r)^2)^2+(0.4340277778e-3*(2.-1.*r))*(-.2000000000*r^2-0.3333333333e-1*(2.-1.*r)^2)+0.6250000000e-2*(2.-1.*r)^2)-1.*x*h^2*(0.9390024038e-5*(-.2000000000*r^2-0.3333333333e-1*(2.-1.*r)^2)^2+0.4340277778e-3*r*(-.2000000000*r^2-0.3333333333e-1*(2.-1.*r)^2)+0.6250000000e-2*r^2))*(1.+h)*(2.-

$\overline{U}(x,\alpha)$ =-h*(x^2*(2-r)-(1/5)*x^6*r^2-(1/30)*x^6*(2-r)^2)-(1+h)*h*(x^2*(2-r)-(1/5)*x^6*r^2-(1/30)*x^6*(2-r)^2)+h*(h^2*(0.9390024038e-5*((1/5)*r^2-(1/30)*(2-r)^2)^2+(0.4340277778e-3*(2-r))*((1/5)*r^2-(1/30)*(2-r)^2)+0.6250000000e-2*(2-r)^2)+x*h^2*((1/13)*((1/5)*r^2-(1/30)*(2-r)^2)^2*x^13+(2/9)*r*((1/5)*r^2-(1/30)*(2-r)^2)*x^9+(1/5)*x^5*r^2)+x*h^2*(0.9390024038e-5*((1/5)*r^2-(1/30)*(2-r)^2)^2+0.4340277778e-3*r*((1/5)*r^2-(1/30)*(2-r)^2)+0.6250000000e-2*r^2))+(1+h)*(-(1+h)*h*(x^2*(2-r)-(1/5)*x^6*r^2-(1/30)*x^6*(2-r)^2)+h*(h^2*(0.9390024038e-5*((1/5)*r^2-(1/30)*(2-r)^2)^2+(0.4340277778e-3*(2-r))*(-(1/5)*r^2-(1/30)*(2-r)^2)+0.6250000000e-2*(2-r)^2)+x*h^2*((1/13)*(-(1/5)*r^2-(1/30)*(2-r)^2)^2*x^13+(2/9)*r*(-(1/5)*r^2-(1/30)*(2-r)^2)*x^9+(1/5)*x^5*r^2)+x*h^2*(0.9390024038e-5*(-(1/5)*r^2-(1/30)*(2-r)^2)^2+0.4340277778e-3*r*(-(1/5)*r^2-(1/30)*(2-r)^2)+0.6250000000e-2*r^2)))+h*(0.6801333060e-5*h^6*(-.2000000000*r^2-0.3333333333e-1*(2.-

1.*r)^2)^4*x^30+0.5259697569e-4*h^6*r*(-.2000000000*r^2-0.3333333333e-1*(2.-1.*r)^2)^3*x^26+(0.2164502165e-2*(-(.1538461538*(-(1.*(1.+h))*h*(-.2000000000*r^2-0.3333333333e-1*(2.-1.*r)^2)-.2000000000*h^3*r^2))*h^3*(-.2000000000*r^2-0.3333333333e-1*(2.-1.*r)^2)^2+0.4938271605e-1*h^6*r^2*(-.2000000000*r^2-0.3333333333e-1*(2.-1.*r)^2)^2))*x^22+(0.3267973856e-2*((.1538461538*(1.+h))*h^4*(2.-1.*r)*(-.2000000000*r^2-0.3333333333e-1*(2.-1.*r)^2)^2-(.4444444444*(-(1.*(1.+h))*h*(-.2000000000*r^2-0.3333333333e-1*(2.-1.*r)^2)-.2000000000*h^3*r^2))*h^3*r*(-.2000000000*r^2-0.3333333333e-1*(2.-1.*r)^2)))*x^18+0.6410256412e-3*h^4*(h^2*(0.9390024038e-5*(-.2000000000*r^2-0.3333333333e-1*(2.-1.*r)^2)^2+(0.4340277778e-3*(2.-1.*r))*(-.2000000000*r^2-0.3333333333e-1*(2.-1.*r)^2)+0.6250000000e-2*(2.-1.*r)^2)-1.*x*h^2*(0.9390024038e-5*(-.2000000000*r^2-0.3333333333e-1*(2.-1.*r)^2)^2+0.4340277778e-3*r*(-.2000000000*r^2-0.3333333333e-1*(2.-1.*r)^2)+0.6250000000e-2*r^2))*(-.2000000000*r^2-0.3333333333e-1*(2.-1.*r)^2)^2*x^16+(0.5494505494e-2*((.4444444444*(1.+h))*h^4*(2.-1.*r)*r*(-.2000000000*r^2-0.3333333333e-1*(2.-1.*r)^2)+(-(1.*(1.+h))*h*(-.2000000000*r^2-0.3333333333e-1*(2.-1.*r)^2)-.2000000000*h^3*r^2)^2))*x^14+0.3367003367e-2*h^4*(h^2*(0.9390024038e-5*(-.2000000000*r^2-0.3333333333e-1*(2.-1.*r)^2)^2+(0.4340277778e-3*(2.-1.*r))*(-.2000000000*r^2-0.3333333333e-1*(2.-1.*r)^2)+0.6250000000e-2*(2.-1.*r)^2)-1.*x*h^2*(0.9390024038e-5*(-.2000000000*r^2-0.3333333333e-1*(2.-1.*r)^2)^2+0.4340277778e-3*r*(-.2000000000*r^2-0.3333333333e-1*(2.-1.*r)^2)+0.6250000000e-2*r^2))*r*(-.2000000000*r^2-0.3333333333e-1*(2.-1.*r)^2)*x^12-(0.2222222222e-1*(1.+h))*h*(2.-1.*r)*(-(1.*(1.+h))*h*(-.2000000000*r^2-0.3333333333e-1*(2.-1.*r)^2)-.2000000000*h^3*r^2)*x^10-0.3571428571e-1*h*(h^2*(0.9390024038e-5*(-.2000000000*r^2-0.3333333333e-1*(2.-1.*r)^2)^2+(0.4340277778e-3*(2.-1.*r))*(-.2000000000*r^2-0.3333333333e-1*(2.-1.*r)^2)+0.6250000000e-2*(2.-1.*r)^2)-1.*x*h^2*(0.9390024038e-5*(-.2000000000*r^2-0.3333333333e-1*(2.-1.*r)^2)^2+0.4340277778e-3*r*(-.2000000000*r^2-0.3333333333e-1*(2.-1.*r)^2)+0.6250000000e-2*r^2))*(-(1.*(1.+h))*h*(-.2000000000*r^2-0.3333333333e-1*(2.-1.*r)^2)-.2000000000*h^3*r^2)*x^8+0.3333333333e-1*(1.+h)^2*h^2*(2.-1.*r)^2*x^6+.1666666667*h^2*(h^2*(0.9390024038e-5*(-.2000000000*r^2-0.3333333333e-1*(2.-1.*r)^2)^2+(0.4340277778e-3*(2.-1.*r))*(-.2000000000*r^2-0.3333333333e-1*(2.-1.*r)^2)+0.6250000000e-2*(2.-1.*r)^2)-1.*x*h^2*(0.9390024038e-5*(-.2000000000*r^2-0.3333333333e-1*(2.-1.*r)^2)^2+0.4340277778e-3*r*(-.2000000000*r^2-0.3333333333e-1*(2.-1.*r)^2)+0.6250000000e-2*r^2))*(1.+h)*(2.-1.*r)*x^4+.5000000000*h^2*(h^2*(0.9390024038e-5*(-.2000000000*r^2-0.3333333333e-1*(2.-1.*r)^2)^2+(0.4340277778e-3*(2.-1.*r))*(-.2000000000*r^2-0.3333333333e-

Fig(1a)Exact solution $\underline{u}(x,\alpha)$ Fig(1b) Approximate solution $u(x,\alpha)$

Fig(2a)Exact solution $\overline{u}(x,\alpha)$ Fig(2b) Approximate solution $\overline{u}(x,\alpha)$

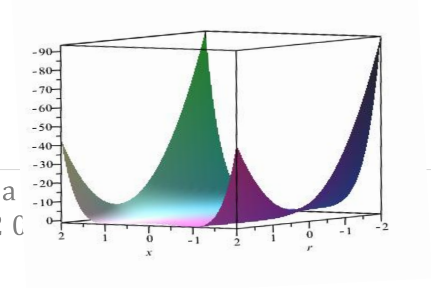
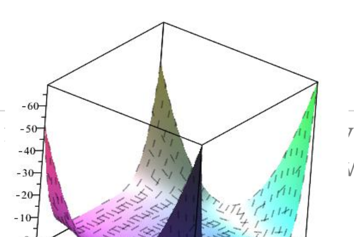



Table (1) computation between the exact and HAM $h = -1$ in formula 2 and determine the absolute error

x	$exact\ \underline{u}(x,r)$	$HAM\underline{u}(x,r)$ $\alpha = 0.1$	$exact\ \underline{u}(x,r)$	$HAM\underline{u}(x,r)$ $\alpha = 0.5$	$exact\ \underline{u}(x,r)$	$\mathrm{HAM}\underline{u}(x,r)$ $\alpha = 1$
0	0.00000	0.0000000	0.00000	0.00000000	0.00000	0.000000000
0.2	0.04000	0.0477480	0.02000	0.02082064	0.04000	0.039294998
0.4	0.01600	0.0167550	0.08000	0.07969799	0.16000	0.1569610105
0.6	0.03600	0.0263958	0.18000	0.17088774	0.36000	0.3544127825
0.8	0.06400	0.0594177	0.32000	0.26787734	0.64000	0.5662234183
1	0.10000	0.1360818	0.50000	0.48953632	1.00000	0.9882424058

x	$exact\ \overline{u}(x,r)$	$HAM\overline{u}(x,r)$ $\alpha = 0.1$	$exact\ \overline{u}(x,r)$	$HAM\overline{u}(x,r)$ $\alpha = 0.5$	$exact\overline{u}(x,r)$	$HAM\overline{u}(x,r)$ $\alpha = 1$
0	0.00000	0.000000000	0.0000	0.0000000000	0.0000	0.000000000
0.2	0.07600	0.068505806	0.0600	0.0576338330	0.0400	0.035410480
0.4	0.30400	0.280978546	0.2400	0.2260091416	0.1600	0.151048729
0.6	0.68400	0.655780340	0.5400	0.5168401528	0.3600	0.358551838
0.8	1.12160	1.161509992	0.9600	0.8996547609	0.6400	0.639295323
1	1.90000	1.875435586	1.5000	1.5511618405	1.0000	0.999216891

Example 2

Now we discus this example

We will discuss Now the case of formula 2 , $\underline{k\left(x,t,F_{1\alpha}\left(t,u(t,\alpha)\right)\right)} \geq 0$, $\overline{G\left(t,\int_a^t F_{2\alpha}\left(t,s,u(s,\alpha)\right)ds\right)} \geq 0$

$0 \leq t \leq \frac{1}{2}$, and $\underline{k\left(x,t,F_{1\alpha}\left(t,u(t,\alpha)\right)\right)} < 0$, $\overline{G\left(t,\int_a^t F_{2\alpha}\left(t,s,U(s,\alpha)\right)ds\right)} < 0$, $1/2 \leq t \leq x$, c=1/2

$$\underline{u}(x,\alpha) = \underline{f}(x,\alpha) + \lambda \int_a^c k\left(x,t,\underline{F}_{1\alpha}\left(t,\underline{u}(t,\alpha)\right)\right)dt + \lambda \int_c^x k(x,t,G\left(t,\int_a^t \overline{F}_{2\alpha}\left(t,s,\overline{u}(s,\alpha)\right)ds\right)dt$$

$$\overline{u}(x,\alpha) = \overline{f}(x,\alpha) + \lambda \int_a^c k(x,t,G\left(t,\int_a^t \overline{F}_{2\alpha}\left(t,s,\overline{u}(s,\alpha)\right)ds\right)dt + \lambda \int_c^x k\left(x,t,\underline{F}_{1\alpha}\left(t,\underline{u}(t,\alpha)\right)\right)dt$$

Where

$$k\left(x,t,\underline{F}_{1\alpha}\left(t,\underline{u}(t,\alpha)\right)\right) = xt(\underline{u}(x,\alpha))^2$$

$$G\left(t,\int_a^t \overline{F}_{2\alpha}\left(t,s,\overline{u}(s,\alpha)\right)ds\right) = xt.\int_a^t (\overline{u}\,(s,\alpha))^{2}ds$$

$$\underline{f}(x,\alpha) = \left(\frac{5}{4}-\frac{1}{4}r\right)e^{x} - 0.05760255714r^{2} + 0.5135255714r - 1.377563928 + \frac{1}{64}x^{2}r^{2} + \frac{3}{32}x^{2}r + \frac{9}{64}x^{2} - \frac{1}{64}e^{2x}r^{2}x + \frac{1}{128}e^{2x}r^{2} - \frac{3}{32}e^{2x}rx + \frac{3}{64}e^{2x}r - \frac{9}{64}xe^{2x} + \frac{9}{128}e^{2x}$$

$$\overline{f}(x,\alpha) = \left(\frac{3}{4}+\frac{1}{4}r\right)e^{x} + 2.088501429 + 0.08104005716r^{2} - 0.8729005716r - \frac{1}{32}e^{2x}r^{2} + \frac{5}{16}e^{2x}r - \frac{25}{32}e^{2x}$$

the intimal condition is $\underline{u}_0(x,\alpha) = \overline{u}_0(x,\alpha) = 0$ $, and\ \ 0 \leq \alpha \leq 1$

the exact solution $\underline{u}(x,\alpha) = \left(\frac{5}{4}-\frac{r}{4}\right)ep(x)\ and\ \overline{u}(x,\alpha) = (\frac{3}{4}+\frac{r}{4})exp(x)$

$$\underline{u}_1(x,\alpha) = -h\left(\left(\frac{5}{4}-\frac{1}{4}r\right)e^{x} - 0.05760255714r^{2} + 0.5135255714r - 1.377563928 + \frac{1}{64}x^{2}r^{2} + \frac{3}{32}x^{2}r + \frac{9}{64}x^{2} - \frac{1}{64}e^{2x}r^{2}x + \frac{1}{128}e^{2x}r^{2} - \frac{3}{32}e^{2x}rx + \frac{3}{64}e^{2x}r - \frac{9}{64}xe^{2x} + \frac{9}{128}e^{2x}\right)$$

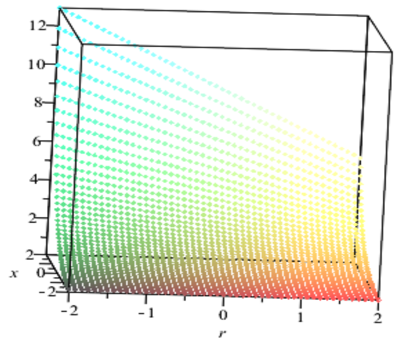
$$\overline{u}_1(x,\alpha) = -h\left(\left(\frac{3}{4}+\frac{1}{4}r\right)e^x + 2.088501429 + 0.08104005716r^2 - 0.8729005716r - \frac{1}{32}e^{2x}r^2 + \frac{5}{16}e^{2x}r - \frac{25}{32}e^{2x}\right)$$

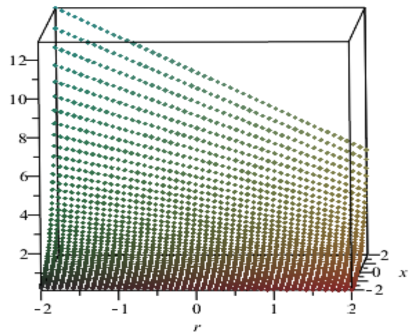
$\underline{u}_2(x, \alpha)$

$$
\begin{aligned}
=&\\
&-h\left(\left(\frac{5}{4}-\frac{1}{4}r\right)e^{x}-0.05760255714\,r^{2}+0.5135255714\,r-1.377563928+\frac{1}{64}x^{2}r^{2}\right.\\
&+\frac{3}{32}x^{2}r+\frac{9}{64}x^{2}-\frac{1}{64}e^{2x}r^{2}x+\frac{1}{128}e^{2x}r^{2}-\frac{3}{32}e^{2x}rx+\frac{3}{64}e^{2x}r-\frac{9}{64}xe^{2x}\\
&\left.+\frac{9}{128}e^{2x}\right)-(1+h)\cdot\left(h\cdot\left(\left(\frac{5}{4}-\frac{r}{4}\right)\cdot\exp(x)-0.05369630714\,r^{2}+0.5369630714\,r\right.\right.\\
&-1.342407678-0.03515625000-0.003906250000\,r^{2}-0.02343750000\,r+\frac{1}{64}x^{2}r^{2}\\
&+\frac{3}{32}x^{2}r+\frac{9}{64}x^{2}-\frac{1}{64}e^{2x}r^{2}x+\frac{1}{128}e^{2x}r^{2}-\frac{3}{32}e^{2x}rx+\frac{3}{64}e^{2x}r-\frac{9}{64}xe^{2x}\\
&\left.\left.+\frac{9}{128}e^{2x}\right)\right)-h\cdot((0.07219843022\,h^{4}+0.1443968604\,h^{3}+0.001285261232\,h^{4}r^{4}\\
&-0.01163777838\,h^{4}r^{3}+0.002570522464\,h^{3}r^{4}+0.01223394163\,h^{4}r^{2}\\
&-0.02327555675\,h^{3}r^{3}+0.001285261232\,h^{2}r^{4}+0.05885643660\,h^{4}r\\
&+0.02446788327\,h^{3}r^{2}-0.01163777838\,h^{2}r^{3}+0.1177128732\,h^{3}r\\
&+0.01223394163\,h^{2}r^{2}+0.05885643660\,h^{2}r+0.07219843022\,h^{2})-(\\
&-0.5804907395\,h^{4}rx^{2}-1.160981480\,h^{3}rx^{2}+0.001144180580\,h^{4}r^{4}x^{2}\\
&+0.002288361161\,h^{3}r^{4}x^{2}-0.04151602174\,h^{4}r^{3}x^{2}-0.08303204345\,h^{3}r^{3}x^{2}\\
&+0.3059602080\,h^{4}r^{2}x^{2}+0.6119204160\,h^{3}r^{2}x^{2}-0.5804907395\,h^{2}rx^{2}\\
&+0.001144180580\,h^{2}r^{4}x^{2}-0.04151602174\,h^{2}r^{3}x^{2}+0.3059602080\,h^{2}r^{2}x^{2}\\
&-1.215369394\,h^{4}rx^{3}-2.430738788\,h^{3}rx^{3}+0.002189163621\,h^{4}r^{4}x^{3}\\
&+0.004378327243\,h^{3}r^{4}x^{3}-0.04715994147\,h^{4}r^{3}x^{3}-0.09431988297\,h^{3}r^{3}x^{3}\\
&+0.3668199860\,h^{4}r^{2}x^{3}+0.7336399723\,h^{3}r^{2}x^{3}-1.215369394\,h^{2}rx^{3}\\
&+0.002189163621\,h^{2}r^{4}x^{3}-0.04715994147\,h^{2}r^{3}x^{3}+0.3668199860\,h^{2}r^{2}x^{3}\\
&-0.04052002858\,e^{x}h^{2}r^{3}+0.3148902001\,e^{x}h^{2}r^{2}+0.2651001429\,e^{x}h^{2}r\\
&-0.04052002858\,e^{x}h^{4}r^{3}-0.08104005716\,e^{x}h^{3}r^{3}+0.3148902001\,e^{x}h^{4}r^{2}\\
&+0.6297804001\,e^{x}h^{3}r^{2}+0.2651001429\,e^{x}h^{4}r+0.5302002858\,e^{x}h^{3}r\\
&+3.132752144\,e^{x}h^{2}x+3.132752144\,e^{x}h^{4}x+6.265504287\,e^{x}h^{3}x-0.7721616465\,h^{2}x^{2}\\
&-0.7721616465\,h^{4}x^{2}-1.544323292\,h^{3}x^{2}+1.453946073\,h^{2}x^{3}+1.453946073\,h^{4}x^{3}\\
&+2.907892146\,h^{3}x^{3}+0.04052002858\,e^{x}h^{2}r^{3}x-0.3148902001\,e^{x}h^{2}r^{2}x\\
&-0.2651001429\,e^{x}h^{2}rx+0.04052002858\,e^{x}h^{4}r^{3}x+0.08104005716\,e^{x}h^{3}r^{3}x\\
&-0.3148902001\,e^{x}h^{4}r^{2}x-0.6297804001\,e^{x}h^{3}r^{2}x-0.2651001429\,e^{x}h^{4}rx\\
&-0.5302002858\,e^{x}h^{3}rx-0.3805275670\,h^{2}re^{2\cdot x}-0.00001525878906\,h^{2}r^{4}e^{4\cdot x}\\
&+0.001157407408\,h^{3}r^{3}e^{3\cdot x}+0.0003051757812\,h^{4}r^{3}e^{4\cdot x}+0.0006103515625\,h^{3}r^{3}e^{4\cdot x}\\
&+0.0006331254465\,h^{4}r^{4}e^{2\cdot x}-0.002288818359\,h^{4}r^{2}e^{4\cdot x}+0.007629394531\,h^{2}re^{4\cdot x}\\
&-0.002893518519\,h^{4}re^{3\cdot x}-0.005787037037\,h^{3}re^{3\cdot x}-0.004577636719\,h^{3}r^{2}e^{4\cdot x}\\
&-0.01315079018\,h^{4}r^{3}e^{2\cdot x}-0.02630158035\,h^{3}r^{3}e^{2\cdot x}+0.001266250893\,h^{3}r^{4}e^{2\cdot x}\\
&-0.004050925926\,h^{4}r^{2}e^{3\cdot x}-0.008101851852\,h^{3}r^{2}e^{3\cdot x}+0.0005787037037\,h^{2}r^{3}e^{3\cdot x}\\
&+0.0003051757812\,h^{2}r^{3}e^{4\cdot x}+0.09252741072\,h^{2}r^{2}e^{2\cdot x}-0.3805275670\,h^{4}re^{2\cdot x}\\
&-0.7610551340\,h^{3}re^{2\cdot x}-0.00001525878906\,h^{4}r^{4}e^{4\cdot x}-0.00003051757812\,h^{3}r^{4}e^{4\cdot x}\\
&+0.0005787037037\,h^{4}r^{3}e^{3\cdot x}+0.007629394531\,h^{4}re^{4\cdot x}+0.01525878906\,h^{3}re^{4\cdot x}\\
&+0.09252741072\,h^{4}r^{2}e^{2\cdot x}+0.1850548215\,h^{3}r^{2}e^{2\cdot x}-0.002893518519\,h^{2}re^{3\cdot x}\\
&+0.0006331254465\,h^{2}r^{4}e^{2\cdot x}-0.004050925926\,h^{2}r^{2}e^{3\cdot x}-0.002288818359\,h^{2}r^{2}e^{4\cdot x}\\
&-0.01315079018\,h^{2}r^{3}e^{2\cdot x}-0.1302083333\,h^{2}xe^{3\cdot x}+0.07629394530\,h^{3}xe^{4\cdot x}\\
&-0.6751958705\,h^{2}xe^{2\cdot x}-0.2604166667\,h^{3}xe^{3\cdot x}+0.03814697265\,h^{4}xe^{4\cdot x}\\
&-0.1302083333\,h^{4}xe^{3\cdot x}+0.03814697265\,h^{2}xe^{4\cdot x}-0.6751958705\,h^{4}xe^{2\cdot x}\\
&-1.350391742\,h^{3}xe^{2\cdot x}+1.522110268\,h^{3}rxe^{2\cdot x}+0.00006103515625\,h^{4}r^{4}xe^{4\cdot x}\\
&+0.0001220703125\,h^{3}r^{4}xe^{4\cdot x}-0.001736111111\,h^{4}r^{3}xe^{3\cdot x}\\
&-0.03051757812\,h^{4}rxe^{4\cdot x}-0.06103515625\,h^{3}rxe^{4\cdot x}-0.1850548214\,h^{4}r^{2}xe^{2\cdot x}\\
&-0.3701096430\,h^{3}r^{2}xe^{2\cdot x}+0.008680555557\,h^{2}rxe^{3\cdot x}-0.001266250893\,h^{2}r^{4}xe^{2\cdot x}\\
&+0.01215277778\,h^{2}r^{2}xe^{3\cdot x}+0.009155273438\,h^{2}r^{2}xe^{4\cdot x}+0.02630158036\,h^{2}r^{3}xe^{2\cdot x}\\
&+0.7610551340\,h^{2}rxe^{2\cdot x}+0.00006103515625\,h^{2}r^{4}xe^{4\cdot x}\\
&-0.003472222223\,h^{3}r^{3}xe^{3\cdot x}-0.001220703125\,h^{4}r^{3}xe^{4\cdot x}\\
&-0.002441406250\,h^{3}r^{3}xe^{4\cdot x}-0.001266250893\,h^{4}r^{4}xe^{2\cdot x}\\
&+0.009155273438\,h^{4}r^{2}xe^{4\cdot x}-0.03051757812\,h^{2}rxe^{4\cdot x}+0.008680555557\,h^{4}rxe^{3\cdot x}\\
&+0.01736111111\,h^{3}rxe^{3\cdot x}+0.01831054688\,h^{3}r^{2}xe^{4\cdot x}+0.02630158036\,h^{4}r^{3}xe^{2\cdot x}\\
&+0.05260316070\,h^{3}r^{3}xe^{2\cdot x}-0.002532501786\,h^{3}r^{4}xe^{2\cdot x}+0.01215277778\,h^{4}r^{2}xe^{3\cdot x}\\
&+0.02430555556\,h^{3}r^{2}xe^{3\cdot x}-0.001736111111\,h^{2}r^{3}xe^{3\cdot x}-0.001220703125\,h^{2}r^{3}xe^{4\cdot x}\\
&-0.1850548214\,h^{2}r^{2}xe^{2\cdot x}+0.7610551340\,h^{4}rxe^{2\cdot x}+2.620606047\,h^{4}\\
&+5.241212094\,h^{3}+0.6751958708\,h^{3}e^{2\cdot x}+0.04340277778\,h^{2}e^{3\cdot x}\\
&-0.01907348632\,h^{3}e^{4\cdot x}+0.3375979352\,h^{2}e^{2\cdot x}+0.08680555556\,h^{3}e^{3\cdot x}\\
&-0.009536743162\,h^{4}e^{4\cdot x}+0.04340277778\,h^{4}e^{3\cdot x}-0.009536743162\,h^{2}e^{4\cdot x}\\
&+0.3375979352\,h^{4}e^{2\cdot x}-0.0006724386462\,h^{4}r^{4}+0.05322886062\,h^{4}r^{3}\\
&-0.001344877292\,h^{3}r^{4}-0.4079153381\,h^{4}r^{2}+0.1064577212\,h^{3}r^{3}\\
&-0.0006724386462\,h^{2}r^{4}+0.1283958359\,h^{4}r-0.8158306763\,h^{3}r^{2}\\
&+0.05322886062\,h^{2}r^{3}+0.2567916720\,h^{3}r-0.4079153381\,h^{2}r^{2}+0.1283958359\,h^{2}r\\
&-3.132752144\,e^{x}h^{2}-3.132752144\,e^{x}h^{4}-6.265504287\,e^{x}h^{3}+2.620606047\,h^{2}))
\end{aligned}
$$

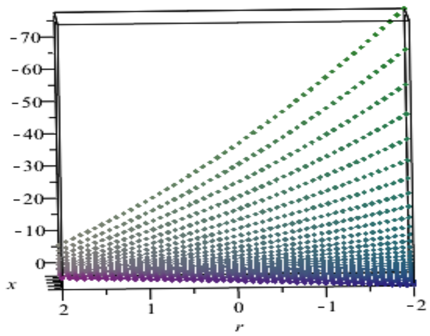
$$
\begin{aligned}
&\overline{u}_2(x,\alpha) = \\
&-h\cdot\left(\left(\frac{3}{4}+\frac{r}{4}\right)\cdot\exp(x) - 0.03515625000 - 0.003906250000\,r^2 - 0.02343750000\,r\right.\\
&\quad + 2.123657679 - 0.8494630716\,r + 0.08494630716\,r^2 - \frac{1}{32}\,e^{2x} r^2 + \frac{5}{16}\,e^{2x} r\\
&\quad \left.-\frac{25}{32}\,e^{2x}\right) + (1+h)\cdot\left(-h\cdot\left(\left(\frac{3}{4}+\frac{r}{4}\right)\cdot\exp(x) - 0.03515625000 - 0.003906250000\,r^2\right.\right.\\
&\quad - 0.02343750000\,r + 2.123657679 - 0.8494630716\,r + 0.08494630716\,r^2 - \frac{1}{32}\,e^{2x} r^2\\
&\quad \left.\left.+\frac{5}{16}\,e^{2x} r - \frac{25}{32}\,e^{2x}\right)\right) - h\big(0.00005457198874\,h^2 r^4 - 0.0004419213461\,h^2 r^3\\
&\quad + 0.006837471598\,h^2 r^2 - 0.01770428784\,h^2 r + 0.1406821266\,h^2\\
&\quad + 0.00004882812500\,h^2 r^4 x^5 + 0.0005859375000\,h^2 r^3 x^5 + 0.002636718750\,h^2 r^2 x^5\\
&\quad + 0.005273437500\,h^2 r x^5 + 0.003318054589\,h^2 r^4 x - 0.05916077214\,h^2 r^3 x\\
&\quad + 0.4224109222\,h^2 x r^2 - 1.414828607\,h^2 x r + 0.01234548423\,h^2 r^2 x^3\\
&\quad - 0.0006000266369\,h^2 r^4 x^3 + 0.001749064881\,h^2 r^3 x^3 - 0.03795472318\,h^2 r x^3\\
&\quad + 0.01317627857\,h^2 e^x r^3 + 2.300720892\,h^2 e^x r + 0.3515625000\,h^2 e^x x^2\\
&\quad - 0.7031250000\,h^2 x e^x - 0.4163941786\,h^2 e^x r^2 + 0.6073054632\,h^2 e^{2.x}\\
&\quad + 0.003089904785\,h^2 e^{4.x} + 0.09765625000\,h^2 e^{3.x} + 0.00003814697266\,h^2 e^{4.x} r^4\\
&\quad + 0.0004577636719\,h^2 e^{4.x} r^3 - 0.0006558993303\,h^2 r^4 e^{2.x} + 0.005553284821\,h^2 r^3 e^{2.x}\\
&\quad - 0.002170138889\,h^2 e^{3.x} r^3 - 0.002170138889\,h^2 e^{3.x} r^2 + 0.04557291667\,h^2 e^{3.x} r\\
&\quad + 0.06295182010\,h^2 e^{2.x} r^2 - 0.3430648973\,h^2 e^{2.x} r + 0.004943847656\,h^2 x^2 e^{4.x}\\
&\quad - 0.007415771484\,h^2 x e^{4.x} - 0.01977539062\,h^2 x^3 e^{2.x} + 0.03955078125\,h^2 x^2 e^{2.x}\\
&\quad - 0.1171875000\,h^2 x e^{3.x} + 0.1541691461\,h^2 x e^{2.x} + 0.002059936523\,h^2 e^{4.x} r^2\\
&\quad + 0.004119873047\,h^2 e^{4.x} r - 2.740784820\,h^2 e^x + 1.897682376\,h^2 x\\
&\quad + 0.003955078125\,h^2 x^5 - 0.1291466182\,h^2 x^3 + 0.1640625000\,h^2 e^x r x^2\\
&\quad - 0.3281250000\,h^2 e^x r x - 0.007812500000\,h^2 e^x r^3 x^2 + 0.01562500000\,h^2 e^x r^3 x\\
&\quad - 0.007812500000\,h^2 e^x r^2 x^2 + 0.01562500000\,h^2 e^x r^2 x + 0.002604166667\,h^2 r^2 x e^{3.x}\\
&\quad - 0.05468750000\,h^2 r x e^{3.x} + 0.002604166667\,h^2 r^3 x e^{3.x}\\
&\quad + 0.0004117587053\,h^2 r^4 x e^{2.x} - 0.008482972321\,h^2 r^3 x e^{2.x}\\
&\quad - 0.00009155273438\,h^2 r^4 x e^{4.x} - 0.001098632812\,h^2 r^3 x e^{4.x}\\
&\quad - 0.004943847656\,h^2 r^2 x e^{4.x} - 0.009887695312\,h^2 r x e^{4.x} - 0.04488541385\,h^2 r^2 x e^{2.x}\\
&\quad + 0.004197709772\,h^2 r x e^{2.x} - 0.0002441406250\,h^2 r^4 x^3 e^{2.x}\\
&\quad + 0.0004882812500\,h^2 r^4 x^2 e^{2.x} - 0.002929687500\,h^2 r^3 x^3 e^{2.x}\\
&\quad + 0.005859375000\,h^2 r^3 x^2 e^{2.x} - 0.01318359375\,h^2 r^2 x^3 e^{2.x}\\
&\quad + 0.02636718750\,h^2 r^2 x^2 e^{2.x} + 0.006591796875\,h^2 r x^2 e^{4.x} - 0.02636718750\,h^2 r x^3 e^{2.x}\\
&\quad + 0.05273437500\,h^2 r x^2 e^{2.x} + 0.00006103515625\,h^2 r^4 x^2 e^{4.x}\\
&\quad + 0.0007324218750\,h^2 r^3 x^2 e^{4.x} + 0.003295898438\,h^2 r^2 x^2 e^{4.x} - 0.0006675088745\,h^2 r^4\\
&\quad - 0.005379409797\,h^2 r^3 + 0.3413186192\,h^2 r^2 - 2.066205222\,h^2 r + 1.960534772\,h^2\big)
\end{aligned}
$$

Fig(3a)Exact solution $\underline{u}(x,\alpha)$Fig(3b) Approximate solution $\underline{u}(x,\alpha)$





Fig(4a)Exact solution $\overline{u}(x,\alpha)$Fig(4b) Approximate solution $u(x,\alpha)$



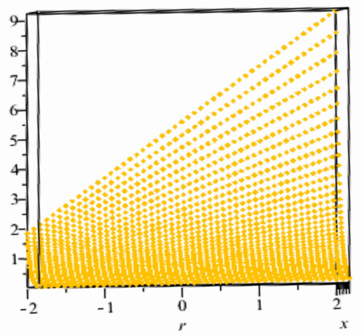


Table (2) computation between the exact and HAM $h=-1$ and determine the absolute error

x	$exact\ \underline{u}(x,r)$	$HAM\underline{u}(x,r)$ $\alpha=0.1$	$exact\ \underline{u}(x,r)$	$HAM\underline{u}(x,r)$ $\alpha=0.5$	$exact\ \underline{u}(x,r)$	$\mathrm{HAM}\underline{u}(x,r)$ $\alpha=1$
0	1.225000	1.2250000	1.125000	1.1250000	1.000000	1.000000000
0.2	1.496218	1.4283208	1.374078	1.3989047	1.221403	1.183280129
0.4	1.827485	1.7853893	1.673028	1.5618896	1.491825	1.465080107
0.6	2.232096	1.9869999	2.049884	1.9880279	1.822119	1.763887097
0.8	2.726287	2.2975165	2.503734	2.0827974	2.225541	2.180635898
1	3.329852	3.2589450	3.058067	2.9980588	2.718282	2.689145141

x	exact $\overline{u}(x,r)$	HAM$\overline{u}(x,r)$ $\alpha = 0.1$	exact $\overline{u}(x,r)$	HAM$\overline{u}(x,r)$ $\alpha = 0.5$	exact$\overline{u}(x,r)$	HAM$\overline{u}(x,r)$ $\alpha = 1$
0	0.775000	0.775000000	0.875000	0.87500000	1.000000	1.00000000
0.2	0.946587	1.006239408	1.068727	1.157998195	1.221403	1.253657499
0.4	1.156164	1.207942563	1.305347	1.342539991	1.491825	1.511335517
0.6	1.412142	1.394015112	1.594354	1.489358232	1.822119	1.785015292
0.8	1.724794	1.662404563	1.947348	1.886852525	2.225541	2.198045673
1	2.106668	1.998260457	2.378497	2,287056234	2.718282	2.667938815

4. CONCLUSION

The proposed method is a powerful procedure for solving new type fuzzy nonlinear Volterra integral equation . The examples analyzed illustrate the ability and reliability of the method presented in this paper and severals that the one is very simple and effective. The obtained solutions , in comparison with the exact solution admit a remarkable accuracy. Results indicate that the convergence rate is very fast, and lower approximations can achieve high accuracy

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