A NOTE ON SOLVABILITY OF FINITE GROUPS

Rola. A. Hijazi

Department of Mathematics, Faculty of Science 14466, King Abdulaziz University, Jeddah 21424, Saudi Arabia

Abstract. Let $G$ be a finite group. A subgroup $H$ of $G$ is said to be $c$-normal in $G$ if there exists a normal subgroup $K$ of $G$ such that $G = HK$ and $H \cap K \leq H_G$, where $H_G$ is the largest normal subgroup of $G$ contained in $H$. In this note we prove that if every Sylow subgroup $P$ of $G$ has a subgroup $D$ such that $1 < |D| < |P|$ and all subgroups $H$ of $P$ with $|H| = |D|$ are $c$-normal ($S$-permutable) in $G$, then $G$ is solvable. This results improve and extend classical and recent results in the literature.

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1 INTRODUCTION

All groups considered in the sequel will be finite. Most of the notation is standard and can be found in Huppert [10].

The relationship between the properties of the Sylow subgroups of a group $G$ and its structure has been investigated by a number of authors. In particular, Gaschütz and Itô [10, p. 436, Satz 5.7] proved that a group $G$ is solvable if all its minimal subgroups are normal (a subgroup of prime order is called a minimal subgroup). Buckley [5] proved that a group of odd order is supersolvable if all its minimal subgroups are normal. Srinivasan [14] got the supersolvability of $G$ under the assumption that the maximal subgroups of all Sylow subgroups are $S$-permutable in $G$ (a subgroup which permutes with all Sylow subgroups of a group $G$ is called $S$-permutable in $G$; see Kegel [11]). Recall that a subgroup $H$ of a group $G$ is said to be $c$-normal in $G$ if there exists a normal subgroup $K$ of $G$ such that $G = HK$ and $H \cap K \leq H_c$, where $H_c = \text{Core}_G(H)$ is the largest normal subgroup of $G$ contained in $H$. This concept was introduced by Wang [15] in 1996 and has been studied extensively by many authors. In fact, Wang extended the above results by proving that a group $G$ is supersolvable when all minimal subgroups and the cyclic subgroups of order 4 are $c$-normal in $G$ or the maximal subgroups of all Sylow subgroups of $G$ are $c$-normal in $G$. In 2000, Ballester-Bolinches et al. [4] introduced the concept of $c$-supplementation of a finite group which is weaker than $c$-normality. A subgroup $H$ of a group $G$ is said to be $c$-supplement in $G$ if there exists a subgroup $K$ of $G$ such that $G = HK$ and $H \cap K \leq H_c$. By using this concept, Ballester-Bolinches et al. [4] proved that a group $G$ is solvable if and only if every Sylow subgroup of $G$ is $c$-supplemented in $G$. Moreover, as applications, they proved that if all minimal subgroups and the cyclic subgroups of order 4 of a group $G$ are $c$-supplemented in $G$, then $G$ is supersolvable. In 2008, Asaad and Ramadan [2] dropped the assumption that every cyclic subgroup of order 4 is $c$-supplemented in $G$ and proved that: If every minimal subgroup of $G$ is $c$-supplemented in $G$, then $G$ is solvable. In 2012, Asaad [1] achieved interesting results about the structure of the group $G$ when certain subgroups of prime power orders are $c$-supplemented in $G$. In 2014, Heliel [9] continued the above mentioned studies and obtained results improved and generalized the results of Hall [7-8], Ballester-Bolinches and Guo [3], Ballester-Bolinches et al. [4] and Asaad and Ramadan [2] as follows:

Theorem A. If each subgroup of prime odd order of a group $G$ is $c$-supplemented in $G$, then $G$ is solvable.

Theorem B. Let $G$ be a group. Then $G$ is solvable if and only if every Sylow subgroup of odd order of $G$ is $c$-supplemented in $G$.

In connection with the above two Theorems, the following conjecture is posed at the end of Heliel [9].

Conjecture. Let $G$ be a finite group such that every non-cyclic Sylow subgroup $P$ of odd order of $G$ has a subgroup $D$ such that $1 < |D| < |P|$ and all subgroups $H$ of $P$ with $|H| = |D|$ are $c$-supplemented in $G$. Is $G$ solvable?

In the same year 2014, Li et al. [12] presented a counterexample to show that the answer of this conjecture is negative and also gave a generalization of Theorems A and B. Based on the above mentioned results, the main goal of this note is to prove the following results:

Theorem C. Suppose that each Sylow subgroup $P$ of $G$ has a subgroup $D$ such that $1 < |D| < |P|$ and all subgroups $H$ of $P$ with $|H| = |D|$ are $S$-permutable in $G$. Then $G$ is solvable.

Theorem D. Suppose that each Sylow subgroup $P$ of $G$ has a subgroup $D$ such that $1 < |D| < |P|$ and all subgroups $H$ of $P$ with $|H| = |D|$ are $c$-normal in $G$. Then $G$ is solvable.

Remark. The research on $c$-normal subgroups has formed a series, which is similar to the series of $S$-permutable subgroups. However, the two series are independent of each other.

2 Proofs

First we give an improvement of Gaschütz and Itô result that was mentioned in the introduction as follows:

Theorem 3.1. Suppose that each Sylow subgroup $P$ of a finite group $G$ has a subgroup $D$ such that $1 < |D| < |P|$ and all subgroups $H$ of $P$ with $|H| = |D|$ are normal in $G$. Then $G$ is solvable.
Proof. Assume that the result is false and let $G$ be a counterexample of minimal order. If all minimal subgroups of $G$ are normal in $G$, then $G$ is solvable by Gaschütz and Itô result [10, p. 436, satz 5.7], a contradiction. Thus there exists a subgroup $H$ of $G$ of prime order, say $p$, such that $L$ is not normal in $G$. Let $P$ be a Sylow $p$-subgroup of $G$ such that $1 \leq P$. Then there exists a subgroup $H$ of $P$ such that $L \leq H < P$ with $[H] = [D]$. By the hypothesis, $H$ is normal in $L$ and since $L$ is not normal in $G$, we have $L \leq H < P$. Clearly, $\Phi(H)$ is characteristic in $H$ and since $H \triangleleft G$, we have $\Phi(H) \triangleleft G$. If $\Phi(H) \neq 1$, then $G/\Phi(H)$ satisfies the hypothesis of the theorem and so $G/\Phi(H)$ is solvable by the minimal choice of $G$. Hence $G$ is solvable as the class of solvable groups is a saturated formation, a contradiction. Thus $\Phi(H) = 1$ and $H$ is elementary abelian $p$-group [6, p. 174, Theorem 1.3]. In fact, $[H] > p$ and so $H$ is noncyclic. We argue that $[P/H] \neq p$. If not, $[P] = p[H]$ and $P$ is noncyclic. Then $P$ contains a subgroup $N$ such that $[P : N] = p$ and $N \neq H$. By hypothesis, $H$ and $N$ are both normal in $G$ and so $H \cap N$ is normal in $G$. Then, by Schur-Zassenhaus Theorem [6, p. 221, Theorem 1.2], there exists a subgroup $K$ of $G$ such that $G = PK$ and $P \cap K = 1$. But $K$ is solvable by the minimal choice of $G$, then $G$ is solvable, a contradiction. Thus $[P/H] = p^n$, where $n \geq 2$. Let $L_p/H$ be a subgroup of $P/H$ of order $p$. Then $[L_p] = [P/H]$ and since $L_p$ is noncyclic as above, we have $L_p \leq G$ and so $L_p/H < G/H$. Hence $G/H$ is solvable by the minimal choice of $G$ and so $G$ is solvable, a final contradiction completing the proof of the theorem.

Proof of Theorem C. Assume that the result is false and let $G$ be a counterexample of minimal order. Then, by Theorem 3.1, there exists a subgroup $H$ of $G$ with $[H] = [D]$ such that $H$ is not normal in $G$. By the hypothesis, $K$ is $S$-permutably complemented in $G$. By [13, Lemma A], $O^p(G) \leq N_G(H)$ and since $H$ is not normal in $G$, we have $N_G(H) < G$. Let $M$ be a maximal subgroup of $G$ such that $N_G(H) \leq M < G$. Then $M$ is normal in $G$ and $[G/M] = p$ (recall that $M$ is a Sylow $p$-subgroup of $G$). Clearly, $P \cap M$ is a Sylow $p$-subgroup of $M$ and $H < P \cap M$. Hence if $1 \leq H < P \cap M$, $M$ satisfies the hypothesis of the Theorem and so $M$ is solvable by the minimal choice of $G$ and consequently so $G$ is solvable, a contradiction. Thus we may assume that $H = P \cap M$, so $[P : H] = p$, that is, $H < P$. Hence $G = P, O^p(G) \leq N_G(H) \leq M < G$, a contradiction completing the proof of the Theorem.

Proof of Theorem D. Assume that the result is false and let $G$ be a counterexample of minimal order. Then, by Theorem 3.1, there exists a subgroup $H$ of $G$ such that $[H] = [D]$ and $H$ is not normal in $G$. Without loss of generality we may assume that $H < P$, where $P$ is a Sylow $p$-subgroup of $G$ for some prime $p$ dividing the order of $G$. Then, by the hypothesis, $H$ is $c$-normal in $G$; that is, there exists a normal subgroup $K$ of $G$ such that $G = HK$ and $H \cap K < H_G$. As $H$ is not normal in $G$, we have $H_G < H$. Hence if $H_G \neq 1$, $G/H_G$ satisfies the hypothesis of the theorem by [15, Lemma 2.1], and so $G/H_G$ is solvable and since $H_G$ is of prime power order, it follows that $G$ is solvable, a contradiction. Thus we may assume that $H_G = 1$. Since $G/K \leq H$ and $H < P$, where $P$ is a Sylow $p$-subgroup of $G$, it follows that there exists a subgroup $M$ of $G$ such that $K \leq M, M < G$ and $[G/M] = p$. Clearly, $P \cap M$ is a Sylow $p$-subgroup of $M$. Hence if $[D] = [H] < [P \cap M]$, $M$ satisfies the hypothesis of the theorem by [15, Lemma 2.1], and so $M$ is solvable by the minimal choice of $G$. But $[G/M] = p$, that is, $G/M$ is solvable, then $G$ is solvable, a contradiction. So we may assume that $[P \cap M] = [D]$. Then by the hypothesis, $[P \cap M] = [D]$. Set $P \cap M = L$. By [15, Lemma 2.1], $L$ is $c$-normal in $M$, that is there exists a normal subgroup $N$ of $M$ such that $M = LN$ and $L \cap N \leq L_M$. Hence if $L_M = 1$, $N$ is a normal $p'$-Hall subgroup of $N$. Clearly, $N$ is a $p'$-Hall subgroup of $G$ and $n$ satisfies the hypothesis of the theorem and so $N$ is solvable by the minimal choice of $G$. Then $M$ is solvable and so $G$ is solvable, a contradiction. Thus we may assume that $L_M \neq 1$. Hence if
\[ L_M = L = P \cap M \triangleleft M, \quad M = LN, \quad N \triangleleft M \quad \text{and} \quad L \cap N = 1 \] by Schur-Zassenhaus Theorem [6, p. 221, Theorem 1.2]. As above, \( N \) is solvable and so \( G \) is solvable, a contradiction. Thus \( 1 \neq L_M < L \). Now we consider the normal closure of \( L_M \), that is, \( L^G_M = \langle L^G_M : g \in G \rangle \). Since \( G = MH \), we have \( L^G_M = L^M_M = L^M_M \leq P \) (where \( m \in M \) and \( h \in H \)) and so \( L^G_M \leq P \). Hence if \( L^G_M = P \), once again Schur-Zassenhaus Theorem implies that \( G = PK \), \( P \cap K = 1 \) and \( K \) is solvable by the minimal choice of \( G \) and so \( G \) is solvable, a contradiction. Thus we may assume that \( L^G_M < P \). Hence if \( |L^G_M| \neq |D| \), \( G/L^G_M \) is solvable by the minimal choice of \( G \) and so \( G \) is solvable, a contradiction. Now we may assume that \( |L^G_M| = |D| \). Since \( |P \cap M| = |D| \) and \( |P/P \cap M| = p \) and \( L^G_M < P \), we should have \( |L^G_M| = |D| \). Also, \( L^G_M \neq P \cap M \) (otherwise, \( G \) is solvable, a contradiction). Then \( G = L^G_M M \) and \( L^G_M \cap M \triangleleft G \) and \( |L^G_M \cap M| < |D| \).

Hence \( G/(L^G_M \cap M) \) is solvable by the minimal choice of \( G \) and so \( G \) is solvable, a final contradiction completing the proof of the theorem.

References