BEST CO-APPROXIMATION AND BEST SIMULTANEOUS CO-APPROXIMATION IN INTUITIONISTIC FUZZY NORMED LINEAR SPACES

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ABSTRACT

The main purpose of this paper is to study the t-best co-approximation and t-best simultaneous co-approximation in intuitionistic fuzzy normed spaces. We develop the theory of t-best co-approximation and t-best simultaneous co-approximation in quotient spaces. This new concept is employed by us to improve various characterisations of t-co-proximinal and t-co-Chebyshev sets.

Keywords

t-norm, t-conorm; intuitionistic fuzzy normed linear space; open(closed) ball and bounded set in intuitionistic fuzzy normed linear space.
1. INTRODUCTION

The theory of a fuzzy sets was firstly introduced by Zadeh [14] in 1965 and thereafter several authors applied it to different branches of pure and applied mathematics. On the other hand, the notion of fuzzyness has a wide application in many areas of science and engineering.


In 1986, Atanassov [2] introduced the concept of intuitionistic fuzzy sets. Park [8] first introduced the concept of intuitionistic fuzzy metric space and Saadati and Park [9] introduced the concept of intuitionistic fuzzy normed space, while the notion of intuitionistic fuzzy n-normed linear space was introduced by S. Vijayabalaji, N. Thillaigovindan and Y. Bae [13].

In this paper we study the set of all t-best co-approximation and t-best simultaneous co-approximation in intuitionistic fuzzy normed linear spaces and we develop the theory of t-best co-approximation and t-best simultaneous co-approximation in quotient spaces. This new concept is employed us to improve various characterizations of t-co-proximinal and t-co-Chebyshevsets.

2. PRELIMINARIES

**Definition 2.1.** [11] A binary operation $\cdot : [0,1] \times [0,1] \rightarrow [0,1]$ is said to be a continuous t-norm if the following axioms are satisfied:

(a) $\cdot$ is associative and commutative.
(b) $\cdot$ is continuous.
(c) $a \cdot 1 = a$ for all $a \in [0,1].$
(d) $a \cdot b \leq c \cdot d$ whenever $a \leq c$ and $b \leq d$ for each $a, b, c, d \in [0,1].$

**Definition 2.2.** [11] A binary operation $\diamond : [0,1] \times [0,1] \rightarrow [0,1]$ is said to be a continuous t-conorm if the following axioms are satisfied:

(a) $\diamond$ is associative and commutative
(b) $\diamond$ is continuous
(c) $a \diamond 0 = a$ for all $a \in [0,1]$
(d) $a \diamond b \leq a \diamond c$ whenever $a \leq c$ and $b \leq d$ for each $a, b, c, d \in [0,1].$

**Remark 2.3.** [11]

(1) For any $r_1, r_2 \in (0,1)$ with $r_1 > r_2$, there exists $r_3, r_4 \in (0,1)$ such that $r_1 \cdot r_3 \geq r_2$ and $r_1 \geq r_2 \diamond r_4$.

(2) For any $r_5 \in (0,1)$, there exists $r_6, r_7 \in (0,1)$ such that $r_6 \cdot r_7 \geq r_5$ and $r_7 \diamond r_7 \leq r_5$.

**Definition 2.4.** [9] The 5-tuple $(X, \mu, v, \cdot, \diamond)$ is said to be an intuitionistic fuzzy normed linear space (IFNLS) if $X$ be a linear space over the field $F$ (R or C), $\cdot$ is a continuous t-norm, $\diamond$ is a continuous t-conorm, and $\mu$, $v$ fuzzy sets on $X \times (0, \infty)$ satisfy the following conditions for every $x, y \in X$ and $s, t > 0$:

(1) $\mu(x, t) + v(x, t) \leq 1$
(2) $\mu(x, t) > 0$
(3) $\mu(x, t) = 1$ if and only if $x = 0$
(4) $\mu(a x, t) = \mu(x, t) \cdot |a|$ for each $a \neq 0$
(5) $\mu(x+t, s+t) \geq \mu(x, t) \cdot \mu(y, s)$
(6) $\lim_{t \to \infty} \mu(x, t) = 1$
(7) $v(x, t) < 1$
(8) $v(x, t) = 0$ if and only if $x = 0$
Lemma 2.5.[9]: Let $(X, \mu, \nu, \tilde{\sigma})$ be an intuitionistic fuzzy normed linear space then:

(i) $\mu(x+y, t) = \mu(x, t) \cap \mu(y, t)$ and $\nu(x-y, t) = \nu(y, t) \cap \nu(x, t)$ for every $t > 0$ and $x, y \in X$.

(ii) $\mu(x, t)$ and $\nu(x, t)$ are non-decreasing and non-increasing with respect to $t$, respectively.

Example 2.6. [9]: Let $(X, ||.||)$ be a normed linear space, and let $a \ast b = \min(a, b)$ and $a \odot b = \max(a, b)$

for all $a, b \in [0, 1]$. For all $x \in X$ and $t > 0$, $\mu(x, t) = \frac{t}{t+|\beta|}$ and $\nu(x, t) = \frac{1}{t+|\beta|}$

Then $(X, \mu, \nu, \tilde{\sigma})$ is an intuitionistic fuzzy normed linear space.

Definition 2.7.[9]: Let $(X, \mu, \nu, \tilde{\sigma})$ be an intuitionistic fuzzy normed linear space. We define an open ball $B(x, r, t)$ with the center $x \in X$ and the radius $0 < r < 1$, as $B(x, r, t) = \{y \in X : \mu(x-y, t) > 1-r, \nu(x-y, t) < r\}$ for every $t > 0$ also a subset $A \subseteq X$ is called open if for each $x \in A$, there exist $t > 0$ and $0 < r < 1$ such that $B(x, r, t) \subseteq A$. Let $\tau_{(\mu, \nu)}$ denote the family of all open subset of $X$. $\tau_{(\mu, \nu)}$ is called the topology induced by intuitionistic fuzzy norm.

Definition 2.8.[9]: Let $(X, \mu, \nu, \tilde{\sigma})$ be an intuitionistic fuzzy normed linear space. For $t > 0$, we define closed ball $B[x, r, t]$ with center $x \in X$ and radius $0 < r < 1$ as $B[x, r, t] = \{y \in X : \mu(x-y, t) \geq 1-r, \nu(x-y, t) \leq r\}$.

Definition 2.9.[9]: Let $(X, \mu, \nu, \tilde{\sigma})$ be IFNLS. A subset $G$ of $X$ is called intuitionistic fuzzy-bounded set (IF-bounded) if there exists $t > 0$ and $0 < r < 1$ such that $\mu(x, t) \geq 1-r$ and $\nu(x, t) \leq r$ for all $x \in G$.

3. t-BEST CO-APPROXIMATION IN INTUITIONISTIC FUZZY NORMED LINEAR SPACES

Definition 3.1: Let $(X, \mu, \nu, \tilde{\sigma})$ be IFNLS and $G$ be a nonempty subset of $X$. An element $x_0 \in G$ is called an intuitionistic fuzzy $t$-best co-approximation to $x$ from $G$ (IF-$t$-best co-approximation) if for $t > 0$, $\mu(x_0 - g, t) \geq \mu(x - g, t)$ and $\nu(x_0 - g, t) \leq \nu(x - g, t)$ for all $g \in G$. The set of all IF-$t$-best co-approximation to $x$ from $G$ will be denoted by $IF - R^G_t(x)$.

Remark 3.2: The set $IF - R^G_t(x)$ of all IF-$t$-best co-approximation to $x$ from $G$ can be written as:

$IF - R^G_t(x) = \{g_0 \in G : \mu(x_0 - g_0, t) \geq \mu(x - g_0, t) \text{ and } \nu(x_0 - g_0, t) \leq \nu(x - g_0, t) \text{ for all } g \in G\}$

Definition 3.3: Let $(X, \mu, \nu, \tilde{\sigma})$ be IFNLS and $G$ be a nonempty subset of $X$. The set of the $t$-co-metric complement, Define as: for $t > 0$ and $0 < r < 1$ such that $\mu(x, t) \geq 1-r$ and $\nu(x, t) \leq r$ for all $x \in G$.

Proposition 3.4: Let $(X, \mu, \nu, \tilde{\sigma})$ be IFNLS and $G$ be a subspace of $X$. Then for all $x \in X$, $g_0 \in IF - R^G_t(x)$ if and only if $x - g_0 \in IF - \tilde{G}$ for $t > 0$.

Proof: $(\Rightarrow)$ Suppose that $g_0 \in IF - R^G_t(x)$, $x \in X$

$\Rightarrow \mu(x_0 - g_0, t) \geq \mu(x - g_0, t) \text{ and } \nu(x_0 - g_0, t) \leq \nu(x - g_0, t)$, $\forall g \in G$.

Let $g_1 = g + g_0$, $\forall g \in G$, $\Rightarrow g_1 \in G$.

$\Rightarrow \mu(x_0 - g_1, t) \geq \mu(x - g_1, t)$ and $\nu(x_0 - g_1, t) \leq \nu(x - g_1, t)$

Since $\mu(g_1 - g_0, t) = \mu(g_0 - g_1, t)$

$\Rightarrow \mu(x - g_0 - g_0, t) \geq \mu(x - g_0, t)$

$\Rightarrow \mu(g_0, t) \geq \mu(x - g_0, t)$

$\Rightarrow \mu(g_0, t) \geq \mu(x - g_0, t)$

Similarly, we get $\nu(g_0, t) \leq \nu(x - g_0, t)$, $\forall g \in G$.

$\Rightarrow x - g_0 \in IF - \tilde{G}$.

$(\Leftarrow)$ Assume $x - g_0 \in IF - \tilde{G}$

$\Rightarrow \mu(g, t) \geq \mu(x - g_0, t)$ and $\nu(g, t) \leq \nu(x - g_0, t)$, $\forall g \in G$.
Let \( g_1 = g - g_0 \Rightarrow g_1 \in G \)
\[
\mu(g_1, t) \geq \mu(g_1 - (x - g_0), t) \quad \text{and} \quad \nu(g_1, t) \leq \nu(g_1 - (x - g_0), t)
\]
\[
\Rightarrow \mu(g - g_0, t) \geq \mu(g - g_0 - (x - g_0), t) \quad \text{and} \quad \nu(g - g_0, t) \leq \nu(g - g_0 - (x - g_0), t)
\]
\[
\Rightarrow \mu(g - g_0, t) \geq \mu(g - x, t) \quad \text{and} \quad \nu(g - g_0, t) \leq \nu(g - x, t), \forall g \in G
\]
Hence \( g_0 \in IF - R_{\mu}(x) \) if and only if \( x - g_0 \in IF - G \).

**Definition 3.5:** Let \((X, \mu, \nu, \circ, \bullet)\) be an IFNLS and \(G\) be a nonempty subset of \(X\). If for every \(x \in X\) has at least one IF-t-best co-approximation in \(G\), then \(G\) is called an intuitionistic fuzzy-t-co-proximinal set (IF-t-co-proximinal set).

**Definition 3.6:** Let \((X, \mu, \nu, \circ, \bullet)\) be an IFNLS and \(G\) be a nonempty subset of \(X\). If for every \(x \in X\) has exactly one IF-t-best co-approximation in \(G\), then \(G\) is called an intuitionistic fuzzy-t-co-Chebyshev set (IF-t-co-Chebyshev set).

**Definition 3.7:** Let \((X, \mu, \nu, \circ, \bullet)\) be an IFNLS. A subset \(G\) is said to be convex set if \((1 - \lambda)x + \lambda g_0 \in G\) whenever \(g_0 \in G, x \in X\) and \(0 < \lambda < 1\).

**Theorem 3.8:** Let \((X, \mu, \nu, \circ, \bullet)\) be an IFNLS and \(G\) is a nonempty subset of \(X\), if \(g_0 \in IF - R_{\mu}^t(x)\) and \((1 - \lambda)x + \lambda g_0 \in G\) for \(0 < \lambda < 1, t > 0\), then \((1 - \lambda)x + \lambda g_0 \in IF - R_{\mu}^t(x)\).

**Proof:** Let \(g_0 \in IF - R_{\mu}^t(x)\) and \((1 - \lambda)x + \lambda g_0 \in G\) for \(0 < \lambda < 1, t > 0\)
\[
\Rightarrow \mu(g_0 - g, t) \geq \mu(g - x, t) \quad \text{and} \quad \nu(g_0 - g, t) \leq \nu(g - x, t), \forall g \in G ....(1)
\]
Therefore, for a given \(t > 0\), take the natural number \(n\) such that \(t > \frac{1}{n}\)

By assumption and definition 2.4., we have
\[
\mu(((1 - \lambda)x + \lambda g_0) - g, t) = \mu(((1 - \lambda)x - \lambda g + \lambda g_0) - g, t) = \mu((1 - \lambda)x - (1 - \lambda)g + \lambda(g_0 - g), t) = \mu((1 - \lambda)(x - g) + \lambda(g_0 - g), t)
\]
\[
\geq \mu\left(x - g, \frac{1}{2(1 - \lambda)n}\right) \ast \mu\left(g_0 - g, \frac{t}{2}\right)
\]
\[
\geq \mu\left(x - g, \frac{1}{2(1 - \lambda)n}\right) \ast \mu\left(x - g, \frac{1}{2\lambda n}\right) = \lim_{n \to \infty} \mu\left(x - g, \frac{1}{2\lambda n}\right) = \mu(x - g, t) \quad [\text{since (1) and } t > \frac{1}{n}]
\]
and for a given \(t > 0\), take the natural number \(n\) such that \(t < \frac{1}{n}\)
\[
u(((1 - \lambda)x + \lambda g_0) - g, t) = \nu((1 - \lambda)x - \lambda g + \lambda g_0) - g, t) = \nu((1 - \lambda)x - (1 - \lambda)g + \lambda(g_0 - g), t) = \nu((1 - \lambda)(x - g) + \lambda(g_0 - g), t)
\]
\[
\leq \nu\left(x - g, \frac{1}{2(1 - \lambda)n}\right) \ast \nu\left(g_0 - g, \frac{t}{2}\right)
\]
\[
\leq \nu\left(x - g, \frac{1}{2(1 - \lambda)n}\right) \ast \nu\left(x - g, \frac{1}{2\lambda n}\right) = \lim_{n \to \infty} \nu\left(x - g, \frac{1}{2\lambda n}\right) = \nu(x - g, t) \quad [\text{since (1) and } t < \frac{1}{n}]
\]
Thus \((1 - \lambda)x + \lambda g_0 \in IF - R_{\mu}^t(x)\).

**Corollary 3.9:** Let \((X, \mu, \nu, \circ, \bullet)\) be an IFNLS. If \(G\) is convex subset of \(X\), then \(IF - R_{\mu}^t(x)\) is convex subset of \(X\).

**Proof:** Let \(G\) is convex subset of \(X\) and \(g_0 \in IF - R_{\mu}^t(x), for every x \in X\) and \(0 < \lambda < 1\)
Since \(G\) is convex subset of \(X\)
By theorem 3.8, we get
\[(1 - \lambda)x + \lambda g_0 \in IF - R_{\mu}^t(x)
\]
Hence \(IF - R_{\mu}^t(x)\) is convex subset of \(X\).
Theorem 3.10: Let \((X, \mu, v, \delta)\) be an IFNLS and \(G\) be a subspace of \(X\), then:

(i) \(IF - R_{ag}^{[\alpha]}(ax) = IF - aR_{g}^{[\alpha]}(x)\) for every \(x \in X\), \(a \in \mathbb{R}\(0\))\).

(ii) \(IF - R_{g+y}^{[\alpha]}(x+y) = IF - R_{g}^{[\alpha]}(x) + y\) for every \(x, y \in X\), \(a \in \mathbb{R}\(0\))

Proof: (i) \(g_0 \in IF - R_{ag}^{[\alpha]}(ax)\)
\[
\Rightarrow g_0 \in aG, \mu(g_0 - g, \alpha(t) \geq \mu(ax - g, |\alpha t) \text{ and } v(g_0 - g, |\alpha t) \leq v(ax - g, |\alpha t), \forall g \in aG
\]
\[
\Rightarrow \mu\left(\frac{1}{\alpha}g_0 - g, t\right) \geq \mu(x - \frac{1}{\alpha}g, t) \text{ and } v\left(\frac{1}{\alpha}g_0 - g, t\right) \leq v\left(x - \frac{1}{\alpha}g, t\right), \forall \frac{1}{\alpha} g \in G
\]
\[
\Rightarrow \mu\left(\frac{1}{\alpha}g_0 - g_1, t\right) \geq \mu(x - g_1, t) \text{ and } v\left(\frac{1}{\alpha}g_0 - g_1, t\right) \leq v(x - g_1, t), \forall g_1 = \frac{1}{\alpha} g \in G
\]
\[
\Rightarrow \frac{1}{\alpha}g_0 \in IF - R_{0}^{[\alpha]}(x) \Leftrightarrow g_0 \in IF - aR_{0}^{[\alpha]}(x)
\]
Hence \(IF - R_{ag}^{[\alpha]}(ax) = IF - aR_{g}^{[\alpha]}(x)\)

(ii) \(g \in IF - R_{g+y}^{[\alpha]}(x+y)\)
\[
\Rightarrow \mu(g - (g+y), t) \geq \mu((x+y) - (g+y), t) \text{ and } v(g - (g+y), t) \leq v((x+y) - (g+y), t), \forall g + y \in G + y
\]
\[
\Rightarrow \mu(g, t) - g, t \geq \mu(x, t) \text{ and } v(g, t) \leq v(x, t), \forall g \in G
\]
\[
\Rightarrow g - y \in IF - R_{0}^{[\alpha]}(x) \Leftrightarrow g \in IF - R_{0}^{[\alpha]}(x) + y
\]
Hence \(IF - R_{g+y}^{[\alpha]}(x+y) = IF - R_{g}^{[\alpha]}(x) + y\)

Corollary 3.11: Let \((X, \mu, v, \delta)\) be an IFNLS and \(G\) be a subspace of \(X\), then the following statements are hold:

(i) \(G\) is IF-t-co-proximinal set (resp. IF-t-co-Chebyshev set) if and only if \(\alpha G\) is IF-\(|\alpha|\)t-co-proximinal set (resp. IF-\(|\alpha|\)t-co-Chebyshev set) for any scalar \(\alpha \in \mathbb{R}(0)\)

(ii) \(G\) is IF-t-co-proximinal set (resp. IF-t-co-Chebyshev set) if and only if \(G + y\) is IF-t-co-proximinal set (resp. IF-t-co-Chebyshev set) for every \(y \in X\).

Proof: (i) \(G\) is IF-t-co-proximinal \(\Leftrightarrow IF - R_{0}^{[\alpha]}(x) \neq \emptyset\)
by Theorem 3.10
\[
\Rightarrow IF - aR_{g}^{[\alpha]}(x) \neq \emptyset
\]
\[
\Rightarrow IF - R_{ag}^{[\alpha]}(ax) \neq \emptyset
\]
Then \(\alpha G\) is IF-t-co-proximinal set.

Similarly, we get
\[
\alpha G\] is IF-t-co-Chebyshev set.

(ii) \(G\) is IF-t-co-proximinal set \(\Leftrightarrow IF - R_{0}^{[\alpha]}(x) \neq \emptyset\)
\[
\Rightarrow IF - R_{g}^{[\alpha]}(x) + y \neq \emptyset \Leftrightarrow IF - R_{g+y}^{[\alpha]}(x+y) \neq \emptyset
\]
Then \(G + y\) is IF-t-co-proximinal set.

Similarly, we get
\(G + y\) is IF-t-co-Chebyshev set.

4. t-CO-PROXIMALITY AND t-CO-CHEBYSHEVITY IN QUOTIENT SPACES

Definition 4.1.[3]: Let \((X, \mu, v, \delta)\) be an IFNLS and \(M\) is a closed subspace of \(X\), for \(t > 0\), we define
\[
\emptyset(x + M, t) = \sup\{\mu(x + y, t) : y \in M\}
\]
\[
\varphi(x + M, t) = \inf\{v(x + y, t) : y \in M\}
\]
where \(x + M = \{x + m : m \in M\}\).

Theorem 4.2. [3]: Let \((X, \mu, v, \delta)\) be an IFNLS and \(M\) is a closed subspace of \(X\), \(\emptyset(x + M, t)\) and \(\varphi(x + M, t)\) are defined in Definition 4.1, and \(X/M = \{x + M : x \in X\}\). Then \((X|M, \emptyset, \varphi, \delta)\) is an intuitionistic fuzzy normed linear space.
Theorem 4.3: Let \((X, \mu, v, \nu, \alpha)\) be an IFNLS and \(M\) is a closed subspace of \(X\) and \(G \supseteq M\) a subspace of \(X\). If \(G\) is an IF-t-co-proximinal set of \(X\), then \(G|M\) is an IF-t-co-proximinal set of \(X|M\).

**Proof:** Let \(G\) is an IF-t-co-proximinal set of \(X\).
\[
\exists g_0 \in G, x \in X \text{ such that } \mu(g_0 - g, t) \geq \mu(x - g, t) \text{ and } v(g_0 - g, t) \leq v(x - g, t), \forall g \in G
\]
\[
\mu(g_0 - m + m - g, t) \geq \mu(x - m + m - g, t) \text{ and } v(g_0 - m + m - g, t) \leq v(x - m + m - g, t), \forall m \in M
\]
\[
\mu((g_0 + m) - (g + m), t) \geq \mu((x + m) - (g + m), t) \text{ and } v((g_0 + m) - (g + m), t) \leq v((x + m) - (g + m), t), \forall g \in G
\]
\[
G_0 + M \in IF - R_{G|M}^t(x + M)
\]
\[
IF - R_{G|M}^t(x + M) \neq \emptyset
\]
\[
G|M\text{ is an IF-t-co-proximinal set of } X|M.
\]

**Corollary 4.4:** Let \((X, \mu, v, \nu, \alpha)\) be an IFNLS and \(M\) is a closed subspace of \(X\) and \(G \supseteq M\) a subspace of \(X\). If \(G|M\) is an IF-t-co-proximinal set of \(X|M\), then \(G\) is an IF-t-co-proximinal set of \(X\).

**Proof:** Let \(G|M\) is an IF-t-co-proximinal set with \(X|M\).
\[
\exists IF - R_{G|M}^t(x + M) \neq \emptyset
\]
\[
G_0 + M \in IF - R_{G|M}^t(x + M)
\]
\[
\mu(g_0 - g, t) \geq \mu(x - g, t) \text{ and } v(g_0 - g, t) \leq v(x - g, t), \forall g \in G
\]
\[
\mu((g_0 + m) - (g + m), t) \geq \mu((x + m) - (g + m), t) \text{ and } v((g_0 + m) - (g + m), t) \leq v((x + m) - (g + m), t), \forall m \in M
\]
\[
G_0 \in IF - R^t_{G|M}(x)
\]
\[
IF - R^t_{G|M}(x) \neq \emptyset
\]
\[
G\text{ is an IF-t-co-proximinal set of } X.
\]

**Theorem 4.5:** Let \((X, \mu, v, \nu, \alpha)\) be an IFNLS and \(M\) is a closed subspace of \(X\) and \(G \supseteq M\) a subspace of \(X\). If \(G|M\) is an IF-t-Co-Chebyshev set with \(X|M\), then \(G\) is an IF-t-Co-Chebyshev set with \(X\).

**Proof:** Let \(G|M\) is an IF-t-Co-Chebyshev set with \(X|M\) and \(G\) has two distinct \(t\)-best co-approximation of \(x \in X\) such as \(y_1, y_2\) in \(X\).
\[
\exists \mu(y_1 - g, t) \geq \mu(x - g, t) \text{ and } v(y_1 - g, t) \leq v(x - g, t), \forall g \in G
\]
also \(\mu(y_2 - g, t) \geq \mu(x - g, t) \text{ and } v(y_2 - g, t) \leq v(x - g, t), \forall g \in G
\]
\[
\mu((y_1 + m) - (g + m), t) \geq \mu((x + m) - (g + m), t) \text{ and } v((y_1 + m) - (g + m), t) \leq v((x + m) - (g + m), t), \forall m \in M
\]
also \(\mu((y_2 + m) - (g + m), t) \geq \mu((x + m) - (g + m), t) \text{ and } v((y_2 + m) - (g + m), t) \leq v((x + m) - (g + m), t), \forall m \in M
\]
\[
y_1 + M, y_2 + M \in IF - R^t_{G|M}(x + M)
\]
since \(y_1 \neq y_2 \Rightarrow y_1 + M \neq y_2 + M
\]
\[
IF - R^t_{G|M}(x + M) is not IF - t - co - Chebyshev, this contradiction(\#).
\]
\[
y_1 = y_2
\]
Then \(G\) is an IF-t-Co-Chebyshev set with \(X\).

**Definition 4.6.6:** Let \((X, \mu, v, \nu, \alpha)\) be an IFNLS and \(M\) is a closed subspace of \(X\), for \(t > 0\) and \(x \in X\) the distance between \(x\) and \(M\) define as :
\[
d_\mu(x, M, t) = \sup \{ \mu(x - y, t) : y \in M \} \text{ and } d_v(x, M, t) = \inf \{ v(x - y, t) : y \in M \}.
\]

**Theorem 4.7:** Let \(M\) and \(G\) are two subspaces of \((X, \mu, v, \nu, \alpha)\) such that \(M \subset G\) and \(x + G \in X/G\), \(g_1 \in G\). If \(g_1\) is IF-t-best co-approximation to \(x\) from \(G\), then \(g_1 + M\) is an IF-t-best co-approximation to \(x + M\) from the quotient space \(G/M\).
**Proof:** Suppose that \( g_1 \) is IF-t-best co-approximation to \( x \) from \( G \) and \( g_1 + M \) is not IF-t-best co-approximation to \( x + M \) from the quotient space \( G/M \).

\[
\exists \, \hat{g}_1 + M \in G/M \text{ such that } \mu(\hat{g}_1 + M - (g_1 + M), t) < \mu(x + M - (\hat{g}_1 + M), t) \quad \text{and} \\
v(\hat{g}_1 + M - (g_1 + M), t) > v(x + M - (\hat{g}_1 + M), t)
\]

\[
\Rightarrow \mu(\hat{g}_1 + g_1, M + t) < \mu(x - \hat{g}_1 + m, t) \quad \text{and} \\
v(\hat{g}_1 + g_1, M + t) > v(x - \hat{g}_1 + m, t)
\]

Since \( d_\mu(x, M, t) = \sup(\mu(x-y, t) : y \in M) \) and \( d_\mu(x, M, t) = \inf(v(x-y, t) : y \in M) \)

\[
\Rightarrow \sup(\mu(x - \hat{g}_1 + m, t)) > \inf(\mu(x - \hat{g}_1 + m, t) \quad \text{and} \\
\Rightarrow d_\mu(x, \hat{g}_1 + M, t) > d_\mu(\hat{g}_1 - g_1, M, t) \quad \text{and} \\
d_\mu(x, \hat{g}_1 + M, t) > d_\mu(\hat{g}_1 - g_1, M, t)
\]

This implies that there exists \( g \in M \) such that

\[
\mu(x - \hat{g}_1 + g, t) > d_\mu(\hat{g}_1 - g_1, M, t) \quad \mu(x - \hat{g}_1 + g, t) < d_\mu(\hat{g}_1 - g_1, M, t) \quad \mu(x - (g + \hat{g}_1), t) > v(x - (g + \hat{g}_1), t)
\]

\[
\Rightarrow \exists g + \hat{g}_1 \in G \text{ such that } \mu((g + \hat{g}_1) - g_1, t) < \mu(x - (g + \hat{g}_1), t) \quad \text{and} \\
v((g + \hat{g}_1) - g_1, t) > v(x - (g + \hat{g}_1), t)
\]

\[
\Rightarrow g_1 \text{ is not an IF-t-best co-approximation to } x \text{ from } G \text{, this contradiction with hypothesis } (#).
\]

Then \( g_1 + M \) is an IF-t-best co-approximation to \( x + M \) from the quotient space \( G/M \). \( \blacksquare \)

### 5.1-BEST SIMULTANEOUS CO-APPROXIMATION IN INTUITIONISTIC FUZZY NORMED LINEAR SPACES

**Definition 5.1:** Let \((X, \mu, v, *, \otimes)\) be an IFNLS and \( G \) be a subset of \( X \), \( M \) be IF-bounded subset in \( X \). An element \( g_0 \in G \) is called IF-t-best simultaneous co-approximation to \( M \) from \( G \), if for \( t > 0 \),

\[
\mu(g_0 - g, t) \geq \inf(\mu(m - g, t) : m \in M) \quad \text{and} \quad v(g_0 - g, t) \leq \sup( v(m - g, t) : m \in M ) \text{ for all } g \in G.
\]

The set of all IF-t-best simultaneous co-approximation to \( M \) from \( G \), will be denoted by \( IF - S_0^t(M) \) and define as follows:

\[
IF - S_0^t(M) = \{ g_0 \in G : \mu(g_0 - g, t) \geq \inf_{m \in M} \mu(m - g, t) \quad \text{and} \quad v(g_0 - g, t) \leq \sup_{m \in M} v(m - g, t), \forall g \in G \}
\]

**Definition 5.2:** Let \( G \) be a subset of \((X, \mu, v, *, \otimes)\). It is called IF-t-best simultaneous co-proximinal subset of \( X \), if for each IF-bounded subset \( M \) in \( X \), there exists at least one IF-t-best simultaneous co-approximation from \( G \) to \( M \).

**Definition 5.3:** Let \( G \) be a subset of \((X, \mu, v, *, \otimes)\). It is called IF-t-best simultaneous co-Chebyshev subset of \( X \), if for each IF-bounded subset \( M \) in \( X \) there exists a unique IF-t-best simultaneous co-approximation from \( G \) to \( M \).

**Theorem 5.4:** Let \((X, \mu, v, *, \otimes)\) and \( G \) be a subset of \( X \). If \( M \) is IF-bounded subset of \( X \), \( +, *, \otimes \) satisfying the condition \( a \otimes b \geq a, a \otimes b \leq a, \forall a, b \in [0, 1] \), then \( IF - S_0^t(M) \) is IF-bounded subset of \( X \).

**Proof:** Let \( M \) is IF-bounded subset of \( X \) and \( g_0 \in IF - S_0^t(M) \)

\[
\Rightarrow \exists \text{ there exist } 0 < r < 1 \text{ such that } \mu(x, t) \geq 1 - r, v(x, t) \leq r, \forall x \in M, t > 0 \quad \text{and} \\
\mu(g_0 - g, t) \geq \inf_{m \in M} \mu(m - g, t) \text{ and } v(g_0 - g, t) \leq \sup_{m \in M} v(m - g, t), \forall g \in G
\]

\[
\Rightarrow \text{ for every } g \in G, m \in M, \mu(g_0, 3t) = \mu(g_0 - m + m, 3t) \geq \mu(g_0 - m, 2t) \ast \mu(m, t)
\]

\[
\geq \mu(g_0 - g + g - m, 2t) \ast (1 - r)
\]

\[
\geq \mu(g_0 - g, t) \ast \mu(g - m, t) \ast (1 - r)
\]

\[
\geq \inf_{m \in M} \mu(m - g, t) \ast \mu(m - g, t) \ast (1 - r)
\]

\[
\geq \inf_{m \in M} \mu(m - g, t) \ast (1 - r)
\]

\[
\geq 1 - r_0 \text{ for some } 0 < r_0 < 1
\]

\[
\Rightarrow v(g_0, 3t) = v(g_0 - m + m, 3t) \leq v(g_0 - m, 2t) \otimes v(m, t)
\]

\[
\leq v(g_0 - g + g - m, 2t) \otimes r
\]

\[
\leq v(g_0 - g, t) \otimes v(g - m, t) \otimes r
\]

\[
\leq \sup_{m \in M} v(m - g, t) \otimes v(m - g, t) \otimes r
\]

\[
\leq \sup_{m \in M} v(m - g, t) \otimes r
\]

\[
\leq r_0 \text{ for some } 0 < r_0 < 1
\]

Then \( IF - S_0^t(M) \) is an IF-bounded subset of \( X \). \( \blacksquare \)
Theorem 5.5: Let \((X, \mu, \nu, \lambda)\) FNLS and \(M\) is IF-bounded subset of \(X\). If \(G\) is a convex subset of \(X\) and \(*, \lambda\) satisfying the condition \(a * b \geq a, a \& b \leq a, \forall a, b \in [0,1]\), then IF - \(S_{\lambda}^M(G)\) is a convex subset of \(X\).

Proof: Suppose that \(G\) is a convex subset of \(X\)
\[
(1 - \lambda)x + \lambda g_0 \in G
\]
for every \(g_0 \in G, x \in X\) and \(0 < \lambda < 1\)

Therefore, for a given \(t > 0\), take \(n \in N\) such that \(t > \frac{1}{n}\), we get
\[
\mu((1 - \lambda)m + \lambda g_0) - g, \frac{t}{n}
\]
\[
= \mu((1 - \lambda)m - \lambda g + \lambda g_0) - g, \frac{1}{n}
\]
\[
= \mu((1 - \lambda)m - \lambda g + \lambda g_0 - g, \frac{t}{2n})
\]
\[
\geq \mu(\frac{m - g}{2(1 - \lambda)} + \frac{t}{2n}) = \lim_{n \to \infty} \inf_{m \in M} \mu(\frac{m - g}{2n}) = \inf_{m \in M} \mu(m - g, t)
\]
and for a given \(t > 0\), take \(n \in N\) such that \(t < \frac{t}{n}\), we get
\[
v((1 - \lambda)m + \lambda g_0) - g, \frac{t}{n}
\]
\[
v((1 - \lambda)m - \lambda g + \lambda g_0) - g, \frac{1}{n}
\]
\[
v((1 - \lambda)m - \lambda g + \lambda g_0 - g, \frac{t}{2n})
\]
\[
\leq \sup_{m \in M} v(\frac{m - g}{2(1 - \lambda)} - \frac{t}{2n}) = \sup_{m \in M} v(m - g, t)
\]

Theorem 5.6: Let \(G\) is a subset of \((X, \mu, \nu, \lambda)\) and \(M\) is IF-bounded in \((X, \mu, \nu, \lambda)\). Then the following assertions are hold for \(t > 0\):

1. IF - \(S_{\lambda}^M(x + M) = IF - S_{\lambda}^M(x)\) , \(\forall x \in X\) .
2. IF - \(S_{\lambda}^{|a|}^M(aM) = IF - a S_{\lambda}^M(M)\) , \(\forall a \in R\) .

Proof: (1) let \(g_0 \in IF - S_{\lambda}^M(M + x)\)
\[
\Rightarrow \forall \epsilon > 0, \ \mu(g_1 - x - g_0, \epsilon) = \inf_{m \in M} \mu(m - (g_1 - x - g_0), \epsilon) \Rightarrow \mu(g_1 - (g_0 + x), \epsilon) \geq \inf_{m \in M} \mu(m - (g_1 - x - g_0), \epsilon) \Rightarrow \mu(\frac{g_1 - (g_0 + x)}{2(1 - \lambda)}, \epsilon) \geq \inf_{m \in M} \mu(m - (g_1 - x - g_0), \epsilon) \Rightarrow \mu(\frac{g_1 - (g_0 + x)}{2(1 - \lambda)}, \epsilon) \geq \inf_{m \in M} \mu(m - (g_1 - x - g_0), \epsilon)
\]

Similarly, we get \(\sup_{m \in M} v(m + x - g_1, \epsilon) \leq \sup_{m \in M} v(m + x - g_1, \epsilon)\)

Then \(g_0 + x \in IF - S_{\lambda}^M(M + x)\)

Let \(g_0 + x \in IF - S_{\lambda}^M(M + x)\)
\[
\Rightarrow \forall \epsilon > 0, \ \mu(g_1 - x - g_0, \epsilon) = \inf_{m \in M} \mu(m + x - (g_1 + x), \epsilon) \Rightarrow \inf_{m \in M} \mu(m + x - (g_1 + x), \epsilon)
\]

Similarly, we get
\[
\sup_{m \in M} v(m + x - g_1, \epsilon) \leq \sup_{m \in M} v(m + x - g_1, \epsilon), \forall x \in X
\]

Then \(IF - S_{\lambda}^{|a|}^M(aM) = IF - a S_{\lambda}^M(M)\) , \(\forall a \in R\)

Proof: clearly equality holds for \(a = 0\)

Let \(a \neq 0\), \(g_0 \in IF - S_{\lambda}^{|a|}^M(aM)\) if and only if
such that \( \mu(g_0 - g, |\alpha|t) \geq \inf_{m \in M} \mu(m - g, |\alpha|t) \) and
\[
v(g_0 - g, |\alpha|t) \leq \sup_{m \in M} v(m - g, |\alpha|t), \forall g \in G
\]
\[
\Leftrightarrow \mu\left(\frac{1}{\alpha}g_0 - \frac{1}{\alpha}g, t\right) \geq \inf_{m \in M} \mu\left(m - \frac{1}{\alpha}g, t\right) \quad \text{and} \quad v\left(\frac{1}{\alpha}g_0 - \frac{1}{\alpha}g, t\right) \leq \sup_{m \in M} v\left(m - \frac{1}{\alpha}g, t\right), \forall \frac{1}{\alpha}g \in G
\]
\[
\Leftrightarrow \mu\left(\frac{1}{\alpha}g_0 - g_1, t\right) \geq \inf_{m \in M} \mu\left(m - g_1, t\right) \quad \text{and} \quad v\left(\frac{1}{\alpha}g_0 - g_1, t\right) \leq \sup_{m \in M} v\left(m - g_1, t\right), \forall g_1 = \frac{1}{\alpha}g \in G
\]
\[
\frac{1}{\alpha}g_0 \in IF - S^\text{int}_{\text{inf}}(M) \iff g_0 \in IF - \alpha S^\text{int}_{\text{inf}}(M)
\]

Then \( IF - S^\text{int}_{\text{inf}}(\alpha M) = IF - \alpha S^\text{int}_{\text{inf}}(M) \)

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REFERENCES


