SOME RESULTS OF GENERALIZED LEFT \((\theta, \theta)\)-DERIVATIONS ON SEMIPRIME RINGS

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ABSTRACT
Let \( R \) be an associative ring with center \( Z(R) \). In this paper, we study the commutativity of semiprime rings under certain conditions, it comes through introduce the definition of generalized left \((\theta, \theta)\)-derivation associated with left \((\theta, \theta)\)-derivation, where \( \theta \) is a mapping on \( R \).

Keywords
Semiprime ring; left derivation; generalized left derivation; left \((\theta, \theta)\)-derivation; generalized left \((\theta, \theta)\)-derivation.
INTRODUCTION

This paper consists of two sections. In section one, we recall some basic definitions and other concepts, which be used in our paper. We explain these concepts by examples, remarks. Yass in [1] introduced the definition of generalized left derivation associated with left derivation on a ring R. In section two, we will introduce the definition of generalized left \((\theta, \phi)\) -derivation associated with left \((\theta, \phi)\)-derivation on a ring R. And we will give some related results in commutativity of semiprime ring, where \(\theta\) is a mapping on R.

1. BASIC CONCEPTS

Definition 1.1: [1] A ring R is called a semiprime ring if for any \(a \in R\), \(aRa = \{0\}\) implies that \(a = 0\).

Definition 1.2: [1] Let R be an arbitrary ring. If there exists a positive integer n such that \(na = 0\), for all \(a \in R\), then the smallest positive integer with this property is called the characteristic of the ring, by symbols we write \(\text{char } R = n\).

Definition 1.3: [1] A ring R is said to be n-torsion-free where \(n \neq 0\) is an integer if whenever \(na = 0\) with \(a \in R\), then \(a = 0\).

Definition 1.4: [2] Let R be a ring. Define a Lie product \([,\] \) on R as follows
\[
[x,y] = xy - yx , \text{ for all } x,y \in R.
\]

Properties 1.5: [2] Let R be a ring, then for all \(x,y,z \in R\), we have:
1. \([xyz] = [yxz] + [xy]z\)
2. \([xy, z] = [x, yz] + [x, z]y\)
3. \([x+y, z] = [x, z] + [y, z]\)
4. \([x+y, z] = [x, y] + [x, z]\)

Definition 1.6: [2] Let R be a ring. Define a Jordan product \(\circ\) on R as follows
\[
a \circ b = ab + ba , \text{ for all } a,b \in R .
\]

Properties 1.7: [2] Let R be a ring. Then for all \(x,y,z \in R\), we have
1. \(x_0(yz) = (x_0 y)z - y(x_0 z) = y[x_0, z] + [x, y]z\)
2. \((x_0 y)_0 z = x(y_0 z) - [x, z]y = (x_0 y)z + x[y, z]\)

Definition 1.8: [2] Let R be a ring, the center of R denoted by \(Z(R)\) and is defined by: \(Z(R) = \{ x \in R : rx = xr , \text{for all } r \in R \}\).

Definition 1.9: [3] Let R be a ring. An additive mapping \(d : R \rightarrow R\) is called a left derivation if \(d(xy) = xd(y) + yd(x)\), for all \(x,y \in R\) and we say that \(d\) is a Jordan left derivation if \(d(x^2) = 2xd(x)\), for all \(x \in R\).

Example 1.10: [3] Let R be a commutative ring and let \(a \in R\) such that \(xay = 0\), for all \(x,y \in R\) such that \(x \neq y\).

Define a map \(d : R \rightarrow R\) as follows \(d(x) = xa + ax\).

Then \(d\) is a left derivation of \(R\).

It is clear that \(d\) is an additive mapping.

Now, we have to show that \(d\) satisfies

\[
d(xy) = xd(y) + yd(x) , \text{for all } x, y \in R .
\]

\[
d(xy) = xya + axy = xay + xay = 0 , \text{for all } x, y \in R . \text{ And}
\]

\[
xd(y) + yd(x) = x(ya + ay) + y(xa + ax) = xay + xay + xay + xay = 0 , \text{for all } x, y \in R .
\]

Hence \(d(xy) = xd(y) + yd(x)\), for all \(x, y \in R\).

Then \(d\) is a left derivation of \(R\).
Remark 1.11 : [3] It is easy to see that every left derivation on a ring $R$ is a Jordan left derivation. However, in general, a Jordan left derivation need not to be a left derivation.

Example 1.12 : [3] Let $R$ be a commutative ring and let $a \in R$, such that $xax = 0$, for all $x \in R$, but $xay \neq 0$, for some $x$ and $y$, $x \neq y$. Define a map $d : R \to R$, as follows: $d(x) = xa + ax$

Then $d$ is a Jordan left derivation, but not a left derivation.

Definition 1.13 : [1] Let $R$ be a ring, an additive mapping $F : R \to R$ is called a generalized left derivation associated with left derivation if there exists a left derivation $d : R \to R$, such that:

$F(xy) = x F(y) + y d(x)$, for all $x, y \in R$.

Definition 1.14 : [4] Let $R$ be a ring. An additive mapping $d : R \to R$ is called a left ($\theta$) derivation, where $\theta : R \to R$ is a mapping of $R$, if

$d(xy) = \theta(x) d(y) + \theta(y) d(x)$, for all $x, y \in R$ and we say that $d$ is a Jordan left ($\theta$) derivation if $d(x^2) = 2 \theta(x) d(x)$, for all $x \in R$.

2. GENERALIZED LEFT ($\theta$, $\theta$)-DERIVATIONS

Definition 2.1 : Let $R$ be a ring. An additive mapping $F : R \to R$ is called a generalized left ($\theta$, $\theta$)-derivation associated with left ($\theta$, $\theta$)-derivation, where $\theta : R \to R$ is a mapping of $R$, if there exists a left ($\theta$, $\theta$)-derivation $d : R \to R$, such that $F(xy) = \theta(x) F(y) + \theta(y) d(x)$, for all $x, y \in R$.

Example 2.2 : Consider the ring:

$R = \begin{pmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{pmatrix} : a, b, c \in S$, where $S$ is a ring.

Define $F, d : R \to R$, by:

$F\left(\begin{array}{ccc} 0 & 0 & 0 \\ a & 0 & 0 \\ b & c & 0 \end{array}\right) = \left(\begin{array}{ccc} 0 & 0 & 0 \\ a & 0 & 0 \\ b & 0 & 0 \end{array}\right)$, for all $a, b, c \in S$. And

$d\left(\begin{array}{ccc} 0 & 0 & 0 \\ a & 0 & 0 \\ b & c & 0 \end{array}\right) = \left(\begin{array}{ccc} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{array}\right)$, for all $a, b, c \in S$.

Suppose that $\theta : R \to R$ is a mapping such that

$\theta\left(\begin{array}{ccc} 0 & 0 & 0 \\ a & 0 & 0 \\ b & c & 0 \end{array}\right) = \left(\begin{array}{ccc} 0 & 0 & 0 \\ a & 0 & 0 \\ b & 0 & 0 \end{array}\right)$, for all $a, b, c \in S$.

It is clear that $d$ is a left ($\theta$, $\theta$)-derivation of $R$.

Then $F$ is a generalized left ($\theta$, $\theta$)-derivation associated with left ($\theta$, $\theta$)-derivation $d$.

Theorem 2.3 : Let $R$ be a 2-torsion free semiprime ring. If $R$ admits a generalized left ($\theta$, $\theta$)-derivation $F$ associated with left ($\theta$, $\theta$)-derivation $d$, where $\theta$ is an automorphism of $R$, then $d(x) \in Z(R)$, for all $x \in R$.

Proof: We have:

$F(x^2y) = \theta(x^2) F(y) + \theta(y) d(x^2)$, for all $x, y \in R$ (1)
That is:
\[ F(x^2y) = \theta(x^2)F(y) + 2\theta(x)\theta(y)d(x), \text{ for all } x, y \in R \] (2)

On the other hand, we find that:
\[ F(xy) = \theta(x)F(xy) + \theta(x)\theta(y)d(x), \text{ for all } x, y \in R \] (3)

That is:
\[ F(x^2y) = \theta(x^2)F(y) + 2\theta(x)\theta(y)d(x), \text{ for all } x, y \in R \] (4)

Comparing (2) and (4), we obtain:
\[ 2[\theta(y), \theta(x)]d(x) = 0, \text{ for all } x, y \in R \] (5)

Since \( R \) is 2-torsion free, we have:
\[ [\theta(y), \theta(x)]d(x) = 0, \text{ for all } x, y \in R \] (6)

This can be written as:
\[ [\theta(x), \theta(y)]d(x) = 0, \text{ for all } x, y \in R \] (7)

Linearizing (7) on \( x \), we find that:
\[ [\theta(x), \theta(y)]d(w) + [\theta(w), \theta(y)]d(x) = 0, \text{ for all } x, y, w \in R \] (8)

Replacing \( y \) by \( yz \) in (7) and using (7), we have:
\[ [\theta(x), \theta(y)]\theta(z)d(x) = 0, \text{ for all } x, y, z \in R \] (9)

Replacing \( \theta(z) \) by \( d(w)\theta(z) \) in \( (9) \), we have:
\[ [\theta(x), \theta(y)]d(w)\theta(z) [\theta(z), \theta(y)]d(x) = 0, \text{ for all } x, y, z, w \in R \] (10)

Comparing (8) and (10), we obtain:
\[ [\theta(x), \theta(y)]d(w)\theta(z) [\theta(z), \theta(y)]d(x) = 0, \text{ for all } x, y, z, w \in R \] (11)

Since \( R \) is semiprime, we obtain:
\[ [\theta(x), \theta(y)]d(w) = 0, \text{ for all } x, y, w \in R \] (12)

By \([1,\text{Lemma 2.1.16}]\), we get \( d(w) \in Z(R) \), for all \( w \in R \).

**Theorem 2.4:** Let \( R \) be a 2-torsion free semiprime ring. If \( R \) admits a generalized left \( (\theta, \theta) \)-derivation \( F \) associated with left \( (\theta, \theta) \)-derivation \( d \), where \( \theta \) is an automorphism of \( R \), such that \( [d(x),F(y)] = [\theta(x), \theta(y)] \), for all \( x, y \in R \). Then \( R \) is commutative.

**Proof:** We have:
\[ [d(x),F(y)] = [\theta(x), \theta(y)] \], for all \( x, y \in R \) (1)

By Theorem 2.3, (1) gives:
\[ [\theta(x), \theta(y)] = 0 \], for all \( x, y \in R \) (2)

\[ \theta(x)\theta(y) = \theta(y)\theta(x) \], for all \( x, y \in R \) (3)

Thus \( R \) is commutative.

**Theorem 2.5:** Let \( R \) be a 2-torsion free semiprime ring. If \( R \) admits a generalized left \( (\theta, \theta) \)-derivation \( F \) associated with left \( (\theta, \theta) \)-derivation \( d \), where \( \theta \) is an automorphism of \( R \), such that \( [d(x),F(y)] = \theta(x)\theta(y) \), for all \( x, y \in R \). Then \( R \) is commutative.

**Proof:** We assume:
\[ [\theta(x) \circ F(y)] = \theta(x) \circ \theta(y), \quad \text{for all } x, y, \in R \tag{1} \]

By Theorem 2.3, (1) gives:
\[ \theta(x) \circ \theta(y) = 0, \quad \text{for all } x, y, \in R \tag{2} \]

Replace \( \theta(y) \) by \( \theta(yr) \) in (2), to get:
\[ \theta(x) \circ \theta(yr) = 0, \quad \text{for all } x, y, r, \in R \tag{3} \]

This can be rewritten as:
\[ (\theta(x) \circ \theta(y)) \circ \theta(r) \circ \theta(y) \circ \theta(r) \circ \theta(x) \circ \theta(r) = 0, \quad \text{for all } x, y, r, \in R \tag{4} \]

Using (2), (4) gives:
\[ \theta(y) \circ \theta(x) \circ \theta(r) = 0, \quad \text{for all } x, y, r, \in R \tag{5} \]

Left multiplication of (5) by \( \theta(x) \circ \theta(r) \), and since \( R \) is semiprime, we get:
\[ [\theta(x), \theta(r)] = 0, \quad \text{for all } x, r, \in R \tag{6} \]

Hence, \( R \) is commutative.

**Theorem 2.6**: Let \( R \) be a 2-torsion free semiprime ring. If \( R \) admits a generalized left \( (\theta, \theta) \)-derivation \( F \) associated with left \( (\theta, \theta) \)-derivation \( d \), where \( \theta \) is an automorphism of \( R \), such that \( d(x) \circ F(y) \circ \theta(y) = \theta(x) \circ F(y), \quad \text{for all } x, y, \in R \). Then \( R \) is commutative.

**Proof**: We have:
\[ d(x) \circ F(y) \circ \theta(y), \quad \text{for all } x, y, \in R \tag{1} \]

By Theorem 2.3, (1) gives:
\[ 2d(x) \circ F(y) \circ \theta(y), \quad \text{for all } x, y, \in R \tag{2} \]

Replacing \( y \) by \( yz \) in (2), we obtain:
\[ 2d(x) \circ F(y \circ \theta(z)) \circ \theta(y \circ \theta(z)) \circ \theta(y \circ \theta(z)) = \theta(x) \circ \theta(y \circ \theta(z)), \quad \text{for all } x, y, z \in R \tag{3} \]

Again, by Theorem 2.3, (3) gives:
\[ 2 \theta(y) \circ d(x) \circ F(z) \circ \theta(y) \circ d(x) \circ F(z) = \theta(y) \circ d(x) \circ F(z) \circ \theta(y \circ \theta(z)), \quad \text{for all } x, y, z \in R \tag{4} \]

Since \( \theta(y) \circ d(x) \circ F(z) \circ \theta(y) \circ d(x) \circ F(z) = \theta(y) \circ d(x) \circ F(z) \circ \theta(y \circ \theta(z)) + [\theta(x) \circ \theta(y) \circ \theta(z)], \) then from (2) and (4), we obtain:
\[ 2d(x) \circ \theta(x) \circ d(y) = [\theta(x) \circ \theta(y) \circ \theta(z)], \quad \text{for all } x, y, z \in R \tag{5} \]

In particular, for \( z = x \), (5) gives:
\[ 2d(x) \circ \theta(x) \circ d(y) = [\theta(x) \circ \theta(y) \circ \theta(x)], \quad \text{for all } x, y \in R \tag{6} \]

Replacing \( y \) by \( ry \) in (6), we obtain:
\[ 2d(x) \circ \theta(x) \circ \theta(r) \circ d(y) + 2d(x) \circ \theta(x) \circ \theta(r) \circ d(y) = [\theta(x) \circ \theta(ry) \circ \theta(x)], \quad \text{for all } x, y, r \in R \tag{7} \]

Again, using Theorem 2.3, (7) reduces to:
\[ 2d(x) \circ \theta(x) \circ \theta(y) \circ \theta(r) + 2d(x) \circ \theta(x) \circ \theta(y) \circ \theta(r) = [\theta(x) \circ \theta(ry) \circ \theta(x)], \quad \text{for all } x, y, r \in R \tag{8} \]

From (6) and (8), we obtain:
\[ [\theta(x) \circ \theta(y) \circ \theta(x) \circ \theta(r)] + [\theta(x) \circ \theta(ry) \circ \theta(x)] = [\theta(x) \circ \theta(ry) \circ \theta(y)] \circ \theta(x), \quad \text{for all } x, y, r \in R \]
for all $x, y, r \in R$  \hspace{1cm} (9)

This implies that:

$$[[\theta(x) , \theta(y)] \theta(x), \theta(r)] + [\theta(x) , \theta(r)] [\theta(x) , \theta(y)] = 0,$$

for all $x, y, r \in R$  \hspace{1cm} (10)

Replacing $r$ by $sr$ in (10), we obtain:

$$\theta(s) \left[ [\theta(x) , \theta(y)] \theta(x), \theta(r) \right] + \left[ [\theta(x) , \theta(y)] \theta(x), \theta(s) \right] \theta(r) \left[ [\theta(x) , \theta(y)] = 0, \right.$$

for all $x, y, r, s \in R$  \hspace{1cm} (11)

From (10) and (11), we obtain:

$$[[\theta(x) , \theta(y)] \theta(x), \theta(s)] \theta(r) + [\theta(x) , \theta(s)] \theta(r) [\theta(x) , \theta(y)] = 0,$$

for all $x, y, r, s \in R$  \hspace{1cm} (12)

Replacing $r$ by $rt$ in (12), we have:

$$[[\theta(x) , \theta(y)] \theta(x), \theta(s)] \theta(rt) + [\theta(x) , \theta(s)] \theta(rt) [\theta(x) , \theta(y)] = 0,$$

for all $x, y, r, s, t \in R$  \hspace{1cm} (13)

Again, right multiplying of (12) by $\theta(t)$, we have:

$$[[\theta(x) , \theta(y)] \theta(x), \theta(s)] \theta(rt) + [\theta(x) , \theta(s)] \theta(rt) [\theta(x) , \theta(y)] \theta(t) = 0,$$

for all $x, y, r, s, t \in R$  \hspace{1cm} (14)

Subtracting (14) from (13), we obtain:

$$[\theta(x), \theta(s)] \theta(rt) [\theta(t), \theta(y)] = 0, \text{ for all } x, y, r, s, t \in R \hspace{1cm} (15)$$

Replacing $y$ by $yx$ in (15), we have:

$$[\theta(x), \theta(s)] \theta(rt) [\theta(t), \theta(y)] = 0, \text{ for all } x, y, r, s, t \in R \hspace{1cm} (16)$$

Now, (10) can be rewritten as:

$$[\theta(r), \theta(y)] \theta(x) + [\theta(x), \theta(y)] \theta(r) = 0, \text{ for all } x, y, r \in R \hspace{1cm} (17)$$

For $r = t$ in (17), we have:

$$[\theta(t), \theta(y)] \theta(x) + [\theta(x), \theta(y)] \theta(t) = 0, \text{ for all } x, y, t \in R \hspace{1cm} (18)$$

Left multiplying of (18) by $[\theta(x), \theta(s)] \theta(r)$, we obtain:

$$[\theta(x), \theta(s)] \theta(r) [\theta(t), \theta(y)] [\theta(x), \theta(y)] = 0, \text{ for all } x, y, r, s, t \in R \hspace{1cm} (19)$$

From (16) and (19), we obtain:

$$[\theta(x), \theta(s)] \theta(r) [\theta(t), \theta(y)] [\theta(x), \theta(y)] = 0, \text{ for all } x, y, r, s, t \in R \hspace{1cm} (20)$$

Replacing $\theta(r)$ by $[\theta(x), \theta(y)] \theta(r)$ in (20), we have:

$$[\theta(x), \theta(s)] \theta(r) [\theta(t), \theta(y)] [\theta(x), \theta(y)] = 0,$$

for all $x, y, r, s, t \in R$  \hspace{1cm} (21)

For $s = t$ in (21) and since $R$ is semiprime ring, we obtain:
\[ [\theta(x), \theta(t)] [\theta(x), \theta(y)] = 0, \text{ for all } x, y, t \in R \quad (22) \]

Replacing \( \theta(t) \) by \( \theta(yt) \) in (22) and using (22), we have:

\[ [\theta(x), \theta(y)] \theta(t) [\theta(x), \theta(y)] = 0, \text{ for all } x, y, t \in R \quad (23) \]

Since \( R \) is semiprime ring, (23) gives:

\[ [\theta(x), \theta(y)] = 0, \text{ for all } x, y \in R \quad (24) \]

Thus \( R \) is commutative.

**Theorem 2.7**: \( R \) be a 2-torsion free semiprime ring. If \( R \) admits a generalized left \( (\theta, \theta) \)-derivation \( F \) associated with left \( (\theta, \theta) \)-derivation \( d \), where \( \theta \) is an automorphism of \( R \), such that \( d(x) \circ F(y) = [\theta(x), \theta(y)] \), for all \( x, y \in R \).

Then \( R \) is commutative.

**Proof**: We assume:

\[ d(x) \circ F(y) = [\theta(x), \theta(y)], \text{ for all } x, y \in R \quad (1) \]

By Theorem 2.3, (1) gives:

\[ 2d(x)F(y) = [\theta(x), \theta(y)], \text{ for all } x, y \in R \quad (2) \]

Replacing \( y \) by \( yz \) in (2) we obtain:

\[ 2d(x) \theta(y) F(z) + 2d(x) \theta(z) d(y) = \theta(y) [\theta(x), \theta(z)] + [\theta(x), \theta(y)] \theta(z), \]

for all \( x, y, z \in R \quad (3) \]

Again, Theorem 2.3, (3) gives:

\[ 2 \theta(y) d(x) F(z) + 2d(x) \theta(z) d(y) = \theta(y) [\theta(x), \theta(z)] + [\theta(x), \theta(y)] \theta(z), \]

for all \( x, y, z \in R \quad (4) \]

From (2) and (4), we obtain:

\[ 2d(x) \theta(z) d(y) = [\theta(x), \theta(y)] \theta(z), \text{ for all } x, y, z \in R \quad (5) \]

In particular, (5) gives:

\[ 2d(x) \theta(y) d(y) = [\theta(x), \theta(y)] \theta(x), \text{ for all } x, y \in R \quad (6) \]

This implies (see how relation (24) was obtained from relation (6) in the proof of Theorem 2.6) that \( R \) is commutative.

**REFERENCES**


