Methodology to favour the assimilation of theorems

Armando Morales Carballo[1], Edgardo Locia Espinoza[1], Melvis Ramírez Barragán[1] and José María Sigarreta Almira[1]

[1]Universidad Autónoma de Guerrero, México

armando280@hotmail.com, lociae999@hotmail.com, rbmelvis1789@gmail.com, josemariasigarretalmira@hotmail.com

Abstract

In this investigation, a methodology based on Social Constructivism is proposed to favor the assimilation of theorems through the process of Problem Solving in the students of the Higher Technical University level of the Autonomous University of Guerrero, Mexico. The theoretical-qualitative validation of the methodology was carried out by consulting experts who contributed to systematize the link between the theory and the methodological elements and its implication in the design of specific activities for the treatment of the theorem of variable change.

Keywords: Assimilation, change of variable, definite integral, methodology.

Introduction

Llorens and Santonja (1997), document the following didactic and cognitive conflicts that originate during the teaching-learning process of the Defined Integral in college students. Generally, students identify "integral" with "primitive"; "defined" integrals are identified with Barrow's rule, even when it cannot be applied and the concept of area is not integrated with the concept of integral. Certainly, students have heard that there is a relationship between (definite) integrals and area; but there is no proper union between them, henceforth a purely algebraic interpretation of the integral persists.

Taking in to account investigations around the study of the theorems of integral calculus, it has been identified that in most cases their treatment does not prioritize the conceptual elements, it is reduced to procedural elements. In this direction, (Aldana & González, 2009, 2012; Llorens & Santonja, 1997; Muñoz, 2007, 2010) it is declared that the procedures to calculate integrals using the integration methods, are carried out, fundamentally, through repetitive exercises and in a separate way with their conceptual base. López (2010) states that this approach, privilege is given to the algorithmic, achieving that students can often integrate, but do not understand what integral is, nor what its usefulness is.

Mejía (2009) reports on his studies on the perception of conservation notions, comparison and quantification of the area by university students, how students argue and perceive that the area is conserved and quantified. One of the activities he proposed to the students was to solve the integral $\int_0^1 x(x^2 + 1)^2 dx$ and identified that the students established that the conservation of the area is related to the method of integration used, in this case the change of variable. In addition, it was identified that when students use variable change, they do not usually wonder about the nature of the new variable.

Morales (2013) establishes that previous situations show that defined integral activities which are generally practiced on students, the meanings of the change of variable are not emphasized. Furthermore, he emphasizes that another important factor that influences the difficulties that students present with regard to the change of variable in the resolution of the defined integral, as much as the formation of the professor, are the textbooks used by them (Apostle, 2001; Haaser, La Salle & Sullivan, 2003; Leithold, 1998; Piskunov, 1977) which present the technique without focusing on geometric analysis to understand its meaning, so the explanation is centered on analytical elements.

In that direction, two treatments are identified: the first one, the most used, resolves the integral as if it were indefinite and then it is evaluated in the limits of integration; considering null the constant of integration,

situation that does not favor the mathematical sense of the change of variable; and the second consists on changing the limits of integration when the variable is changed, this procedure is based on the theorem of the change of variable, in this property is the argument that under a change of variable both the function and the interval of integration are transformed, and therefore the region changes form, but not the value of its area.

In general, regarding the assimilation of mathematical contents associated with Integral Calculus, investigations shows the multiple factors that make this process difficult, among them the lack of didactic strategies, methodologies and models; as well as professors' misconceptions and beliefs (Aldana & González, 2012; López, 2010; Muñoz, 2007, 2010; Ríos & Cantoral, 2019). In particular, regarding the change of variable in the solution of the integral defined Cabañas-Sánchez (2011); Díaz, Cruz, Velázquez & Molina (2019); Llorens & Santonja (1997); Paschos & Faumak (2006) point out the following difficulties: students use the change of variable to solve the defined integral without attending to the necessary conditions that allow them to make such change and they present insufficiencies to argue the geometric meaning of the change of variable, they have difficulties to relate the concept of defined integral with the area, as well as they do not see the relation between the Theorem of the Change of Variable with the Fundamental Theorem of Calculus.

Several investigations highlight the importance of the methodology to teach theorems, emphasizing their "teaching to search, formulate and demonstrate" as an essential element that favors the processes of assimilation of mathematical contents, that allows the student the foundation and argumentation of procedural and application skills (Cardozo, Molina & Ortiz, 2015; Ricardo & Moya, 2007). In the present document, the problems associated with the insufficient assimilation of the theorems of integral calculus are attended and the objective is the creation of a holistic methodology to favor the assimilation of the theorem of the change of variable in students of the Technical Superior University level of the Autonomous University of Guerrero, Mexico.

Materials and Methods

▪ **Theoretical-methodological basis**

This investigation is based on the contributions of Social Constructivism, which has its theoretical bases in developed studies by Lev Semyonovich Vygotsky (1896-1934) and collaborators and Jean Piaget (1896-1980) and collaborators. From this theoretical perspective, it is considered that the process of assimilation of the human psyche is given on the basis of social experience and furthermore, it is considered that the social relations that occur at a given moment and under a given cultural condition contribute to the development of man's knowledge. Some researchers (Ferreiro, 2003, 2012; García, Ortiz, Martínez & Tintorer, 2009) highlight the importance of human activity mediated by historical and cultural influences. Thus, assimilation begins when subjects get close to the objects of knowledge through a system of actions and operations, that is, when the subject approaches to the objects of knowledge subjects do so through the mental structures they have acquired in his performance and social experience, but it is also necessary for subjects to act on the objects through the tools or socio-cultural signs.

Under this approach, knowledge is developed from a need in practice, a socio-cultural interaction and internalization by the subject but it must also be updated, that is, it is the result of reflections, actions and operations of individuals in various contexts. Thus (Coll, Mauri, & Onrubia, 2008; Cubero, 2005; García, et al., 2009; Londoño, 2008; Marín, 2015) emphasize the relations that exist between learning, understood as the process of construction of meanings and giving sense to the contents and teaching as the process that helps in a systematic and sustained way to the learning process, this is possible by the sequences of joint activities, which are carried out between professors and students for periods in which they practice with activities according to the content.

Social Constructivism is based on two basic principles: teaching guides development and the development of knowledge occurs in two moments; in the first moment the subject is in the Zone of Current Development (ZCD), formed by all the knowledge that the subject has managed to develop up to that moment; the objective is to reach the Zone of Potential Development (ZPD), in which resides the desired knowledge for students to achieve. This structure allows the dosage of the student's development through stages, in which each stage is designed to achieve a particular objective using various tools that the professor considers necessary to use.

From the positions of Social Constructivism we assume assimilation as a "cognitive process sustained by the mental structures that the individual has acquired in his social relations (previous knowledge, experiences, skills, habits and motivations); which allows the understanding, updating, development and application of a certain knowledge". Thus, for our study the process of assimilation of theorems will be structured by the following stages: understanding, identification, fixation, application and evaluation.

Bransford & Stein (1987); Cala, Buendía & Herrera (2017); Campistrous & Rizo (1999); Pérez & Ramírez (2011); Polya (1999); Santos-Trigo & Moreno-Armella (2016); Sigarreta & Laborde (2004); Silva, Rodríguez & Santillán (2009); Schoenfeld (1985, 2001); Valle & Curotto (2008); Valle, Juárez & Guzmán (2007) have focused their academic efforts on the process of Problem Solving by proposing different techniques and strategies, which refer to a pattern of decisions that must be made to achieve the acquisition, retention and use of information that serves to achieve a certain objective when solving a problem. Based on the above-mentioned investigation, the following problem-solving strategy will be implemented in the present investigation: Action 1. Approach to the problem; Action 2. Deepening of the problem; Action 3. Selection of a working route; Action 4. Application of the selected route and Action 5. Assessment.

With the methodological treatment of theorems, students are expected to understand the mathematical and functional essence of the relationships expressed in them and to be able to apply them in the substantiation of their statements and reasoning, as well as in the resolution of intramathematical and extramathematical problems (Ballester, 2007; Ballester et al., 1992; Cruz-Ramírez, 2006; González & Sigarreta, 2011; Martínez, Infante & Brito, 2017; Morales, Locia, Mederos, Ramírez & Sigarreta, 2018). From the methodological point of view, these authors, for the treatment of theorems in order to favor their assimilation, structure it in two stages: Search of the Theorem and Search of the Proof.

Our goal is to train students to search for and formulate a certain theorem by solving a structured system of problems. For the methodological structuring, specific aspects of the processes for the search of theorems are presented below. The actions and operations of the professor and the students must be directed to carry out the process through the system of specific activities, which allow to establish a certain assumption or conjecture. In this sense, the partial search is directly related to heuristic resources and problem solving (see Ballester, 2007; Ballester et al., 1992; Cruz-Ramírez, García & Sigarreta, 2016; Morales et al., 2018).

▪   **Methodology to favour the assimilation of theorems**

By methodology we mean the "system of actions and operations that allows to organize, structure, develop and evaluate the teaching-learning process of a certain content". For this methodology, the stages are considered: approach, planning, concretion and assessment; it should be noted that in each stage a partial control is carried out. For each stage of the methodology, actions and operations are proposed, aimed at favoring the teaching-learning process to achieve the assimilation of theorems through problem solving. Next, the holistic methodology for the treatment of integral calculus theorems is described.

APPROXIMATION. ACTIONS:

1. To understand the theoretical foundations that guarantee the implementation of the proposed methodology.

2. To know the student and the socio-cultural environment where the activity takes place.

OPERATIONS:

- Analyze and synthesize each theoretical foundation of the proposed methodology from an epistemological point of view.

- To relate the main elements associated with the theoretical foundations considering the socio-cultural conditions under which the activity is developed.

- To diagnose the student integrally, considering the spheres: cognitive, affective-motivational and volitional associated to the context where the activity is developed.

PLANNING. ACTIONS:

3.   Set out the objectives to be achieved in the assimilation of theorems.

4.   Determine the necessary elements for the assimilation of theorems.

5.   To design the system of activities to achieve the proposed objectives.

OPERATIONS:

- Establish a hierarchical order (depending on the importance of the theorem within the mathematical formation and the results of the diagnosis) of the theorems of the subject, according to the selection of the theorem to be treated.

- Elaborate an integral profile of the students from the results of the diagnosis.

- Consider the moment of development of the content (what are the topics that the students need or are in a position to learn).

- To carry out a knowledge update (guidelines for the search, representation and application of the theorems).

- To design a system of activities considering the results of the diagnosis (knowledge, skills, attitudes, habits and values), according to the achievement of the objective.

VALUATION. ACTIONS:

10.   Critical analysis of the system of actions and operations carried out by teachers and students.

11.   Feedback from actions and operations.

12.   Analysis and evaluation of the results of the activities.

OPERATIONS:

- It is worth mentioning that each of the stages of the process has a partial control action. And in this stage of Valuation a global control is made from the partial controls.

- Reflect on the system of actions and operations, are they adequate to achieve the objectives?

- Reflect on the methods applied, procedures, results and errors, in order to critically assess the route used for the assimilation of the theorem.

- To corroborate the achievement of the proposed objectives.

- To reformulate and/or enrich the system of activities based on the experience gained in the process.

Scheme 1. Methodology to favour the assimilation of theorems

A system of activities was designed to favour the assimilation of the theorem of the change of variable in the defined integral. The design consisted of 13 problems with respective sub-sections, 4 associated with the initial diagnosis with the objective of knowing the students' starting level and 9 standard problems (associated with the stages of the assimilation process) to be developed in class. As an example we present some of these problems:

▪   **Activity system**

Diagnosis.

Do you consider that there is a difference between the definite and the indefinite integral?
Objective: To identify the students' conceptual bases.
Solve the following integrals.
$\int (x^2 + 3x + 1)\, dx$. b) $\int_0^\pi \sin x \; dx$.
Objective: To identify the operational and/or procedural bases of the students.
Solve the following integrals.
a) . b) $\int_0^\pi \sin(x) \cos(x)\, dx$

Objective: To analyze whether they recognize the differences and procedures that apply for the resolution.

Solve the integral $\int_0^2 2(x+1)^2 \, dx$ by changing the variable $u = x + 1$ and represent the respective regions and integration functions.

Objective: To corroborate the implicit processes in the application of the variable change theorem.

Type problems to develop in class.

Let $f(x) = 4$ be a constant function. Determine the area below this line, above the horizontal axis and between the lines $x = 0$ and $x = 4$.

Represent the region.

Obtain the area of the region by posing and solving a defined integral.

Objective: To understand the relationship between the defined integral and the area of a given region.

Let $f(x) = ax$ be a given function, with a constant. The following figure shows three cases associated with different values of parameter a, showing that the regions formed by the respective lines vary.
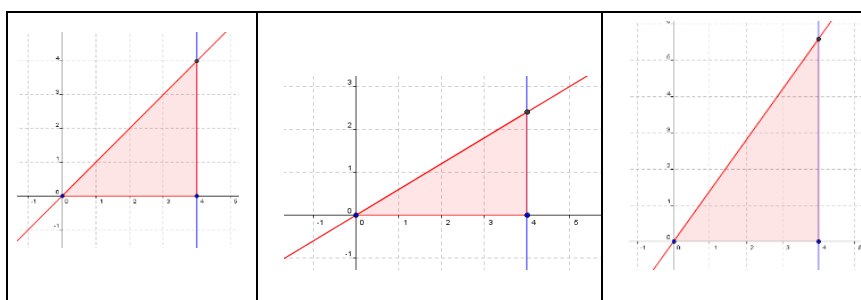

Figure 1.

a) Determine the value of a to verify that the area of the region under the curve $f(x) = ax$, above the horizontal axis and between the lines $x = 0$ and $x = 4$, is 16 square units.

b) Can another region be constructed that is bounded at the top by the $f(x) = ax$ curve, at the bottom by the horizontal axis, and at the side by the $x = x_1, x = x_2$ lines, whose area is 16 square units? Argue your answer and determine if possible the values of a, x_1, x_2.

c) What are the relationships between the previously constructed regions, according to their shape and area respectively? Explain your answers.

Objective: To understand the change in the regions of integration and their relationship with the defined integral.

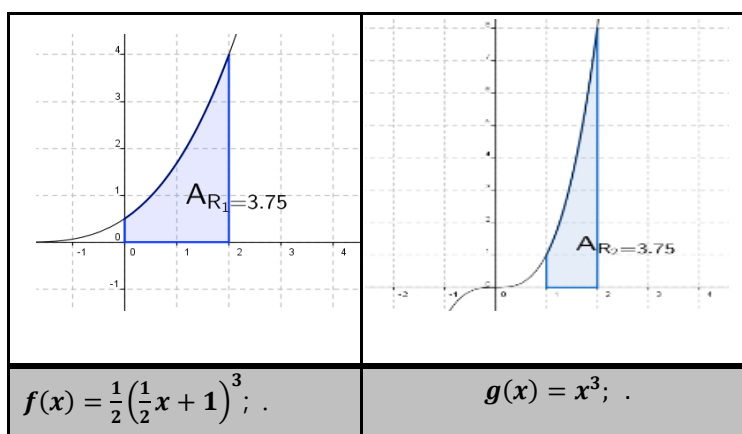7. The following figure shows two regions in the plane with the same area.

$f(x) = \frac{1}{2}\left(\frac{1}{2}x + 1\right)^3; \ .$	$g(x) = x^3; \ .$

Figure 2.

a) How do you justify that, in effect, the areas of the regions are the same?

b) What are the mathematical implications of making a variable change?

c) What effects does the application of the variable change theorem have on the resolution of the integral $\int_0^2 \frac{1}{2}\left(\frac{1}{2}x + 1\right)^3 dx$?

Objective: To set the scope of change of variable in the resolution of defined integrals.

8. Represent the region bounded at the top by the curve of $f(x) = x + 1$, at the bottom by the horizontal axis and laterally by the lines $x = 0, x = 4$.

a)   Graph the region and calculate its area, without using integral calculus.

b)   To calculate the area of the region above, the following integral is proposed $A(R) = \int_0^4 (x + 1)dx$.

Applying the change of variable, $u = x + 1$ with $du = dx$ follows that, $A(R) = \int_0^4 (x + 1)dx = \int_0^4 u\,du = \frac{4^2}{2} = 8$.

Discuss the results obtained.

Objective: To fix the theorem of the change of variable of the defined integral, in particular, to recognize the error associated with the region of integration.

9. Determine the value of the integral $\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \frac{\cos x}{\sin x} dx$. The following graph shows the region under the curve $f(x) = \frac{\cos x}{\sin x}$, above the horizontal axis and between the lines $x = \frac{\pi}{4}$ and $x = \frac{\pi}{2}$.
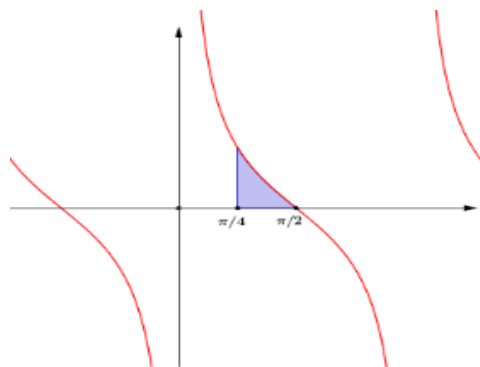

Figure 3.

Objective: To apply the theorem of change of variable in the solution of the defined integral.

10. Prove that $\int_0^2 2(x + 1)^2 dx = \int_1^3 2u^2 du$.

Objective: To apply the variable change theorem as a demonstration method.

11. What is the relationship between $\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \frac{\cos x}{\sin x} dx$ y $\int_{\frac{1}{\sqrt{2}}x}^{1} \frac{1}{x} dx$? In the resolution of an integral defined by the variable change theorem, what variations do you consider occur in the integrating function in the given interval?

Objective: to evaluate the effects produced by the change of variable in the region of integration and in the calculation of the defined integral.

12. Calculate using the universal substitution $u = \tan\frac{x}{2}$, the area of the region shown in Figure 3, limited at the top by the curve $f(x) = \frac{1}{2+\cos x}$, at the bottom by the horizontal axis and at the side by the lines $x = 0, x = 2\pi$.
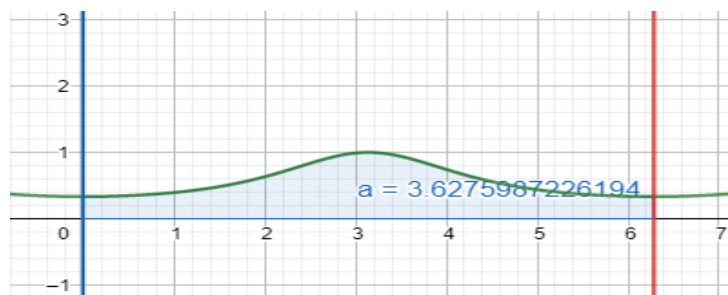
69

Figure 4.

Objective: To assess the necessary conditions for the application of the variable change theorem.

13- What are the necessary conditions for the application of the variable change theorem in the resolution of defined integrals? How would you formulate the variable change theorem for defined integrals?

Objective: To approximate a formulation of the variable change theorem.

In the Concrete Stage where the system of actions and operations is applied to favor the assimilation of theorems through problem solving.

STRATEGY (ACTIONS)	STAGES OF THE ASSIMILATION PROCESS	OPERATIONS
Approach to the problem	Understanding	- Read the problem. - Are all the terms involved in formulating the problem familiar to you? - State the problem in your own words. - Specify what is given and what is sought. - Identify the knowledge that relates to what is given and what is sought. - Is the problem intramathematical or extramathematical? - To which field of knowledge do you associate the problem posed? - What kind of problem are you going to face? - Does it require the use of mathematical knowledge? - Have you seen one formulated in a similar way? - Is it a problem related to your socio-cultural environment?
Deepening the problem	Identification	- Determine the knowledge needed to address the problem. - Have you ever seen one formulated in a similar way? - Establish analogies. - Experiment with other data. - Propose regularities. - Have all the possible cases been checked? - Underline the expressions that you consider to have the greatest semantic value in the problem. - Look for synonyms and antonyms of the terms you consider fundamental.

		- Establish the unknown(s), that is, what is being sought.
		- Determine the data that are given directly in the formulation of the problem.
		- Among which values should it be found?
		- In a second moment, a scheme, diagram, table, etc. can be elaborated.
		- Is there enough or too much data?
		- Are there contradictory elements?
Selecting a working route	Fixation	- Transforming the problem into an equivalent one.
		- How can the data be related to the unknown(s)?
		- What inferences can be made from the data found?
		- Evaluate counterexamples.
		- Use mathematical terminology and symbols correctly.
		- Choose an appropriate language or notation.
		- Delimit what knowledge is related to the elements of the problem.
		- Ground ideas, judgments and arguments.
		- Select the properties or definitions that can be useful.
		- Assume the problem is solved.
		- In which field of knowledge does the problem posed move: arithmetic, algebraic or geometric?
		- Delimit which knowledge system is related to the elements of the problem.
		- Which of them are related to the premise or the thesis of the problem?
		- Could a possible answer be given?
		- What conjectures can you make?
Application of the selected route	Application	- Establish relationships in correspondence with the ways of working.
		- Use cognitive and metacognitive strategies.
		- Execute a plan of action and consciously apply procedures.
		- Make equivalent transformations in the premise and/or the thesis.

		- Analyze the possible ways of solution.
		- Consider particular and general cases.
		- Integrate possible results to be used in different contexts.
		- Can you apply the same working technique to another situation?
		- Make assumptions based on the possible solutions.
		- The method used to solve other problems can be generalized.
Evaluation Reflect and take a critical stance on the system of actions and operations carried out.	Evaluation	- Could you formulate the general idea in the form of a proposal? - Analyse the steps and actions taken, analyse the errors and their possible causes, specify how to avoid the errors. - Explain the process of reasoning that you carried out and give the details about it. - Reflect on the procedures and methods used. - Choose an appropriate language or adequate notation. - Are all the solutions found solutions to the problem? - Explain in your own words how you got to the solution. - Does the answer given make sense in relation to your experience? - Does the proposed solution really respond to the problem in question? - What did you bring to the problem from a social and/or mathematical point of view?

Scheme 2. Assimilation strategy

Results and Discussion

▪ **Theoretical and qualitative validation of the methodology**

Delphi method. Campistrous & Rizo, (2006) and Cruz-Ramirez, (2009) establish that this method consists of analyzing the responses of a group of experts into a questionnaire. Then, an analysis of the possible answers is made. However, since there are no **references** to the methodology at UAGro, we will only use one particular case of the Delphi Method, the so-called Expert Method. The Expert Criteria applied to the methodology, focused on investigating the following aspects:

1.- Planning.

    (a) Elements to be considered in the Validation
        1. Scientific rigour.
        2. Coherence.

3. Adequacy.
(b) Criteria for the Selection of Experts.

2.- Exploration.

(a) Preparation of the Questionnaires.
(b) Application of the questionnaires to the Experts.

3.- Evaluation.

(a) Presentation and analysis of the information.
(b) Criteria for the Selection of Experts:
  (i) Academic and mathematical training.
  (ii) Training with respect to the Teaching of Mathematics.
  (iii) Experience as a teacher and/or investigator.

 (c) Elaboration of the questionnaires.

▪ **Application of Dephi Method.**

For the application of the method, a questionnaire was developed that contains questions that are assigned a value, since each question has the possibility of several answers. The information is then processed using non-parametric experimental design techniques. The questionnaire was applied to eight experts: all of them specialists in Mathematical Analysis and Didactics of Mathematics.

Expert	Question	Answer
1	2	It adheres to a solidly established scientific theory and thus meets the requirements of scientific rigour.
	4	The experiences referred to in the thesis make us suppose that it is possible to do so once the students master the basic knowledge of Integral Calculus.
3	1	The methodology fulfills the expectations, essentially, by being based on a current theory.
	4	Taking into consideration that the students already have the basic knowledge of Integral Calculus, I consider that the approach is the right one.
4	1	From my experience as a teacher, the methodology is coherent with the theories analyzed and the daily practices developed in the classroom.
	4	Yes, I believe that the system of activities is adequate as long as the teacher takes into account the amount and types of problems needed to fulfill the objective of each stage.
6	3	In that sense, it seems to me that the stages are well structured as a system, I consider that the expectations of achieving the objectives are fulfilled.
	8	The system of activities is complete and has the necessary elements for the assimilation of the Theorem of Change of Variable of Integral Calculation.

Scheme 3. Expert responses

Conclusions

As a result of the validation of the methodology and the system of specific activities for the process of assimilation of the variable change theorem, it is concluded that the proposal satisfies the aspects of Scientific Rigour, Coherence and Adaptation for its development in the classroom. This proposal has also been enriched by the experiences resulting from its presentation and discussion in national and international congresses,

workshops, courses, conferences and colloquiums that have been developed during this investigation. The experts emphasize that the studies adheres to an established scientific theory, the approach to its implementation at the indicated level is adequate, that is, the integrative methodological resources that affect teaching and learning are identified, in particular the theorem under study.

References

Aldana, E. & González, M. T. (2012). Análisis de la comprensión del concepto de integral definida en el marco de la Teoría APOE. En Obando, Gilberto (Ed.), *Memorias del 13er Encuentro Colombiano de Matemática Educativa* (pp. 689-705). Medellín: Universidad de Medellín.

Aldana, E. & González, M. T. (2009). Comprensión del concepto de integral definida, el caso de un alumno Universitario. *XIII Simposio de la Sociedad Española de Investigación Matemática (SEIEM),* Santander, España.

Apostol, T. (2001). *Cálculus, volumen 1*. Massachusetts, Unites States: Blaisdell Publishing Company.

Ballester, S. (2007). *Matemática superior I*. La Habana, Cuba: Pueblo y Educación.

Ballester, S., Arango, C., Hernández, S., Almeida, B., Santana, H., & Torres, P. (1992). *Metodología de la enseñanza de la matemática, Tomo II*. La Habana, Cuba: Editorial Pueblo y Educación.

Bransford, J. & Stein, B. (1987). *Solución IDEAL de Problemas*. Barcelona: Labor.

Cabañas, G. (2011). *El papel de la noción de conservación del área en la resignificación de la integral definida. Un estudio socioepistemológico.* Tesis de doctorado, Departamento de Matemática Educativa, Cinvestav-IPN. México.

Cala, A., Buendía, A. M., & Herrera, L. J. (2017). *Métodos y estrategias para la resolución de problemas matemáticos: una revisión desde las investigaciones de la última década.* Proyecto de grado, Medellín, Colombia.

Campistrous, L. & Rizo, C. (1999). *Estrategias de resolución de problemas en la escuela cubana.* Revista Latinoamericana de Investigación en Matemática Educativa, 2(3), 31-45.

Campistrous, L. & Rizo, C. (2006). *El criterio de expertos como método de investigación educativa. Documento elaborado para el Doctorado Curricular*. La Habana, Cuba: Instituto Superior de la Cultura Física "Manuel Fajardo".

Cardozo, S., Molina, O., & Ortiz, A. (2015). Tratamiento de los teoremas de existencia en un libro de geometría plana. *Memorias del Encuentro de Geometría y sus Aplicaciones,* 155-162.

Coll, C., Mauri, M. T., & Onrubia, J. (2008). Analyzing Actual Uses of ICT in Formal Educational Contexts: A Socio-Cultural Approach. *Revista Electrónica de Investigación Educativa, 10 (1),* 1-17.

Cruz-Ramírez, M. (2006). *La enseñanza de la Matemática a través de la resolución de problemas.* Educación Cubana.

Cruz-Ramírez, M. (2009). *El método Delphi en las investigaciones educacionales.* La Habana, Cuba: Academia.

Cruz-Ramírez, M., García, M. & Sigarreta, J. M. (2016). Analogies in Mathematical Problems Solving. *Journal of Science Education, 2 (17),* 84-90.

Cubero, R. (2005). Elementos básicos para un constructivismo social. *Avances en la Psicología Latinoamericana. 1(23),* 43-61

Díaz, Y., Cruz, M., Velázquez, Y., & Molina, S. (2019). Estrategias didácticas para desarrollar el proceso de enseñanza-aprendizaje de los contenidos de las Derivadas de funciones reales de una variable real y aplicaciones. *Épsilon, 103,* 7-23.

Ferreiro, R. (2012). *Cómo Ser Mejor Maestro. El método ELI.* México: Trillas.

Ferreiro, R. (2003). *Estrategias didácticas del aprendizaje cooperativo. Una nueva forma de enseñar y aprender: el constructivismo social.* México: Trillas.

García, H. J., Ortiz, A. M., Martínez, J., & Tintorer, O. (2009). La teoría de la actividad de formación por etapas de las acciones mentales en la resolución de problemas. *Revista Científica Internacional. Inter Sciencie Place, 2(9),* 1-25.

González, J. & Sigarreta, J. M. (2011). Ideas para enseñar. Hacia una generalización del teorema de Pitágoras. *UNIÓN. Revista Iberoamericana de Educación Matemática, 27,* 179-194.

Haaser, N., La Salle, J., & Sullivan, J. (2003). *Análisis matemático, volumen 1. Curso de introducción (1ra ed.).* México: Celigraf, S.A.

Leithold, L. (1998). *El Cálculo (7ª ed.).* Oxford University Pres.

Londoño, G. M. (2008). Aprendizaje colaborativo presencial, aprendizaje colaborativo mediado por computador e interacción: Aclaraciones, aportes, y evidencias. *Revista Educación comunicación Tecnología, 2(4),* 1-22.

López, A. (2010). Propuesta para la enseñanza del concepto de integral un acercamiento visual con Geogebra. *Acta Latinoamericana de Matemática Educativa, 23,* 1351-1358.

Llorens, J., & Santonja, F. (1997). Una Interpretación de las Dificultades en el Aprendizaje del Concepto de Integral. *Divulgaciones Matemáticas, 5(1/2),* 61- 76.

Martínez, J. E., Infante, R., & Brito, M. (2017). Los procesos didácticos para el tratamiento de teoremas matemáticos en el nivel medio superior. *Universidad y Sociedad, 9(2),* 145-153.

Marín, E. (2015). Aprendizaje constructivista para el análisis de estructuras mediante el uso de un entorno virtual. *Revista Tecnocientífica URU,* 41-49.

Mejía, O. (2009). *Percepción de las nociones de conservación, comparación y cuantificación del área por estudiantes universitarios. Un estudio socioepistemológico a través de sus argumentos.* Tesis de Maestría no publicada. Universidad Autónoma de Guerrero. México.

Morales, A., Locia, E., Mederos E., Ramírez, M. & Sigarreta, J. M. (2018). The Theoretical didactic approach to the counterexample in mathematics. *International Journal of Research in Education Methodology, 9 (1),* 1510-1517.

Morales, A. (2013). *Estrategia metodológica para el tratamiento del concepto de límite al infinito.* Tesis de doctorado no publicada, Unidad Académica de Matemáticas, Universidad Autónoma de Guerrero. México.

Muñoz, G. (2010). Hacia un campo de prácticas sociales como fundamento para rediseñar el discurso escolar del cálculo integral. *Revista Latinoamericana de Investigación en Matemática Educativa, 13(4-II),* 283-302.

Muñoz, G. (2007). Rediseño del Cálculo integral escolar fundamentado en la Predicción. En C. Dolores, G. Martínez, R. Farfán, C. Carrillo, I. López & C. Navarro (Eds.), *Matemática Educativa: Algunos aspectos de la socioepistemología y la visualización en el aula* (pp. 27-76). México, Guerrero: Universidad Autónoma de Guerrero.

Paschos, Th. & Faumak, V. (2006). The reflective abstraction in the construction of the concept of the definite integral. A case study. En J. Novotna, H. Moraova, M. Kretke, & N. Stehlikova (Eds.), *Proceedings of the 30th Conference of Internacional Group for the Psychology of Mathematics Education* (pp. 337- 344).

Perez, Y. & Ramírez, R. (2011). Estrategias de enseñanza de la resolución de problemas matemáticos. Fundamentos teóricos y metodológicos. *Revista de Investigación 3(35).*

Piskunov, N. (1977). *Cálculo diferencial e integral (3ª ed.).* Mir Moscú. Traducido del ruso. URSS.

Polya, G. (1999). *Cómo plantear y resolver problemas.* México: Trillas.

Ríos, D. W. & Cantoral, R. A. (2019). Estudio socioepistemólogico acerca de los vínculos entre los teoremas fundamentales de la aritmética, el algebra y Cálculo. *Acta Latinoamericana de Matemática Educativa, 32(4-II),* 134-139.

Ricardo, A. & Moya, D. (2007). Propuesta metodológica al tratamiento de teoremas y sus demostraciones. *EduSol, 7(20),* 1-8.

Santos-Trigo, M., Moreno-Armella, L. & Camacho-Machín, M. (2016). Problem solving and the use of digital technologies within the Mathematical Working Space framework. *ZDM Mathematics Education 48,* 827–842. https://doi.org/10.1007/s11858-016-0757-0.

Sigarreta, J. M. & Laborde. J. M. (2004). Estrategia para la resolución de problemas como recurso para la interacción sociocultural. *Revista PREMISA, 6(20),* 15-29.

Silva, M., Rodríguez, A., & Santillán, O. (2019). Estrategias de resolución de problemas matemáticos empleados por estudiantes de sexto grado de primaria. *X Congreso Nacional de Matemática Educativa.*

Schoenfeld, A. (1985). *Mathematical Problem Solving.* Orlando: Academic Press.

Schoenfeld, A. H. (2001). Objetivos y métodos de la investigación en educación matemática. *La Gaceta, 47(6),* 185-205.

Valle, M. C., Juárez, M. A., & Guzmán, M. E. (2007). Estrategias generales en la resolución de problemas de la olimpiada mexicana de matemáticas. *Revista Electrónica de Investigación Educativa, 9(2).*

Valle, M. C., & Curotto, M. M. (2008). La resolución de problemas como estrategia de enseñanza y aprendizaje. *Revista Electrónica de Enseñanza de las Ciencias, 7(2),* 463-479.