

DOI <https://doi.org/10.24297/ijrem.v11i.8681>

# Understanding University Students About Improper Integral

\*Angie Damián Mojica, \*\* Flor Monserrat Rodríguez Vásquez & \*\*\*Jesús Romero Valencia

Facultad de Matemáticas, Universidad Autónoma de Guerrero, México.

\**adamian@uagro.mx*, \*\**flormor@gmail.com*, \*\*\**jromv@yahoo.com*

## Abstract

The study of the understanding of mathematical concepts is of great interest for research in Mathematics Education. In this sense, an investigation is carried out on the understanding of the concept of improper integral, a basic concept from the first courses of Bachelor of Mathematics for subsequent courses. The objective is to characterize the understanding of this mathematical concept through a proficiency model for this purpose, an instrument was designed that was applied to second-year university students. The research is qualitative exploratory and is based on the framework of the Theory of records of semiotic representations.

**Keywords:** Comprehension, Improper Integral, Semiotic Representations, Competition Model.

## 1 Introduction

The present investigation is about the understanding of the Concept of Improper Integral of university students within the framework of Raymond Duval's theory of semiotic representation. The study of the understanding of the mathematical concepts is a field of great interest for the investigation in Mathematical Education, therefore, in our investigation we deepen in the development of the understanding of this mathematical concept, with students of first and second year of Degree of Mathematics, in an age range of 18 to 22 years. The scope in which the work carried out is located is in the line of research in Didactics of Mathematical Analysis and in the field of Advanced Mathematical Thought. The research that turns the bull into the improper integral, the compression of concepts and finally we will formulate the problem, the question and as well as the research objectives are presented. There is talk of the mathematical object in question, the improper integral, the convergence criteria and how it is found in the Mexican curriculum are mentioned, in particular in the curriculum of the Bachelor's Degree in Mathematics of the Autonomous University of Guerrero. Aspects of the theoretical view with which it was carried out mentioning elements of Advanced Mathematical Thought (PMA) and the Theory of Registries of Semiotic Representation (TRRS) (Duval, 1990, 1996, 2003, 2006) are presented. They mention the three cognitive activities that interest in this research, training, treatment and conversion, which are presented in the competence model (Socas, 2001), used to give evidence of the students' understanding of the concept of improper integral In this sense, aspects of concept compression are also mentioned.



## 1.1 Research background

Several of the investigations have focused on the understanding that students have about the contents of the calculation, however, it is good to first recognize the great variety of important educational research that has been done on other topics related to the calculation, it is essential that be done because calculation is a key part of education in science, technology, engineering and mathematics (STEM).

Mahir (2009) mentions that calculus is one of the fundamental courses of mathematics, provides a basis for more advanced mathematical subjects, therefore, it is crucial for a student to understand the concepts of calculus and be able to apply them. However, several studies have shown that students have difficulties in understanding these concepts, the author suggests that for a mathematical education to be successful it is necessary to balance the conceptual and procedural knowledge of students emphasizing concepts and relationships instead of Problem solving techniques, is the knowledge that is connected to the other pieces of knowledge, and also recognizes the connection, these connections between the pieces are as important as the pieces themselves.

The limit, derived and integral are three of the central ideas in calculation and much of the educational research in calculation focuses on these concepts, the latter is supported by the research of Sofronas, DeFranco, Vinsonhaler, Gorgievski, Schroeder and Hameline (2011) where they present the views of 24 nationally recognized experts in the United States in the field of mathematics in particular of calculus, on the students' understanding of the first-year university calculus, in this study five concepts are identified and Fundamental skills: derivatives, integrals, limits, sequences and series, and approximation. On the other hand Törner, Potari and Zachariades (2014), in their study they carry out a survey of university professors of mathematics in France, Germany, England, Belgium, Italy, Greece and Cyprus on topics such as the calculation curriculum, teaching approaches and The use of technology, its results show that the limit, derivative and integral concepts are the central characteristics of the content of the calculation courses in the first years. The first research on calculus learning in the 80's and 90's focused mostly on the identification of misconceptions and interventions towards students that allowed them to overcome the misconceptions. One of the first studies that identifies students' difficulties and misconceptions with integration is described in an article by Orton (1983), which concludes that students often knew what to do, but when asked they did not know why They were doing, states that there was a set of issues related to the understanding of integration as the limit of a sum that was challenging even for the best students.

Sofronas et al. (2011) affirm that the integral is a fundamental mathematical concept with three important subconceptions critical for the understanding of the students and the ability to apply the calculation to a wide variety of problems: the integral as net change or accumulated total change, the integral as Integration area and techniques. Further studies, for example, Mahir (2009) and Rasslan and Tall (2002) support the idea that most students who have completed a calculus course have a very technical vision of integration and cannot explain what it means or how Interpret it in context. On the other hand, Rasslan and Tall (2002) concluded that the students could not write in a meaningful way the definition of the definite integral and had difficulties to interpret the problems when calculating areas and definite integrals in a wider context, in relation to it Sealey (2006) points out that the area under a curve is not a bad representation, it simply states instead that the area under a curve is not enough to understand the integral definition, it also mentions that the students' understanding of the integral is a fundamentally valuable, since integration serves as the basis for many real-world applications and subsequent courses.

The integral concept and the FTC are complex in terms of the amount of things that must be coordinated, and the emerging cognitive models reflect this complexity. Perhaps doing this work can contribute by helping the field coordinate the numerous mental processes, objects and images involved in student work with integration and FTC.

Existing research also raises questions about the role of some aspects of formal theory (for example, Riemann sums, limits) and traditional instructional approaches such as emphasizing the area and definite integrals in supporting and restricting student integration learning, and the FTC. Thompson, Byerley, and Hatfield (2013) reported on efforts to develop a calculus course in order to help students build a reflexive relationship between the concepts of accumulation and exchange rate, symbolize that relationship and then extend it to have wider scope. The approach causes students to get involved in reasoning about the FTC daily.

Camacho and Aguirre (2001), pose some improper integrals to a group of students and teachers; They check that they have great difficulties in calculating their values and many of them are lost in algebraic stones. It is also observed that both teachers and students tend to replace extreme values of the integral and do not calculate the integrals using limit processes, but generalizing the rule of Barrow and directly replacing the ends of the integral in the variable, even when one of the extremes be infinite.

The works of González-Martín and Camacho-Machín (2002a, 2002b, 2003, 2004, 2005), on integrals have continuity with investigations on improper integral where they state that the fact that the improper integral is defined as an extension of the Integral of Riemann, when some conditions are not met, causes students to have difficulties in interpreting this concept and encounter obstacles, they also showed students' preference for the use of algebraic records, the lack of associated meanings and prior knowledge of the concepts of convergence, series and definite integrals were the main difficulties that prevented progress towards the improper integral.

Gonzalez-Martín (2006), carries out a generalization of the definite integrals, where he shows that the students fail to understand this concept adequately or to relate it to other previously studied knowledge such as defined successions, series and integrals, limit in their first year at the University, it builds a didactic engineering based on epistemological analysis, the main objective was, on the one hand, to analyze the processes of advanced mathematical thinking involved in the learning and manipulation of improper integrals considered as generalization of defined integrals, in addition to investigating in the most common obstacles, difficulties and errors that arise in this context, and on the other hand, to develop in the classroom subsequently a previously designed teaching sequence that promotes more meaningful learning, its proposal is characterized, mainly, by combining more balanced records graphic and algebraic, actively using examples and counterexamples that enrich the experiences of students, and by resorting to the use of the CAS (Computer Algebra System) Maple V to promote the visualization and operationalization of some theoretical results. This supports the results obtained by Fey and Wibster (1985), which mention the lack of connections between series and improper integrals in the analysis of university books on the teaching of the integral for the convergence of series.

González-Martín, Nardi and Biza (2011) combine mainly the frameworks of the Theory of Registers of Semiotic Representation of Duval (TRRS) and the Anthropological Theory of the Didactic (TAD), for textbooks on infinite sums at the university level in Quebec and the United Kingdom. By focusing on the role of visualization in mathematical materials, this study showed few graphic representations and few opportunities to work with different records (algebraic, graphic, verbal), few applications or intra-mathematical references to the importance of the concept and few tasks. Conceptually driven to go beyond practice with the application of convergence tests and prepare students for complex issues that employ the concept of the series where it is involved. The authors see the concept of series deeply intertwined with other themes and manifest a more conceptual compromise between this concept and the notion of convergence, to support limited perceptions.

On the other hand Mendo, Castañeda and Tarifa (2018) show evidence of an analysis of the mathematical arguments affected by a group of students when solving a didactic sequence about the improper integral in an environment of use

of technology, with the purpose of identifying the lines of reasoning and argumentation and describe their mathematical thinking, through the analysis of the data, the argumentative model of Toulmin was used, which describes the construction, content and structure of the students' arguments when traveling between different representations during their solution. The integration of three elements (ACODESA methodology, the Toulmin model and the technological environment, specifically Geogebra software) in a cooperative learning, in which first the scientific debate and the self-reflection combined with a technological environment is adapted as the main work hypothesis for the formation of mathematical arguments in students, with emphasis on those who face the improper integral.

Galán-García, Aguilera-Venegas, Rodríguez-Cielos, Itencia-Mc.Killop (2018), in their study mention that the classic way of solving improper integrals is through elementary Calculus (antiderivatives and calculation of limits) or using numerical approaches To solve this situation, advanced calculation techniques can be applied, such as Laplace or Fourier transform and the residual theorem, the authors describe some Advanced Calculation techniques for the calculation of improper integrals of the first type that cannot be solved with procedures standard. These techniques could be easily integrated into almost all Computer Algebra System (CAS), however they detected some shortcomings of these techniques in many of them.

## 1.2 Formulation of the problem, question and objectives.

When conducting a review of the existing literature on research that revolves around calculus, we see the enormous study carried out on the integral defined from various positions and approaches, and we can see that there is little research on the study of improper integrals, which Although it is true it is a generalization of the definite integrals, there is little literature about them, we see that this is due to what the concept includes by itself, that is to say, the integration of other mathematical concepts of university calculus; series, limits, convergence, functions, etc. Although there are investigations that talk about these other concepts in isolation, there are few that integrate them, the latter would favor the understanding of topics such as improper integral in students of first years of University in courses of Calculation remedying, errors and difficulties that they lead to the difficulty in understanding this concept and the mechanization of procedures on it.

The difficulty in understanding that students show of mathematical objects and, more specifically, the difficulties encountered in learning the concept of Improper Integral, how it is presented, and how some of them place greater emphasis on algorithmic procedures, before that in the conceptual understanding of mathematical object and the contributions found in different studies by some researchers, motivate this research. In order to make an approach to the problem, we ask ourselves the following research question: How do undergraduate students understand the notion of improper integral when they travel through the different semiotic presentation records? The above makes us consider the following objectives:

- Characterize the different levels of understanding of the concept of improper integral through a competition model.
- Identify the cognitive activities of treatment and conversion that allow mobilizing the elements and properties of the improper integral in its different registers of semiotic representation.

## 2 Study of the mathematical object

To define the Riemann integral of a certain function  $f(x)$  in an interval  $[a, b]$ , it is necessary that the integration interval be closed and bounded and that the function be bounded within the interval. When one of these two conditions is not met, the improper integral is defined as a generalization of the Riemann integral. This concept, with multiple applications (probabilities, functional norms, Fourier transforms, ...), offers great resistance to university students, who learn it without giving it meaning and restricting themselves to algebraic calculations and the application of convergence criteria (González-Martín, 2002).

In the definition of the Riemann integral as already mentioned, two fundamental conditions were imposed:

1. It is defined in the closed and bounded interval  $[a, b], a < b$
2. It is defined for functions bounded in  $[a, b]$

The notion of integral is extended to the case of unbounded intervals, and to the case of unbounded functions over a bounded interval. These two extensions give rise to improper integral calls of first and second species respectively.

**Improper Integral of First Species (Interval not Bounded).** Sea  $f : [a, +\infty) \rightarrow \mathbb{R}$  we will say that  $f$  is integrable in  $[a, +\infty)$  if it is fulfilled that:

1.  $\forall x \in (a, +\infty), f$  is integrable in  $[a, x]$  and also
2. there is the limit defined by

$$\lim_{x \rightarrow +\infty} \int_a^x f$$

**Notación:** If a function is integrable in the interval  $[a, +\infty)$  then the limit value is called the improper integral of the first kind of  $f$  and is denoted:

$$\int_a^{+\infty} f = \lim_{x \rightarrow +\infty} \int_a^x f$$

### 2.1 Convergence criteria

Let  $f : [a, +\infty) \rightarrow \mathbb{R}$  be a bounded function.

1. It is said that  $\int_a^{+\infty} f$  is **convergent** if and only if,  $f$  is Riemann integrable for every interval  $[a, x]$ , the limit exists

$$\lim_{x_1 \rightarrow +\infty} \int_a^{x_1} f$$

and it's a real number.

In this case we will say that the function  $f$  is Riemann integrable in the interval  $[a, +\infty)$ .

2. It is said that  $f$  is **divergente** if, and only if,  $f$  is integrable Riemann for every interval  $[a, x]$  exists limit

$$\lim_{x_1 \rightarrow +\infty} \int_a^{x_1} f$$

and it's not finite.

3. It is said that  $\int_a^{+\infty} f$  is **oscillating** in the case that  $f$  is not Riemann integrable in an interval  $[a, x]$  or there is no limit.

$$\lim_{x_1 \rightarrow +\infty} \int_a^{x_1} f$$

Observation: The idea behind the improper integrals of the first kind is to integrate up to an arbitrary  $x_1$  point and then make  $x_1$  tend to infinity.

The convergence or divergence of three fundamental types of improper integrals is established below.

Integral	Convergence	Divergence
$\int_a^{+\infty} \frac{1}{x^p} dx, a > 0$	$p > 1$	$p \leq 1$
$\int_a^b \frac{1}{(x-a)^p} dx, b \geq a$	$p < 1$	$p \geq 1$
$\int_a^b \frac{1}{(b-x)^p} dx, b \geq a$	$p < 1$	$p \geq 1$

Table 1: Fundamental improper integrals

## 2.2 The improper integral in the Mexican curriculum

Mexican curriculum examined, for it explored the plans and programs Studies University of Warrior degree in Mathematics (2009), which is divided into two phases: the first stage basic training and the second, the Specialization Phase. The mathematical object under study is in the Learning Unit Calculus II, Stage Basic Training where parallel students study subjects, Analytical Geometry II and Algebra and admission requirements to study Calculus II are to have completed the learning units : Algebra, Elements of Geometry, Analytical Geometry I and Calculation I.

The general objective "is to get students to master the fundamental concepts and calculation techniques of the Riemann integral, as well as to begin in the study of infinite summation algorithms", for that in this course you must contribute to the Students are able to: *"understand the meaning of an integral as a limit"*.

As for the content of the Learning Unit, we see *Improper Integral*, in Unit 1 at the end of the Integration content in elementary terms. First it is the objective of the unit *Analysis of the convergence of improper integrals* and our study content is named in the curriculum *Improper integrals*.

### 3 Theoretical foundation

The research problem we face focuses on the understanding of the concept of improper integral. In this section a review is made about different theoretical references of Advanced Mathematical Thought and the Theory of semiotic representations, as well as the understanding of mathematical concepts.

#### Advanced Mathematical Thinking

Advanced Mathematical Thinking (PMA) has to do with the mental processes of higher Mathematics that are taught and learned in the last years of high school and especially in the university field, and that according to Azcárate and Camacho (2003, p. 136-141), this type of thinking by its nature has characteristic processes among which it stands out: the level of abstraction, formalization of knowledge, representation, definition of concepts and demonstration.

According to the words of Dreyfus (1991), "understanding is a process that takes place in the student's mind" and is the result of "a long sequence of learning activities during which a large number of mental processes occur and interact. " When we refer to cognitive processes involved in advanced mathematical thinking, we think of a series of mathematical processes among which the process of abstraction that consists in the substitution of concrete phenomena by concepts confined in the mind stands out. It cannot be said that abstraction is an exclusive characteristic of higher mathematics, nor are other cognitive processes of mathematical component such as analyzing, categorizing, conjecturing, generalizing, synthesizing, defining, demonstrating, formalizing, but it is evident that these three The latter acquire greater importance in the higher courses: the progressive mathematization implies the need to abstract, define, demonstrate and formalize.

The importance of semiotic representations in mathematical knowledge as tools both to access mathematical objects and to establish correspondences between these objects and different forms of representation. The transformations of the semiotic representations are at the heart of the mathematical way of working and to teach them it is crucial to develop the cognitive abilities of the students to put different representations in the correspondence.

#### Aspects of the Theory of Semiotic Representations

The theory proposes that semiotic representations are indispensable to achieve the learning of mathematical objects. Duval (2006) describes this affirmative conception as follows: "The comprehensive understanding of a conceptual content is based on the coordination of at least two representation records, and this coordination is evidenced by the rapid use and spontaneity of cognitive conversion "(P.166). The coordination of several records of semiotic representation is presented as a crucial aspect for the conceptual learning of mathematical objects. Therefore, Duval (2006) states that it is essential not to confuse mathematical objects with their representation, that is, the object has to be recognized in each of its various representations. This confusion implies an understanding of the concept only from the record in which it has been demonstrated, which does not allow the transfer of the object to other types of representation. Mental representation and semiotic representation According to Duval (2004), in mathematics, we can access mathematical objects through their different forms of representation: in fact, the mathematical activity is carried out in a context of representation. This author also states that in order to achieve the learning of mathematics, fundamental cognitive activities such as conceptualization, reasoning, problem solving and text comprehension are required. These activities used the use of several semiotic systems of representation and expression. The author defines two types of representations. It is under this view that the authors of this work will study understanding.

#### Mental representations

They are conscious representations referring to a set of images and conceptions that an individual may have about an

object, about a situation and about what is associated with them. In addition, they are strictly internal, that is, they cannot be observed publicly. To develop this type of representation, the acquisition and internalization of different systems of semiotic representations and, in particular, natural language is required.

### **Semiotic representations**

They are conscious representations that are expressed through a system of geared signs (icons, symbols, indices), and that are governed according to explicit or implicit rules. These rules are associated and combined, thus transforming expression or representation within them. In addition, they are external, as these representations are observable and can be exposed publicly. On the other hand, they fulfill the cognitive functions of communication, treatment, and objectification, that is, semiotic representations are necessary for the mathematical activity itself, for the treatment of information, for awareness and for understanding. In addition, they are characterized by mobilizing three cognitive activities: training, treatment and conversion.

Likewise, mental representations and semiotic representations are related, and their functions allow the cognitive activity of the subject. Semiotic representations are those productions constituted by the use of signs, they are the means available to an individual to externalize his mental representations. Duval (2004) argues that semiotic representations are indispensable for the development of mathematical activity, as they allow us to carry out treatments and access mathematical objects in particular, because objects are not directly accessible by the senses. Moreover, these are important for the development of mental representations, for cognitive functioning in their activities of conceptual apprehension, reasoning, and comprehension of sentences; as well as for the communication and production of mathematical knowledge.

The three cognitive activities of semiotic representation records

Duval (2006) argues that semiotic representations have the fundamental property of transforming into other representations that they retain, be it all the content of the initial representation or only a part of that content. These transformations do not correspond to the same cognitive activity, they depend on whether the transformation is carried out within the same register, or that it consists in the change of registration. In turn, he argues that a registry of semiotic representation must allow three fundamental cognitive activities of representation: formation of an identifiable representation, treatment and conversion.

**Formation:** The identifiable formation, be it a phrase, a drawing, a formula, or a scheme implies a selection of a set of characters (features and data) of a perceived content that can be represented according to the possibilities of the register. Fact: Its fundamental purpose is to evoke a real representation and express a mental representation.

**Treatment:** These are transformations that produce another representation in the same register, regarding a question, a problem, or a need. In this activity, there is a sequence of several transformations. The treatment is a strictly internal transformation to a register, that is, the sign system in which the representation is expressed is not changed, and uses only the operating possibilities of the system itself, so each register offers specific treatment possibilities. Likewise, cognitive treatment activity occurs when a specific question is answered or a need is met. Therefore, Duval (2004) considers treatments to perform calculations internally in algebraic representation. For example, in the graphic representation register a treatment would be to solve an equation or a system of equations. In addition, in the registry of representation in natural language, consider as paraphrase treatment or reformulations in natural language through which a linguistic expression is transformed into another, either to replace or explain it.

**Conversion:** It is a transformation in which the semiotic system is changed, that is, it is the transformation of the representation of a mathematical object, given in a record, in a representation of this same object, in another record

keeping all or only A part of the content of the initial representation is an external transformation. The conversion would be the result of the conceptual understanding of the mathematical object worked, that is, this cognitive activity is the most important to develop the understanding of mathematical notions.

According to Duval (1999), the characteristic of the conversion is to keep the reference to the same object, but without retaining the explanations of the same properties of that object. In that sense, the representation in the arrival record will not have the same content as its representation in the departure record.

#### Understanding mathematical concepts

Teaching and learning with understanding are usually admitted as desirable and priority objectives in Educational Mathematics, which has led to an increase in initiatives that essentially deal with the development of understanding in the mathematics classroom (Hiebert, Carpenter, Fennema, Fuson, Wearne, Murray, Olivier and Human, 1997; Carpenter, Fennema, Fuson, Hiebert, Human, Murray, Olivier, and Wearne, 1999; NCTM, 2000). Frequently, these initiatives are affected by significant difficulties and limitations when understanding to the full extent is not taken into account (Sierpinska, 2000).

#### Competition model

We will use the competence approach of (Gonzalez-Martin and Camacho-Machín, 2005), in which he defines competence as the coherent articulation of different semiotic registers; Being competent would mean articulating coherently the different representations of the improper integral when facing the resolution of non-routine problems. The above coincides with the definition of compression we are working on. According (Socas, 2001) the different stages of development that occur in cognitive representation systems in the case of algebraic language to the coordination between two registers in our particular case, instead of favoring a single representation, researchers have argued (for example, Brizuela and Earnest, 2008; Duval, 2006) that students should be able to coordinate different representations of the same object and change flexibly between them. In this way, we will no longer talk about levels within the same record, but about levels in the coordination between two records.

In each stage we will distinguish two categories of behavior, obtaining the following categories, which refer to the possible articulations between two registers in matters of improper integration: See Table 2.

Graphical and algebraic representation records are considered, and added that of the natural language, (González-Martín and Camacho Machin, 2005.) See Table 3.

Representations in usual language (H)

Representations in formal (algebraic) language (F)

Representations in graphic visual language (G)

Therefore, the actions we will distinguish will be:

### 3.1 Methodology

The methodological aspects of the investigation, the instrument that was used and the analysis procedure are described. First, we develop the research method used, referring to the field of research, listing the phases of the investigation, presenting the mathematical elements and the different representation records of the concept of Integral Impropropia.

The type of research that was carried out is a qualitative research aimed at understanding, whose objective is to:

Category	Description
Category 1A	The student has imprecise ideas about the improper integral and incoherently mixes different semiotic representations.
Category 1B	The student recognizes the elements of a system of semiotic representation in relation to the improper integral.
Category 2A	The student knows a semiotic representation system and performs transformations within the representation system.
Category 2B	The student correctly performs conversion activities from one semiotic representation system to another; In these conversion activities there is a system that the student controls, and facilitates conversion to the other.
Category 3A	The student articulates two systems of representation. semiotics. He can take any of them to correctly refer to the improper integral object independently of the other. The student manages the two semiotic representation systems autonomously.
Category 3B	The student articulates coherently different semiotic representation systems, exercises control of the semiotic representations he uses. He has knowledge of the improper integral as a structure and can control coherent and incoherent aspects of it

Table 2: Competition model taken from González-Martín and Camacho Machin, 2005.

Description	Actions
Recognition of elements of a system representation semiotic	$R_H, R_F, R_G$
Internal transformations in a semiotic representation register	$T_H, T_F, T_G$
Conversions (external transformations) between semiotic representation systems	$C_{H \rightarrow F}, C_{H \rightarrow G},$ $C_{F \rightarrow G}, C_{G \rightarrow F}$
Coordination between different systems of semiotic representation	$C_{F \leftrightarrow G}, C_{G \leftrightarrow F},$ $C_{H \leftrightarrow F}, C_{H \leftrightarrow G}$
Production of semiotic representations in solving a task	$PS_F, PS_G$

Table 3: Expected actions, taken from González-Martín and Camacho Machin, 2005.

characterize the different levels of understanding of the concept of improper integral through a competence model and identify and interpret the cognitive activities of treatment and conversion that allow mobilizing the elements and properties of the improper integral in its different registers of semiotic representation.

#### Research ambit

This research was conducted in Guerrero, Mexico, specifically at the Autonomous University of Guerrero (UAGro). The students they refer to have an age range between 18-22 years, who are in the second and fourth semester of the degree, belonging to the Faculty of Mathematics and are assigned to the Mathematics Program, the structure of the latter is composed of Three Phases of Formation.

The concept of Integral Impropia is not contemplated in the curricular standards designed by the Ministry of Public Education (SEP) corresponding to secondary education, so this concept is only studied when students who enter the university in particular will participate in the research study this concept from the second semester of the Bachelor of Mathematics. It should be noted that the students at the time of the application had already completed the Calculation

II Learning Unit, in the case of the fourth semester and the second semester students were completing the Calculation II Learning Unit at the time of the application of the instrument.

### Research Phases

This section has to do with each of the stages in which the research was developed, some occur simultaneously, while others are previous steps for the next phase. The general scheme of research we have used according to Bisquerra and Sabariego (2004, p. 90) "consists of a series of stages interconnected in a logical, sequential and dynamic", which will be described.

**Phase 1:** We have considered it as the starting point of the investigation, selecting the research topic that in our case is about the understanding of the Concept of Improper Integral under the theoretical framework of the Theory of Semiotic Representations (Duval, 1993), which has led us to formulate a research problem, associated with some objectives that focus this research on studying the understanding of university students of the concept of Improper Integral, through representation, graphic and numerical systems, adding natural language. Identifying the relationships they establish between the mathematical elements and the modes of representation they are capable of transiting. To respond to this study, a review of the existing literature on previous research on this question was made, an analysis of how the concept of Improper Integral appears in the textbooks that will be used to determine the mathematical elements that set up the concept.

**Phase 2:** At this stage we have the sample of subjects participating in a test, the design of the instrument, the selection of the problems included in the instrument, its validation by experts, its application possibly its modification, if it is applied again. The sample of students were 17 second semester students and 13 fourth semester students, in an age range of 18 to 22 years.

For the preparation of the questionnaire, the mathematical elements and the different registers of representation of the concept of Comprehensive Improvement will be selected, select a collection of problems taking as reference the problems in research journal articles (Orton, 1983; Mundy, 1984). problems with some modifications, which have been used in Doctoral Theses on the Definite and Improper Integral (Turégano, 1994; Calvo, 2001; Depool, 2004 and González-Martín, 2006) and adapted mathematical exercises (Stewart, 2012).

Likewise, the design of the instrument was done according to: the objectives of the research, and the maximum number of problems that it must contain so that they can be answered in the course of a usual university class, not boring and tired.

**Phase 3:** We consider it as the phase of application and analysis of the definitive data collection instruments, and includes the elaboration of the definitive questionnaire, the selection of the subjects participating in the investigation, the application of the questionnaire, the analysis of the questionnaire, the design of the interviews and the application of the interviews. The instrument was answered by all students of the Calculation II and IV courses, with an age range between 18-22 years, second and fourth semesters of the Mathematics Degree at the Autonomous University of Guerrero.

In the application of the questionnaire the teacher of the participating group and the students knew in advance the time and the process of carrying out the test. Also days prior to the application of the questionnaire the researcher went to the classroom to encourage students to participate in the research. The students were informed, verbally and in writing, about the purpose of the research, the topic, the way to respond in addition to the confidentiality of the answers

**Phase 4:** This stage, in our study, corresponds to the joint analysis of the instruments (questionnaire and interview), to obtain the results, the conclusions and discussion about the research, the implications in the teaching of the concept and the future perspectives.

Aware of the interest and richness of a qualitative analysis in an investigation such as ours, the use of questionnaires and videotaped interviews is proposed to conduct our study. This became an even greater need after performing the written test and analyzing the answers obtained. Two instruments will be used to collect the information: final instrument and task-based interview.

For the selection of the students to be interviewed, it will be done according to their answers to the questionnaire. Basically, we rely on the combination of the following two criteria: total test performance and some very significant representations or ways of proceeding. The technique that will be used will be the interview based on tasks Goldin (2000), in which the interviewee not only interacts with the interviewer, but also with the set of tasks entrusted to him (questions, problems or activities). The interviews were videotaped, as we were interested in collecting large amounts of information. We find important not only verbal behaviors, but also nonverbal ones, to make inferences about the mathematical thinking and learning of the interviewees, the above was done taking care of the confidentiality and anonymity of each of the students chosen.

#### Instrument Design

We design questions that have to do with the content of improper integral, the idea is not to address convergence criteria, in fact the students were not asked to calculate the convergence, that is to say only to calculate that integral in general, thus identifying what they associate this. The tasks were designed so that they could transit through several records of semiotic representation. The student was also motivated to compare certain integrals and thus be able to observe the understanding they have about improper integral and the way in which they can communicate it.

A pilot test was carried out on a group of students who, at the time of the application, started the fourth semester of the Bachelor's Degree in Mathematics, from the Autonomous University of Guerrero.

Items that were difficult at the time of resolution were omitted and therefore the students took a long time to try to solve them, mostly leaving them blank. Several items were modified that were confusing at the time of carrying out or that the instruction was not clear.

1. ¿What is an improper integral?

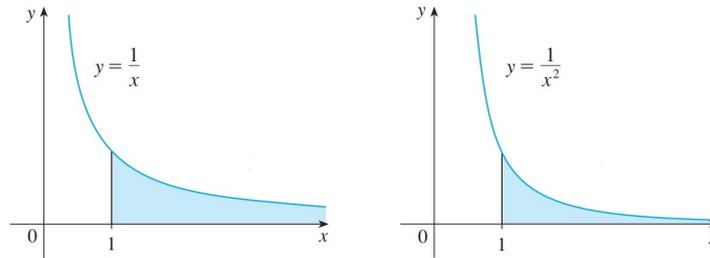
2. ¿How do you interpret the integral  $\int_a^b f(x)dx$  and how would you explain it to a partner? You can help with graphics or any means you consider to complement your answer, justify.

3. ¿How do you interpret the integral  $\int_a^\infty f(x)dx$  and how would you explain it to a partner? You can help with graphics or any means you consider to complement your answer, justify.

4. ¿What is the difference between one integral and the other and how would you explain it??

5. We know that  $\sum_{n=1}^{\infty} \frac{1}{n} = \infty$  y  $\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$ . Make the necessary calculations and interpret the results. What can you say about the value of  $\int_1^{\infty} \frac{1}{x} dx$  and  $\int_1^{\infty} \frac{1}{x^2} dx$ ?

Help yourself with the graphs presented.

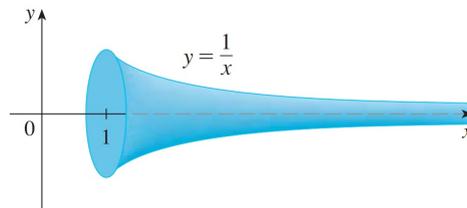


6. If the region

$$R = (x, y) | x \geq 1, 0 \leq y \leq \frac{1}{x}$$

it is rotated around the  $x$  axis, the volume of the resulting solid is finite.

Calculate the surface of Gabriel's trumpet. (The surface is shown in the figure and is known as Gabriel's trumpet.) Help yourself with the attached graphic.



7.

$$\int_0^{\pi} \sec^2 x dx$$

1. Calculate the integral.
2. Do you recognize the type of integral in question? Is it a definite or improper integral? Why? Justify your answer.
3. Can you graphically represent this integral? If so, do it, in case you can't, justify your argument.

## 4 Analysis and results

The analysis of the productions obtained from 4 of the 30 students in different items is shown, the second semester students are identified as  $E1\_1$  if they were student 1 in their first item and so on  $E2\_1, E3\_1, \dots, E17\_1$  the productions of the 17 second semester students of the first item, and so on with the following items. Similarly, fourth semester students are identified from number 18 and letter A, for example  $A18\_1$  if they were the first student with the production of the first item and the following productions  $A19\_1, A20\_1, \dots, A30\_1$ .

For the student  $E3\_1$  the following actions were identified:  $R_H, R_F, R_G$ .

E3: textit What I understand by this is that the graph will continue infinitely. I: How can you see a graph that extends infinitely?

E3: textit Mmm, ... well, it is very easy for me to interpret with graphics.

I: Do you regularly make graphs? E3: textit Yes, on several occasions my teacher asks me to make graphs but sometimes I tell him to sketch with graphs to understand, I like that there is precision in my graphics ...

For the student A27\_1 the following action was identified:  $R_H$ .

I: It catches my attention that you didn't make a graph.

A27: textit Mmm, I didn't see the need to make any graphics.

I: Only for this exercise or in general there is usually no need? A27: textit It is that depending, if the function is rational if I must but if it is another I could not do it.

For the student A19\_2 the actions were identified:  $R_H, R_F, R_G, T_H, T_F, T_G$  for item 2, the shares  $R_H, R_F, R_G, T_H, T_F, T_G$  for item 3 and the action  $R_H$  for item 4.

I: Can you tell me how you explain to a partner?

A19: textit From my previous calculation courses I know that an integral indicates the area under the curve, in this case in the closed interval [a, b], I explain it by graphing the axes, the curve and marking the points ( a, b), in this case shaded, when making the integral would get that value that would be the area which would be a number.

I: In this exercise if you had the need to make a graph? A19: textit Is that as you ask me to explain to a partner, I think it is easier for me and to explain to you through a graphic, an image, so that it can be guided and related, is that sometimes we do not remember how to do things, and seeing the image I can relate it easier with the subject, is that maybe I do not remember the subject but I remember the image and start to relate

For the student E12\_5 the following actions were identified:  $R_H, R_F$  and  $C_{G \rightarrow F}$ .

I: I was struck by the sum you used, can you explain what you did?

E12: textit Mmm, I saw that I was wrong in the results of the sum, it is that the result did not convince me, so I did not finish it because in one it gives me infinity and in the other it does not, but I do not know why , it shouldn't be like that, I knew that when this happens mmm (... silence) and they also explained to me that by definition the limit of  $\frac{1}{x}$  is zero, and I said well if it eventually adds up the sum is getting smaller and smaller, so it will go from 1 up to 0, bone cannot be *infity* I tried to do it but it gives me a result that is not logical.

## 5 Conclusions

From the realization of the investigative activity reported here, it is concluded that: the compression that the students possess does not correspond to a formal definition. Instead, they manifest a series of conceptual variations that in some cases are closer to an intuitive notion; then, based on these results, they would be expected to be considered as the subject of future research.

At the level of semiotic registers of the various available records for them, students prefer the algebraic representation, the idea or the need to transfer an integral expression to one of its limits is infinite and one of its limits is infinite and identifies if a function It is or is not defined in an integration interval. But the fact that they prefer the use of the algebraic record is not synonymous with mastering it perfectly, since the low level of understanding of formal algebraic language was evidenced, an aspect that prevents the student from integrating and constructing different tasks. Forms of representation of the concept of improper integral, students feel comfortable working with the use of integration formulas.

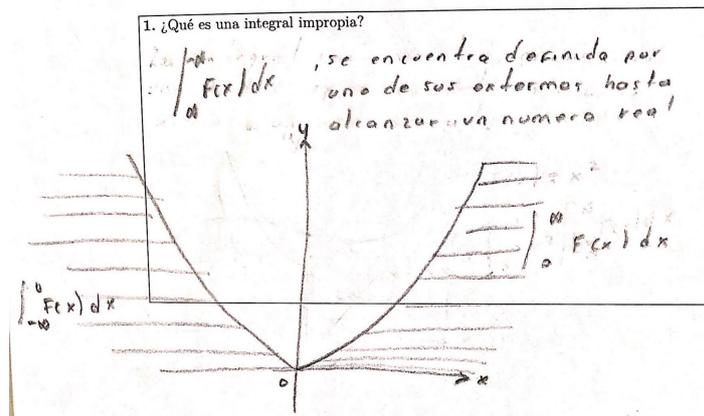


Figure 1: Production of E3\_1

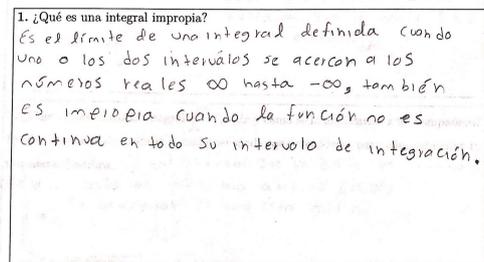


Figure 2: Production of A27\_1

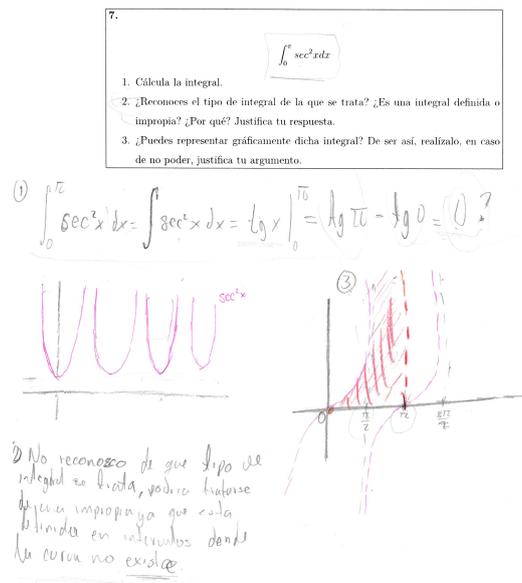
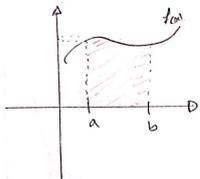


Figure 3: Production of E12\_5

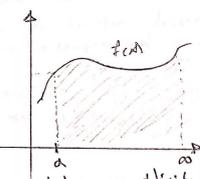
2. ¿Cómo interpretas la integral  $\int_a^b f(x)dx$  y como se la explicarías a un compañero? Puedes ayudarte de gráficas o cualquier medio que consideres para complementar tu respuesta, justifica.

Es una área bajo la curva si  $f(x) > 0$  en el intervalo cerrado  $[a, b]$ .  
 Sea  $F(x)$  la antiderivada de  $f(x)$ , entonces:  
 $\int_a^b f(x) dx = F(b) - F(a)$ .  
 ∴ el área es la diferencia del valor de  $b$  y  $a$  evaluados en la antiderivada.



3. ¿Cómo interpretas la integral  $\int_a^\infty f(x)dx$  y como se la explicarías a un compañero? Puedes ayudarte de gráficas o cualquier medio que consideres para complementar tu respuesta, justifica.

Es área bajo la curva si  $f(x) > 0$  en el intervalo  $[a, \infty)$ .  
 Y como el límite superior es  $\infty$  entonces la integral se lleva a una situación límite. tal. q. r.  
 $\int_a^\infty f(x) dx = \lim_{b \rightarrow \infty} \int_a^b f(x) dx = \lim_{b \rightarrow \infty} (F(b) - F(a))$   
 y aplicamos el límite para  $F(x)$  en  $\infty$ .



4. ¿Qué diferencia hay entre una y otra integral y cómo lo explicarías?

Para la primera es una integral sencilla en donde el intervalo es cerrado para dos números. en cuanto a la otra es un poco más complicado (integral impropia) para su resolución.

Integral definida

Figure 4: Production of A19\_2

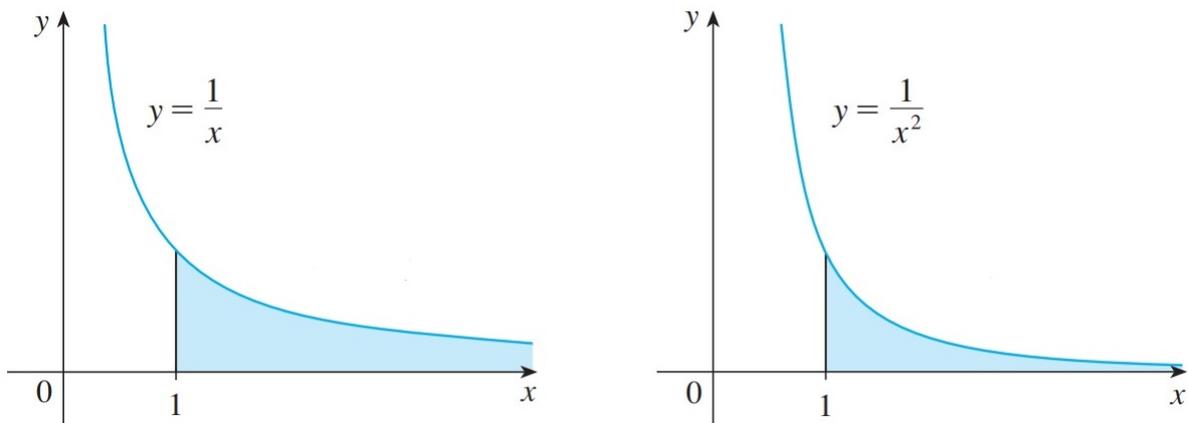


Figure 5: Production of fig5

A convenient knowledge of these variants of the semiotic representation of the function would allow not only the flexibility in the articulation changes between them, but a better understanding of the concept, then this knowledge provides an investigative reference that serves to determine the characteristics that should have the teaching process that teachers develop in the course of Calculus, in order to provide the student with the tools that guarantee the correct appropriation of mathematical concepts so that they will be reflected in the improvement of the various mathematical competence.

We believe that many of the difficulties that we may encounter for students at different levels of the curriculum can be described and explained as a lack of coordination of representation records. In addition, improper construction of a concept can be caused by a lack of articulation between different semiotic records.

It is important to mention that in the classroom the conceptual part of the content of improper integral is not propitiated, as we mentioned before it is presented as a generalization of the Riemann integral and this favors that the students interpret it as an area under the curve and present problems when faced with the infinite and be more familiar with the calculation of convergence criteria of improper integrals. In addition, the transit of semiotic registers is not favored, many times the students, even when they have the graph, discard it because they do not know how to interpret it correctly or do not associate it with another type of representation.

## References

- [1] Bisquerra, R. y Sabariego, M. (2004). El proceso de investigación (Parte I). En R. Bisquerra (Coord.), *Metodología de la investigación educativa*, (pp. 89-125). Madrid: La Muralla
- [2] Brizuela, B. M., y Earnest, D. (2008). Multiple notational systems and algebraic understandings: The case of the "best deal" problem. In J. Kaput, D. Carraher, y M. Blanton (Eds.), *Algebra in the Early Grades* (pp. 273-301). Mahwah, NJ: Lawrence Erlbaum and Associates.
- [3] Camacho, A. y Aguirre M. (2001). Situación didáctica del concepto de límite infinito: análisis preliminar. *Revista Latinoamericana de Investigación en Matemática Educativa RELIME*, 4(3), 237-265
- [4] Depool, R. A. (2004). *La Enseñanza y Aprendizaje del Cálculo Integral en un Entorno Computacional. Actitudes de los Estudiantes Hacia el uso de un Programa de Cálculo Simbólico (PCS)*. Tesis Doctoral. Universidad de La Laguna
- [5] Dreyfus, T. (1991). Advanced in Mathematical Thinking Processes. En D. Tall. (Ed.). *Advanced in Mathematical Thinking*. (pp.25-41). Boston: Kluwer Academic Publishers.
- [6] Duval, R. (1993). Registres de représentation sémiotique et fonctionnement cognitif de la pensée, *Annales de Didactique et de Sciences Cognitives*, 5, IREM de Strasbourg, pp. 37- 65.
- [7] Duval, R. (1996). Quel cognitif retenir en didactique des mathématiques? *Recherches en Didactique des Mathématiques*, Vol. 6, 3, 349-382.
- [8] Duval, R. (1999). Representation, vision and visualization: cognitive functions in mathematical thinking, basic issues for learning. *Actas del PME*, 23, 3-26.
- [9] Duval, R. (2004). Duval, R. (2004). A crucial issue in mathematics education: The ability to change representation register, Regular Lecture en el *10th International Conference on Mathematics Education (ICME10)*, Dinamarca.

- [10] Duval, R. (2006). A Cognitive Analysis of Problems of Comprehension in a Learning of Mathematics. *Educational Studies in Mathematics*, 61, 103-131. <http://dx.doi.org/10.1007/s10649-006-0400-z>
- [Galán-García, Aguilera-Venegas, Galán-García, Rodríguez-Cielos y Atencia-Mc.Killop, 2016] Galán-García, J.L., Aguilera-Venegas, G., Galán-García, M.A., Rodríguez-Cielos, P., y Atencia-Mc.Killop., (2016). *Improving CAS capabilities: New rules for computing improper integrals*. Appl. Math. Comput. In Press (Corrected Proof) doi: 10.1016/j.amc.2016.12.024.
- [11] Goldin, G. A. (2000). A Scientific Perspective on Structured, Task-Based Interviews in Mathematics Education Research, en *Handbook of Research Design in Mathematics and Science Education* (Kelly, A. E. y Lesh, R. A., eds.), Lawrence Erlbaum Associates, London, pp. 517-546.
- [12] González-Martín, A.S. y Camacho-Machín, M. (2002a). The improper integral. An exploratory study with First-Year University Students, *Proceedings of the Second International Conference on the Teaching of Mathematics (ICTM2)*, Hersonisos (Grecia), CD Proceedings (Wiley, J., ed.).
- [13] González-Martín, A.S. y Camacho-Machín, M. (2002b). The comprehension of the concept of improper integral in Mathematics students: representation and transference, *Proceedings of the 54th CIEAEM Conference*, Granada: Ed. Proyecto Sur.
- [14] González-Martín, A.S. y Camacho-Machín, M. (2003). Non-routine situations and new technologies in University teaching. The improper integral, *Proceedings of the 55th CIEAEM Conference*, Płock (Polonia).
- [15] González-Martín, A.S. y Camacho-Machín, M. (2004). What is First-Year Mathematics students actual understanding about improper integration?, *International Journal of Mathematical Education in Science and Technology*, 35 (1), 73-89.
- [16] González-Martín, A.S. y Camacho-Machín, M. (2005). Sobre la comprensión en estudiantes de Matemáticas del concepto de integral impropia. Algunas dificultades, obstáculos y errores. *Enseñanza de las Ciencias*, 23 (1), 81-96.
- [17] González-Martín, A.S., (2006). *La generalización de la integral definida desde las perspectivas numérica, gráfica y simbólica utilizando entornos informáticos. Problemas de enseñanza y de aprendizaje*. Tesis Doctoral. Servicio de Publicaciones de la universidad de La Laguna. ISBN: 84-7756-679-8.
- [18] González-Martín, A. S., Nardi, E., y Biza, I. (2011). Conceptually-driven and visually-rich tasks in texts and teaching practice: The case of infinite series. *International Journal of Mathematical Education in Science and Technology*, 42(5), 565–589.
- [19] Hiebert, J., Carpenter, T. P., Fennema, E., Fuson, K. C., Wearne, D., Murray, H., Olivier, A., y Human, P. (1997). *Making sense: teaching and learning mathematics with understanding*. Portsmouth, NH: Heinemann.
- [20] Mahir, N. (2009). Conceptual and procedural performance of undergraduate students in integration. *International Journal of Mathematical Education in Science and Technology*, 40(2), 201–211.
- [21] Mendo L., Castañeda A. y Tarifa L. (2017). El desarrollo de argumentos matemáticos en estudiantes universitarios. *Atenas*, 3(39). 1-17.
- [22] Orton, A. (1983). Students' understanding of integration, *Educational Studies in Mathematics*, 14 (1), 1-18.
- [23] Rasslan, S., y Tall, D. (2002). Definitions and images for the definite integral concept. In A. Cockburn y E. Nardi (Eds.), *Proceedings of the 26th conference of the International Group for the Psychology of Mathematics Education* (Vol. 4, 89–96). Norwich, United Kingdom: PME.

- [24] Socas, M. (2001). *Investigación en didáctica de la matemática vía modelos de competencia. Un estudio en relación con el lenguaje algebraico*. Departamento de Análisis Matemático. Universidad de La Laguna (sin publicar).
- [25] Sofronas, K. S., DeFranco, T. C., Vinsonhaler, C., Gorgievski, N., Schroeder, L., y Hamelin, C. (2011). What does it mean for a student to understand the first-year calculus? Perspectives of 24 experts. *The Journal of Mathematical Behavior*, 30(2), 131–148.
- [26] Sealey, V. (2006). Definite integrals, Riemann sums, and area under a curve: What is necessary and sufficient. In S. Alatorre, J. L. Cortina, M. Sáiz, y A. Méndez (Eds.), *Proceedings of the 28th annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education* (Vol. 2, pp. 46). Mérida, México: Universidad Pedagógica Nacional.
- [27] Thompson, P. W., y Silverman, J. (2008). The concept of accumulation in calculus. In M. Carlson y C. Rasmussen (Eds.), *Making the connection: Research and practice in undergraduate mathematics, MAA Notes* (Vol. 73, 43–52). Washington, DC: Mathematical Association of America.
- [28] Thompson, P. W., Byerley, C., y Hatfield, N. (2013). A conceptual approach to calculus made possible by technology. *Computers in the Schools*, 30(1–2), 124–147.
- [29] Törner G, Potari D, Zachariades T. (2014). Calculus in European classrooms: curriculum and teaching in different educational and cultural contexts. *The International Journal on Mathematics Education*, 46(4), 549–560. DOI: 10.1007/s11858-014-0612-0.