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Algebra Textbooks' and Teachers' Methods: Simplifying rational expressions

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ABSTRACT

The purpose of this paper is to examine the methods used in algebra textbooks and by secondary school algebra teachers to illustrate simplifying rational expressions, since teachersreport that many students fail to develop accurate facility with this topic.A sample of 10 algebra textbooks was examined, 33 teachers were surveyed, and 6 teachers were interviewed. The results reveal that the vast majority of textbooks and teachers in the sample useda procedurallyfocused method rather than a conceptuallyfocused method. Teachers gravitated toward the quick coverage offered by a procedural method showing students exactly what to do. Moreover, they even knew the possible disadvantages of using such a procedural method and the advantages offered by a conceptual method which offered a clearer explanationof the reasons behind the process. Despite the fact that teachers reported that rational expression simplification was a significant challenge for students to learn, the advantages offered by a conceptual method with the prospect of improved student knowledge wasunheeded by most. Interviews with experienced algebra teachers revealed low expectations for student learning which may have a negative impact on student performance.A teaching strategy is suggested,based on the results of this paper and prior research,to include both procedural and conceptual methods for the simplification of rational expressions to enhance student success.

Keywords

Algebraic and equivalent expressions;Procedural and conceptual knowledge;Concept image;Teachers'perspectives and skills.

Academic Discipline

Education; Mathematics; Learning and Teaching; Algebra.

SUBJECT CLASSIFICATION

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INTRODUCTION

Developing student facility in transforming rational expressions to equivalent expressions, or simplifying, poses a significant challenge for teachers. Student struggles with transforming algebraic expressions to equivalent expressions have been well documented by research(Cerulliand Mariotti 2000; Demby 1997;Greeno 1980; Harelet al. 2008; Kieran 1992).While transformations involving algebraic expressions have been explored, those associated with rational expressions have been examined less so in the research literature. But, the ability to rewrite rational expressions is included in the Common Core State Standards for Mathematic(NGA Center and CCSSO 2011)which defines what students in the United States should understand and be able to do. Further, simplification of rational expressions is fundamental for later topics including operations involving rational expressions, working with rational equations and functions, as well as for more advanced courses like calculus.

Confusions that occur among students are not always their fault (Tall 1988). Textbooks may have deficiencies and teachers may present pedagogic obstacles that inadvertently impede students' success when working with rational expressions. When textbooks and teachers fail to promote robust concept images, students may develop incomplete knowledge and have a propensity to rely too heavily on a single concept image resulting in common errors (Cunninghamand Roberts2010; Harel et al. 2008). One such student error occurs when they resort to the facetious universal law of cancellation: when any two things look alike, cross them out (Laursen 1978). Errors of this type demonstrate incomplete knowledge held by the student necessary for working successfully with rational expressions.Thompson (2008) proposes that the vast majority of students do not experience ideas that carry through an instructional sequence, and that play into linkages so necessary for students' reasoning. Further, Hiebertand Wearne (1986) argued that procedural errors often occur when students lack linkages between procedural and conceptual knowledge.

While research suggests that teachers rely heavily on textbooks for lesson development, Demby found that "explanations advised in curricula and books for teachers are not so efficient as it is assumed" (1997). Some textbooks provide inadequate definitions, a restricted number of examples, or explanations that de-emphasize meaning by emphasizing procedures over concepts,any of which may impact student learning (Cunningham and Roberts 2010; VinnerandHershkowitz 1980; Mac Gregorand Stacy 1997). Despite such impediments to student understanding of mathematics concepts, teachers report that textbooks are the "most commonly used resource" (Kajanderand Lovric 2009). Although teachers frequently "supplement what they see as inadequacies in the text" (Love and Pimm 1996), many teachers leave what they believe to be well enough alone since a textbook "organizes the mathematics curriculum… [and] it also organizes the work of the classroom" (Love and Pimm 1996).

Because of teachers' reliance on textbooks, this paper examined a sample of algebra textbooks to determine the methods used to illustrate simplification of rational expressions. Textbooks can impact student knowledge, but the instructional practice of algebra teachers can be even more influential. Although much research has focused on student learning of algebra, there is still a need for research on the teaching of algebra (Harel et al. 2008; Kieran 2007). Consequently, algebra teachers weresurveyed and interviewed to determine the methods they use when presenting rational expressions in their classrooms.

2 BACKGROUND

2.1 Motivation

An assignment given to an undergraduate secondary mathematics education class partially motivated this paper. Students from the class were assignedthe task of finding an in-service algebra teacher during their student teaching experience in their placement school who would volunteer to fill out a questionnaire. Questionnaires were collected from 21 in-service teachers with between 3 and 22 years teaching experience from 5 different public schools in central New Jersey in the United States.One question asked them to list the most challenging algebra topic for them to teach as measured by student performance. Three topics were tied for frequency: factoring polynomials, logarithms, and rational expressions. On reviewing a few algebra textbooks, illustrations of rational expressions stood out as being very procedural. A more thorough examination of a larger set of teachers and textbooks seemed warranted since it might provide insight, not only for teaching rational expressions, butalso for teaching other challenging algebra topics.

2.2 Rational expressions

A rational expression can be defined as the quotient $\frac{P(x)}{Q(x)}$ of two polynomials with $Q(x)$ not equal zero. Two sample illustrations for the simplification of a rational expression labeled Method A and Method B appear in Table 1 below.

Table 1.Sample illustrations for the simplification of a rational expression

Method A "If possible, factor the numerator and denominator. Remove common factors and then simplify."

$$\frac{x^2-3x-18}{x+3}=\frac{(x-6)(x+3)}{(x+3)}=\frac{(x-6)\cancel{(x+3)}}{\cancel{(x+3)}}=x-6; \quad (x\neq -3)$$

Method B"Group common factors, substitute for expressions equal to 1 and use the multiplicative property of one."

$$\frac{x^2-3x-18}{x+3}=\frac{(x-6)(x+3)}{(x+3)}=(x-6)\cdot\frac{(x+3)}{(x+3)}=(x-6)\cdot 1=x-6; \quad (x\neq -3)$$

Authors of textbooks select from methods like those shown in Table 1 to best illustrate the simplification of rational expressions. Similarly, teachers make selections when they plan and present simplification for their students. While both methods result in the same simplified expression, MethodA highlights the procedure or process for simplification and Method B makes reference to properties or concepts behind the simplification.

2.3Procedural and conceptual knowledge

This examination of methods used by algebra textbooks and teachers will be informed by the work of Hiebert and Lefevre (1986) who characterize procedural knowledge and conceptualknowledge.This framework links to both incomplete knowledge and student errors. Procedural knowledge refers to knowing the symbol representations and the process or algorithm to solve a problem. Conceptual knowledge refers to knowing the rich connections between concepts behind the methods and why certain mathematical methods can be used. Textbooks and teachers may emphasize a procedural focus, presenting material in a quick straight forward manner showing students exactly what to do, or a conceptual focus which could highlight relationships between concepts (Hiebert and Lefevre 1986) or some combination of both.

Over time teachers of mathematics develop a practice which might focus on procedural knowledge or conceptual knowledge that is influenced by the perceived advantages and disadvantages associated with each. Teachers may view a lesson focused on procedural knowledge as the most efficient use of classtime because it allows for quick coverage of the material, and a lesson focused on conceptual knowledge as too time-consuming. These assumptions are contradicted by Pesek and Kirshner (2000) who found that teaching conceptually, or relationally, is more time efficient than procedural, or instrumental, because of the temporary retention that takes place when students only focus on learning methods without an understanding of why they can use them. The quick coverage provided through procedural instruction does not give students reasoning and conceptual skills necessary to understand higher level mathematics and thus more time must be spent relearning material before moving forward.

However, asking teachers, who feel pressured to cover the multitude of topics in algebra, and who are eager to present topics in the simplest way for their students (Thompson 2008), to present lessons that include a focus on conceptual knowledge may go against teacher instincts. Bergqvist (2005) reported that teachers often underestimate secondary student reasoning skills. Similarly, sometimes teachers feel that meaning is important but that their students are incapable of appreciating it (Harel et al. 2008). Intensifying these low expectations of students, studies have also pointed to the mismatch of teacher knowledge of what students find difficult and actual student difficulties (Cunningham 2005; Hadjidemetriouand Williams 2002; Nathan and Koedinger 2000). Yet, research (Boaler 1998; Hiebert and Lefevre 1986; Skemp 1976) suggests the conceptual knowledge provides a stronger foundation, better problem solving skills, and eventually leads to further success.

Despite these benefits, teachers often stress repeated practice of the same procedural representation rather than introduce a different representation that might be more conceptual. When Demby (1997) studied algebraic transformations, she reported that "Conviction that a single very clear general exposition will work is an illusion." And the same might hold true for textbooks and teachers when presenting the simplification of rational expressions. And, simply transmitting algebraic rules by the teacher, memorizing them by students and practicing them in a mechanical way isnot effective.(Demby 1997).In addition to these considerations, research findings suggest that teachers should prepare sets of tasks of diverse types, suitable for the developing the prior knowledge of the student (Demby 1997; Rittle-Johnson et al. 2001).

While a combination of procedural and conceptual representations might conflict with the idea of repeated practice of a single method, further consideration seems prudent. Researchers have reported that both procedural knowledge and conceptual knowledge are not only beneficial but necessary to successfully understand mathematics (Byrnes and Wasik 1991; Hiebert and Wearne 1986; Rittle-Johnson etal. 2001). They have pointed to the advantages offered by the iterative nature and dynamic interaction of both types of knowledge in learning mathematics. Demby (1997) examined 2400 written responses from 108 students and collected oral descriptions about 1200 algebraic transformations made during interviews. She concluded that skills and meaning should develop together and should reinforce each other. Her research indicates that both procedural and conceptual focused examples could be advantageous for teachers and students alike. This might be especially true when teaching challenging topics like rational expressions.

Others have reported that knowledge of a particular type of problem is often incomplete, and a variety of experiences may initiate knowledge change. Smith and Silver (1989) reported success with reducing student cancellation errors by having students examine correct examples and examples that contained errors. Having students examine both procedural and conceptual examples might also improve reasoning and reduce errors. Furthermore, knowledge change may be enacted by examining the relationship between procedural and conceptual problem representations (Rittle-Johnson et al. 2001). Teachers need to be aware of these findings that may go against teacher intuition,but allows students to develop more complete knowledge which might improve student facility with rational expressions.

2.4Student errors

Besides conceptual and procedural knowledge, Brousseau's (1997) ways of knowingalso frames reasons for incomplete student knowledge and student errors. Brousseau defines several types of obstacles to ways of knowing. An obstacle is a way of knowing that works well in one setting but manifests itself as an error in others. Since they work well in some settings, they are resistant to change –hence the name "obstacle." One of these types of errors is called Didactical obstacles which arise as a result of instructional choices and therefore, are avoidable through the development of alternative instructional approaches. Skemp (1976) defines two types of errors that could be characterized as Didactical obstacles distinguished by the mismatch between the teacher's method and the student's perceived needs. One error [Type One] is when the teacher uses a procedural method when the students need, or are looking for,a conceptual method. Another error [Type Two] is the inverse of Type Onewhere the teacher uses a conceptual method when the students are looking for a procedural one.

Sometimes when a teacher presents a topic with a strong conceptual focus, it will evoke the student refrain "Just show us what to do." This common student refrain is emblematic of a Didactical obstacle of Type Two. Teachers must be resistant to such student request when presenting a conceptual focused method. Also, when teachers, as shown in Method A, emphasize the process of striking through common factors when simplifying rational expressions, they may be inadvertently presenting a Didactical obstacle of Type One.This type might occur when a student needs or wants to know "Why" but the teacher is only showing "How" the procedure works. Gray and Tall (1992) found that the majority of students study the process of how to solve problems based on what they see visually. Further, (Harel et al. 2008) point out that students sometimes pay attention to key words and triggers rather than overall meaning. Studies of cognition point to the importance of concept images and also to the strong influence of visual cues in problem solving (Gutierrez and Jaime 1999; Vinner and Hershkowitz 1980). A student typically remembers prior experiences with diagrams, attributes, and examples associated with the concept, these experiences embody the concept image (Gutierrez and Jaime 1999). The concept image may be the first image, and sometimes the only image, that the student recalls when answering a question or solving a problem. This image is recalled rather than the formal concept definition. In the case of rational expressions, it might not be the concept definitions of terms or factors, or the instruction to divide by the greatest common factor that would be recalled, but rather the singular concept image of striking through expressions that look alike.

Similarly, Carry et al. (1980) found that a "majority of errors occurred when students overgeneralized a valid operation arriving at a single generic deletion operation." Their study focused on transformations of linear equations, but the same might apply to the simplification of rational expressions. Some students who are exposed to classroom presentations that emphasize visually, and in words, the physical act of striking through may retain this as a single deletion operation or concept image. This may force some students to over rely on a single concept image, lacking complete meaning, which could lead to errors similar to the universal law of cancellation. Teachers must consider using both procedural and conceptual methods to avoid Didactical obstacles so that their students encounter a variety of experiences and develop more complete knowledge.

2.5Characterization of methods

While the two simplifications methods shown in Table 1 have both procedural and conceptual components, Method A could be characterized as focused more on promoting procedural knowledge. The instructions emphasize the process, rather than the reason why it works, "If possible, factor the numerator and denominator. Remove common factors and then simplify."In addition, the striking through of the common factors is a strong visual cue that might serve as a trigger for students who might ignore the meaning behind the process of cancelling.

Method B could be characterized as more focused on promoting conceptual knowledge, since the expectation would be that the students recall prior knowledge about rational numbers and recognize the relationships between the two properties and the method used for simplification:

1. If $c \neq 0,$ then $\frac{c}{c} = 1$

2. Multiplicative property of one (or identity): $a \cdot 1 = 1 \cdot a = a$

Being imbedded directly into Method B, these two properties of rational numbers are closely linked to rational expression simplification. Also, these properties are referred to in the instructions for Method B, "Group common factors, substitute for expressions equal to 1 and use the multiplicative property of one." Note that Method B avoids the visual cue of striking through common factors but may contribute to a Didacticle obstacle of Type Two.

It must also be acknowledged that some combination of Method A or Method B could be selected to visually demonstrate rational expression simplification by a teacher for use in the classroom. It is possible that one method might be selected for illustration and the other method might be referred in an oral explanation by the teacher. But, unless both methods are visually illustrated, one method is selected for illustration over the other. And, because the majority of students are strongly influenced based on what they see visually (Gray and Tall 1992) the attention of this paper will be focused on the singular method selected for illustrations by teachers.

Textbook authors and teachers may contemplate the possible advantages and disadvantages associated with each of these methods when presenting the simplification of rational expressions.If textbooks illustrations focus completely on the development of procedural knowledge, some students will be presented with a Type One Didacticle obstacle. And, more importantly, if teachers ignore the benefits of lessons focused on conceptual knowledge, not only will many students continue to make errors, but it will impact student success in future mathematics courses. Additionally, students might benefit from robust concept images that de-emphasize the visual cue of striking through common factors and put more emphasis on ways that add conceptual knowledge to the process of simplification. Since teachers report that working with rational expressions pose a significant challenge to students, and textbooks and teachers combine to exert such a strong influence on student knowledge, the following research questions will be considered.

2.6Research questions

1. Given a sample of algebra textbooks, how many textbooks have illustrations ofrational expression simplification that resemble Method A(procedural)?How many textbooks have illustrations that resemble Method B (conceptual)?How many textbooks have a combination of both methods?

2. Given a sample of algebra teachers, how many indicate that when they teach rational expression simplification that their board-work illustration resembles Method A (procedural)?How many indicate their board work resembles Method B (conceptual)?

3. Given a sample of algebra teachers, what are the most commonly reported pros/advantagesand cons/disadvantages for using Method A (procedural) and for using Method B (conceptual) when presenting rational expression simplification in the classroom?

3 METHODOLOGY

3.1 Algebra textbooks

To examine the methods for rational expressions simplification used in textbooks, 10algebra textbooks were selected which included the three most popular textbooks used throughout the state of New Jersey. Of the 10 textbooks, 7 were published prior to 2012 and were in use in public schools in central New Jersey. The other 3 textbooks were published in2012 and were aligned withthe Common Core State Standards (NGA Center & CCSSO, 2011). These were examined using online e-text replicas of the textbooks provided by the publishers.

3.2 Participants

A list of alumni e-mail addresses (42) was obtained from the Department of Mathematics and Statistics at a state college in central New Jersey. Each e-mail address was from a student who graduated from the state college with bachelor degrees in secondary mathematics education. The addresses were taken from correspondence over the previous three years. The correspondences related to graduate school either as a request for letters of recommendation from faculty or a

request for advice about graduate schools and programs. This group of participants formed a desirable target group that had experienced a quality undergraduate preparation in mathematics content and pedagogy, were likely teaching mathematics, and some of whom had attended or were considering attending graduate school.
Utilizing the addresses of alumni,the volunteer surveys were distributedelectronically.The e-mail explained that the survey was voluntary and only those currently teacher mathematics should complete the survey. The participants were given a 4-week window to respond to the survey. Because of anonymity, the number of schools involved was not determined, in all likelihood the majority of participants worked in different public schools.

Beyond the algebra teachers surveyed, in person interviews were conducted with 6 algebra teachers who had been teaching algebra for between 10 and 31 years in public secondary schools in New Jersey. These interviews were conducted with 4 volunteers who were speakers at a mathematics education conference by one of the authors and with 2 volunteers from the other author's student teaching placement school. The questions replicated those of the 8-question survey. It was hoped that the interviews might provide further insight into their perceived pros/advantages and cons/disadvantages for using Method A and Method B.

3.3 Survey instrument

Since no previous research instrument involving the simplification of rational expressions could be found in the literature, a preliminary survey instrument was developed by the authors.It was examined by two experts in mathematics education with both high school and college teaching experience. Both experts made suggestions for improvements and agreed that the questions asked on the survey corresponded to the research questions asked. Following these improvements, the survey was pretested by being administered to six student teachers of mathematics who were teaching algebra. While these student teachers were not members of the target group of in-service mathematics teachers, this was carried out to suggest possible strengths and weaknesses of the survey.

Using the online software Qualtrics,afinal revised survey was developed with just 8 questions to better assure completion by the participants. The first two itemsdetermine the number of years that algebra teachers had taught mathematics, the number of times that they taught algebra and the year that they most recently taught algebra. Method A and Method B from Table 1 were given and participants were asked when they teach simplification of rational expressions would their explanation and board-work resemble Method A or Method B. This closed question was followed by the open question asking them to list the pros/advantages and cons/disadvantages of each method. The surveys also questioned if they use the word "cancel"or the words "cross out", how they would explain a common error, and if they taught any strategies for students to check their work. A copy of the survey is included in Appendix A.

4 RESULTS

4.1 Textbooks

In the sample of algebra textbooks examined, all 10 showed illustrations of rational expression simplification that resembled Method A (procedural).None of the textbooks showed illustrations that directly resembled Method B. But, some of the textbooks offered explanations in the written text that referenced the connection between rational expressions and rational numbers, but not as clearlyas in Method B. The 10 textbooks contained a range of references: 4 made no reference to rational numbers, 5made casual references (e.g. "Simplifying rational expressions is similar to simplifying numerical fractions.", or "Like rational numbers, rational expressions are in simplest form when their only common factor is 1."), and 1 of the textbooks that used Method A did make reference in the written text to the two rational number properties that characterized Method B (by showing the properties: $\frac{a}{a}=1,\quad \frac{ab}{ac}=1\cdot\frac{b}{c}=\frac{b}{c}$).All of the algebra textbooks made efforts to connect rational expressions to one or more other topics in mathematics including: rational functions, excluded values, and word problems. It was also noted that all 10 of the textbooks showed factors struck through, but none of the textbook used the word "cancel" or the words "cross out" in the section on rational expressions.

4.2 Surveys

Completed surveys were returned by (33) participants, all of whom were experienced algebra teachers working in public secondary schools in New Jersey. Of the participants surveyed (42) , 5 were no longer teaching mathematics at the time of the survey and 4 had e-mail addresses that were no longer active. The 33 algebra teachers surveyed had between 2 and 27 years teaching experience. When asked which method they use when they teach rational expression simplification, 29 (87.8%) indicated that their explanation and board work would resemble Method A, and 4 (12.1%) indicated that their explanation and board work would resemble Method B.

After carefully recording the open responses, the two authors independently categorized the responses to the pros/advantages and cons/disadvantages of using each method. Also, two mathematics education experts were asked to independently categorize the responses. Only those responses that were categorized similarly by all four individuals are reported and appear below in Table 2.

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Table 2. Number and percent of most commonly reported responses (n=33).

Method	A (Procedural)	B (Conceptual)
Pros/advantages of method reported	20(60.6%) Quicker, shorter, or simpler to teach	21(63.6%) Clear explanation of cancelling
Cons/disadvantages of method reported	16 (48.4%) Ignores “why” and the reason behind cancelling	13 (39.3%) Cumbersome, more or additional work to teach

Also, the majority of teachers 24 out of 33 (72.7%) reported using the word “cancel” or “cross out” while showcasing the striking through of common factors when teaching rational expression simplification.

4.3 Interviews

Interviews using the 8-question survey were conducted with a sample of 6 algebra teachers. Their years of experience and their responses to the pros/advantages and cons/disadvantages of using each method are reported.

Algebra Teacher 1. (31 years of teaching)
Reported: “I use Method A because it is good for keeping track of strike-throughs but a bit automatic. Method B is very clear in showing a/a =1 but too cumbersome to use with students.”

Algebra Teacher 2.(28 years of teaching)
Reported:“I use Method A because it is faster and easier to show in my mind. Method B shows explicitly multiplication by 1 but may confuse students asking them to go back to previous fact about the identity.”

Algebra Teacher 3. (30 years of teaching)
Reported:“I use Method A because it shows that fraction reduces to 1 over 1 when canceled. This will help when entire numerator or denominator cancels [except for one]…then the 1 will be left over [for the student] to show in the answer.”

Algebra Teacher 4. (17years of teaching)
Reported:“I use Method A because it is less writing. For Method B the student can clearly see that the fraction reduces to 1.”
Algebra Teacher 5. (12 years of teaching)
Reported:“I use Method A because it is easier to teach and they don’t have a good understanding of the rules. Method B gave a better understanding of multiplication rules but would be harder for students to understand.”

Algebra Teacher 6. (10 years of teaching)
Reported:“I use Method A because it has less steps but doesn’t explain cancelling. Method B explains why we cancel.”

5 DISCUSSION

In the sample of (10) algebra textbooks,all showed illustrations that resembled the procedural focused Method Awhich indicates that these textbooks favoredexamples that showprocess over concepts for rational expression simplification.Only one of the textbooks made a significant connection between rational expressions and rational numbers by referring to the properties included in Method B.Whether teachers enhance such connections or avoid them in their classrooms presentations is not known. Nonetheless, the majority of algebra teachers (87.8%) in the samplereported that their explanations and board work resembled the procedural focused Method A. These findings agree with previous research that indicates textbooks and teachers often de-emphasize meaning by emphasizing procedures over concepts which can impact student learning (Cunninghamand Roberts 2010; Hershkowitz 1987; Vinner and Hershkowitz 1980; Mac Gregor and Stacy 1997).

Because the majority of teachers (63.6%) reported that the advantage of Method B was that it gave a clearer explanation and (48.4%) reported that Method A did not, the advantage of Method B was recognized by a majority but may still not be enough for them to emphasize it in their own classroom.The advantage of Method A was reported as quicker, shorter, or simpler to teach by (60.6%) of the teachers over Method B. Moreover, this advantage, along with the fact that textbook illustrations resembled Method A, appears sufficient to have influenced the majority of teachers to use Method A in their own classroom.

When considering the advantages and disadvantages of the methods,experienced teachers who were interviewed reported that Method B "…would confuse students asking them to go back to previous facts about the identity." or "…would be harder for students to understand." Such responses suggest that these very experienced algebra teachers may have developed a hardened theory of low expectations for student learning which may impact both teaching methods and ultimately student performance.These responses also support previous research that indicates teachers underestimate the reasoning skill of students, believe that students are incapable of appreciating meaning, and often incorrectly diagnosing what students find difficult (Bergqvist 2005;Cunningham 2005; Hadjidemetriouand Williams 2002; Harel et al. 2008). Teacher beliefs and expectations such as these may cause them to reject conceptual methods even though they hold promise of giving students more complete and meaningful knowledge.

While all of algebra textbooks examined showed examples with common factors struck through, none of the books used the word "cancel' or "cross out' in the text. In contrast, 24 of the 33 teachers surveyed (72.7%) reported using these words during classroom presentations. These words when used in the classroom without linkages to the concepts behind the operation may not only trigger but amplify the striking through as a singular concept image retained by students (Gutierrez and Jaime 1999; Harel et al. 2008). And as reported by Vinner and Hershkowitz (1980), the concept image held by some students can be limited to a single prototypical image, and an over-reliance on it can impact their understanding. This over-reliance may result in errors much like those associated with the universal law of cancellation. Introducing rational expression simplification with the conceptual focused Method B avoids striking throughand might allow students to focus on prior knowledge and mightreduce student errors.

6 CONCLUSION

While this paper examines only a small number of algebra textbooks, they all focus primarily on procedural methods when presenting rational expression simplification. In addition, despite the fact that algebra teachers were not observed during teaching, the majority reported focusing on procedures, even though they might recognize advantages offered by conceptual methods. These results are particularly disturbing in that if textbooks focus primarily on procedural methods for the simplification of rational expressions, they can impair student learning for this topic and for higher level mathematics topics. Similarly, if teachers who have experienced a quality undergraduate preparation in mathematics content and pedagogy continue to believe that students are incapable of appreciating meaning and reach for the simplicity that procedural methods provide, developing student facility with simplification will remain a significant challenge for teachers.

During interviews, highly experienced teachers reported that the conceptual focused method "…would confuse students asking them to go back to previous facts…" while the procedural focused method "…was easier to teach."These responses indicate that they feel that their mission is to present topics in the simplest way even when the topic itself can becomplex and a challenging for students to learn. Reinforcing these decisions to use procedural methods might be the teachers' low expectations of students and the request by some students to "Just show me what to do."Relying completely on the procedural method honors this student request and poses the most straightforward manner to teach students,but teachers must know that expecting and encouraging students to make connections to prior knowledge is critical to building robust conceptual knowledge of mathematics.

When these algebra teachers reported that a conceptual method was too cumbersome or required too much writing, they may be thinking about applying it to more complex problems than simplification. Most textbooks follow simplification of rational expressions with problems involving the combining of rational expressions using the four operations. These types of problems could add to the number of steps required if a single conceptual focused methodwasemployed. This might also add to the class time needed to present rational expressions and require textbook changes.

However, if textbooks and teachers introduced challenging topics using a conceptual method and then switched to the more procedural method, much might be gained. This strategy would conform to Demby's recommendation that it is very important for teachers to prepare sets of tasks of diverse types, suitable for developing the prior knowledge of students, and that skills and meaning should develop together and should reinforce each other (1997). Also, there is the advantage offered by the iterative nature and dynamic interaction of both types of knowledge in learning mathematics. Such learning should be an iterative process with students developing both procedural and conceptual knowledge where both improve as one is emphasized (Byrnes and Wasik 1991; Rittle-Johnson et al. 2001). In addition, there is the advantage reported by Pesek and Kirshner (2000) that conceptual methods are more efficient in the long run, and offers a host of benefits

including: builds a stronger foundation,enhances problem solving skills, and leads to future success in mathematics (Boaler 1998; Hiebert and Lefevre 1986; Skemp 1976).
Nevertheless, the strategy to use a conceptual method to illustrate the simplification of rational expressions, in addition to the procedural method, needs further examination by experimental research to validate its impact.Thisstrategy to utilize both a conceptual and a procedural method might avoid Didactical obstacles (Brousseau, 1997) and be beneficial for teaching other challenging topics in algebra. Although these topics can be made easier to teach by focusing only on procedure, errors often occur when students lack linkages between procedural and conceptual knowledge (Heibert and Wearne 1986). In fact, Thompson (2008) states "…that too many mathematics teachers at all levels spend too little time at the outset of teaching a topic on having students become steeped in ideas and meanings that are foundational to it."

When introducing a topic using a conceptual method and telling students directly that the topic is going to be challenging seems more honest than just focusing on the simple procedure. This might encourage students to appreciate the connection to prior knowledge even if that knowledge has to sometimes be reviewed. Such a review might seem inefficient but research indicates that this might not be the case in the long run, if students develop robust concept images and more complete knowledge. Utilization of a conceptual methodmay go against a teacher's instinct, deviate from textbook illustrations, and require more effort, but teachers must realize that avoiding it may sabotage student learning and its potential to enhance student success.

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APPENDIX A

Algebra Teacher Survey

This survey is anonymous and is designed to provide recommendations for the teaching of Algebra.

1. How many years have you been teaching mathematics? ___________________________

2. How many times have you taught Algebra ? _____Year most recently taught Algebra ______

Method A"If possible, factor the numerator and denominator. Remove common factors and then simplify."

$$\frac{x^2-3x-18}{x+3}=\frac{(x-6)(x+3)}{(x+3)}=\frac{(x-6)\cancel{(x+3)}}{\cancel{(x+3)}}=x-6 \qquad (x\neq-3)$$

Method B "Group common factors, substitute for expressions equal to 1 and use the multiplicative property of one"

$$\frac{x^2-3x-18}{x+3}=\frac{(x-6)(x+3)}{(x+3)}=(x-6)\cdot\frac{(x+3)}{(x+3)}=(x-6)\cdot1=x-6 \qquad (x\neq-3)$$

3. When you teach simplification of rational expressions would your explanation and board work resemble Method A or Method B? Circle: Method A or Method B

4. List any Pros or Cons of the two methods from your experience teaching algebra students.
(Pros = reasons for using, Cons = reasons for not using.)

Method A:
Pros:
Cons:

Method B
Pros:
Cons:

5. When you teach simplification of rational expressions do you write or say "cancel" or "cross out"? If so, what words do you use to justify for students cancelling or crossing out?

6. Here is a common error a student might make in simplifying: $\frac{4x+20}{12+4x}=\frac{\cancel{4x}+20}{12+\cancel{4x}}=\frac{20}{12}=\frac{5}{3}$
Howwould you explain to the student the error demonstrated?

7. What strategies, if any, do you give for students to check their answers when simplifying rational expressions? If so, give an example.

8. Comments or suggestions about the teaching or learning of simplification of rational expressions.

Authors' biographies

Dr. Robert F. Cunningham is a Professor of Mathematics & Statistics at The College of New Jersey in the United States. He earned a B.A. in Mathematics from LaSalle University, an M.A. in Mathematics from Villanova University, and an

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