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Analytical functions of generalized complex variables and some applications

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ABSTRACT

The object of this work is to use functions of a generalized complex variable to solve the problems of fluid dynamics and elasticity theory. In this paper, we obtain Cauchy-Riemann conditions, generalized Laplace equation and the generalized Poisson formula for such functions.

Keywords

Cauchy-Riemann conditions, generalized Laplace equation, Poisson formula.

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INTRODUCTION

The generalized complex numbers are divided into types such as elliptic, hyperbolic and parabolic complex numbers [1]. This means the following: let a generalized complex number be in this form $z = x + py$, $p^2 = -\theta_0 + p\theta_1$, where θ_0, θ_1 are real numbers. Then mentioned numbers are divided into types depending on θ_0 and θ_1.

- If $D = \frac{\theta_1^2}{4} - \theta_0 < 0$, such generalized complex numbers refer to the elliptic type,
- If $D = \frac{\theta_1^2}{4} - \theta_0 > 0$ we have the hyperbolic type, and
- If $D = \frac{\theta_1^2}{4} - \theta_0 = 0$, we have the parabolic type.

If it is set that $\theta_0 = 1, \theta_1 = 0$, we obtain usual complex numbers. If $\theta_0 = -1, \theta_1 = 0$, we obtain double numbers, and dual numbers if $\theta_0 = \theta_1 = 0$.

In this paper, we consider the theory of analytical functions $f(z) = u(x,y) + pv(x,y)$ of the generalized complex variable $z = x + py$, satisfying a set of Cauchy-Riemann equations

$$u_x + \theta_1 v_x = v_y,\ u_y + \theta_0 v_x = 0, \tag{1}$$

which is essentially equivalent to Laplace equation

$$\Delta u = \frac{1}{-4D}\left(\theta_0 u_{xx} - \theta_1 u_{xy} + u_{yy}\right) = 0. \tag{2}$$

Similarly, for the imaginary part of the function $v(x,y) = \operatorname{Im} f(z)$ we get

$$\Delta v = \frac{1}{-4D}\left(\theta_0 v_{xx} - \theta_1 v_{xy} + v_{yy}\right) = 0. \tag{3}$$

THE EQUIVALENCE OF THE CAUCHY - RIEMANN CONDITIONS AND $\frac{\partial f}{\partial \bar{z}} = 0$ CONDITION

Suppose we are given a function $f(z) = u(x,y) + pv(x,y)$. x and y variables can be easily expressed by $z = x + py$ and $\bar{z} = x + \theta_1 y - py$

$$x = \frac{\theta_1 - p}{\theta_1 - 2p} z - \frac{p}{\theta_1 - 2p}\bar{z},$$

$$y = \frac{-1}{\theta_1 - 2p} z + \frac{1}{\theta_1 - 2p}\bar{z},$$

where $p^2 = -\theta_0 + p\theta_1$. Therefore, the function $f(z)$ can be formally considered as a function of two variables z and $\bar{z}$. Let find $\frac{\partial f}{\partial \bar{z}}$. For this purpose we should consider differential operators

$$\frac{\partial}{\partial z} = \frac{1}{\theta_1 - 2p}\left[(\theta_1 - p)\frac{\partial}{\partial x} - \frac{\partial}{\partial y}\right], \tag{4}$$

$$\frac{\partial}{\partial \bar{z}} = \frac{1}{\theta_1 - 2p}\left[-p\frac{\partial}{\partial x} + \frac{\partial}{\partial y}\right], \tag{5}$$

with the following property

$$\frac{\partial^p}{\partial z^p}\left(\frac{\partial^q}{\partial \bar{z}^q}\right) = \frac{\partial^q}{\partial \bar{z}^q}\left(\frac{\partial^p}{\partial z^p}\right), (p, q = 0,1,2,\dots).$$

Therefore next form operators are uniquely determined

$$\frac{\partial^{p+q}}{\partial z^p \partial \bar{z}^q} = \frac{1}{(\theta_1 - 2p)^{p+q}}\left[(\theta_1 - p)\frac{\partial}{\partial x} - \frac{\partial}{\partial y}\right]^p \left[-p\frac{\partial}{\partial x} + \frac{\partial}{\partial y}\right]^q.$$

In particular, for $p = q = 1$ we obtain

$$\frac{\partial^2}{\partial z \partial \bar{z}} = \frac{1}{(\theta_1 - 2p)^2}\left[-p(\theta_1 - p)\frac{\partial^2}{\partial x^2} + p\frac{\partial^2}{\partial x \partial y} + (\theta_1 - p)\frac{\partial^2}{\partial y \partial x} - \frac{\partial^2}{\partial y^2}\right] = \frac{1}{-4D}\left(\theta_0 \frac{\partial^2}{\partial x^2} - \theta_1 \frac{\partial^2}{\partial x \partial y} + \frac{\partial^2}{\partial y^2}\right) \tag{6}$$

where $D = \frac{\theta_1^2}{4} - \theta_0$.

In case of $p = q = 2$, the generalized biharmonic operator can be written as

$$\frac{\partial^4}{\partial z^2 \partial \bar{z}^2} = \frac{1}{16D^2}\left(\theta_0^2 \frac{\partial^4}{\partial x^4} - 2\theta_0\theta_1 \frac{\partial^4}{\partial x^3 \partial y} + (\theta_1^2 + 2\theta_0)\frac{\partial^4}{\partial x^2 \partial y^2} - 2\theta_1 \frac{\partial^4}{\partial x \partial y^3} + \frac{\partial^4}{\partial y^4}\right) \tag{7}$$

It follows from here that for $\theta_0 = 1, \theta_1 = 0$ values a simple biharmonic operator is inferred

$$\frac{\partial^4}{\partial z^2 \partial \bar{z}^2} = \frac{1}{16}\left(\frac{\partial^4}{\partial x^4} + 2\frac{\partial^4}{\partial x^2 \partial y^2} + \frac{\partial^4}{\partial y^4}\right).$$

Here $z = x + iy, \bar{z} = x - iy$ and $p^2 = -\theta_0 + p\theta_1 = -1$, i.e $p = i$.

Theorem. Cauchy-Riemann conditions are equivalent to $\frac{\partial}{\partial \bar{z}} = 0$.

If $\frac{\partial}{\partial \bar{z}} = 0$, then $\frac{\partial}{\partial \bar{z}} = \frac{1}{\theta_1 - 2p}\left[-p\left(\frac{\partial u}{\partial x} + p\frac{\partial v}{\partial x}\right) + \frac{\partial u}{\partial y} + p\frac{\partial v}{\partial y}\right] = \frac{1}{\theta_1 - 2p}\left[\frac{\partial u}{\partial y} + \theta_0\frac{\partial v}{\partial x} - p\left(\frac{\partial u}{\partial x} + \theta_1\frac{\partial v}{\partial x} - \frac{\partial v}{\partial y}\right)\right] \equiv 0$. Therefore Cauchy-Riemann conditions are satisfied

$$\frac{\partial u}{\partial x} + \theta_1\frac{\partial v}{\partial x} = \frac{\partial v}{\partial y},$$

$$\frac{\partial u}{\partial y} + \theta_0\frac{\partial v}{\partial x} = 0.$$

In general case, $\int_{AB} f(z)dz$ integral depends on the shape of the path (where $z = x + py$, $p^2 = -\theta_0 + p\theta_1$). We should determine conditions when the integral is independent of the path shape. The answer to this question is in the theorem below.

Cauchy theorem. Let $f(z)$ be a generalized analytical function in a simply connected domain Ω. Then the contour integral of this function along any closed piecewise-smooth contour L, lying completely inside Ω, is equal to zero.

Proof. As the function $f(z) = u(x,y) + pv(x,y)$ is analytical in Ω domain we have

$$\oint_L f(z)dz = \oint_L u(x,y)dx - \theta_0 v(x,y)dy + +p\oint_L v(x,y)dx + \big(u(x,y) + \theta_1 v(x,y)\big)dy = -\iint_G \left(\frac{\partial u}{\partial y} + \theta_0\frac{\partial v}{\partial x}\right)dxdy + +p\iint_G \left(\frac{\partial u}{\partial x} + \right.$$
θ1∂v∂x−∂v∂ydxdy=2p−θ1G∂f∂zdxdy.

It follows from Cauchy-Riemann conditions that $\frac{\partial f}{\partial \bar{z}} = 0$. This condition and $u, v, u_x v_x, u_y, v_y$ functions continuity are enough to make integrals vanish.

CAUCHY-RIEMANN CONDITIONS IN POLAR COORDINATES

We should move from algebraic form of the generalized complex number $z = x + py$, $p^2 = -\theta_0 + p\theta_1$ to its exponential trigonometric form.

$$z = |z|e^{\left(-\frac{\theta_1}{2}+p\right)\varphi} = |z|[T(\varphi) + pS(\varphi)],$$

where

$$e^{\left(-\frac{\theta_1}{2}+p\right)\varphi} = T(\theta_0,\theta_1,\varphi) + pS(\theta_0,\theta_1,\varphi) = \begin{cases} \left[\left(\cos\sqrt{-D}\varphi - \frac{\theta_1}{2\sqrt{-D}}\sin\sqrt{-D}\varphi\right) + p\frac{1}{\sqrt{-D}}\sin\sqrt{-D}\varphi\right], D < 0 \\ \left[\left(1 - \frac{\theta_1}{2}\varphi\right) + p\varphi\right], D = 0 \\ \left[\left(\cosh\sqrt{D}\varphi - \frac{\theta_1}{2\sqrt{D}}\sinh\sqrt{D}\varphi\right) + p\frac{1}{\sqrt{D}}\sinh\sqrt{D}\varphi\right], D > 0 \end{cases} \tag{9}$$

Particularly, for $\theta_0 = 1, \theta_1 = 0$ we have: $p^2 = -1, D = -1$; from here we can obtain Euler's formula $e^{i\varphi} = \cos\varphi + i\sin\varphi$. Now, taking into account a connection formula of a point in the plane between Cartesian and the generalized coordinates, we can write

$$x = rT(\theta_0,\theta_1,\varphi),\ y = rS(\theta_0,\theta_1,\varphi),\ \text{where } r^2 = |z|^2 = z\cdot\bar{z} = x^2 + \theta_1 xy + \theta_0 y^2.$$

Some calculations are necessary in the future. Suppose

$$z = x + py = re^{\left(-\frac{\theta_1}{2}+p\right)\varphi}.$$

Let: $\bar{z} = re^{\left(\frac{\theta_1}{2}-p\right)\varphi}$ and $z\cdot\bar{z} = r^2$. Using calculation formulas of partial derivatives for a composite function of two variables we should find from two last equations $r^2 = x^2 + \theta_1 xy + \theta_0 y^2 = z\cdot\bar{z}$,

$$2r\frac{\partial r}{\partial z} = z,\ \frac{\partial r}{\partial \bar{z}} = \frac{z}{2r} = \frac{1}{2}e^{\left(-\frac{\theta_1}{2}+p\right)\varphi},$$

$$\bar{z} = e^{(\theta_1-2p)\varphi}z,\ 1 = z(\theta_1 - 2p)e^{(\theta_1-2p)\varphi}\frac{\partial\varphi}{\partial\bar{z}}.$$

From here $\frac{\partial\varphi}{\partial\bar{z}} = \frac{\theta_1-2p}{4D}\frac{1}{r}e^{\left(-\frac{\theta_1}{2}+p\right)\varphi}$, $\frac{\partial r}{\partial\bar{z}} = \frac{1}{2}e^{\left(-\frac{\theta_1}{2}+p\right)\varphi}$, где $D = \frac{\theta_1^2}{4} - \theta_0$.

In order to write down Cauchy-Riemann conditions in polar coordinates, we should introduce next differential operator

$$\frac{\partial}{\partial\bar{z}} = \frac{\partial r}{\partial\bar{z}}\frac{\partial}{\partial r} + \frac{\partial\varphi}{\partial\bar{z}}\frac{\partial}{\partial\varphi} = \frac{1}{2}e^{\left(-\frac{\theta_1}{2}+p\right)\varphi}\left(\frac{\partial}{\partial r} + \frac{\theta_1-2p}{2D}\frac{1}{r}\frac{\partial}{\partial\varphi}\right).$$

Then Cauchy-Riemann conditions can be written in the form $\frac{\partial f}{\partial\bar{z}} = 0$, and equivalent to the following system

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$$\begin{cases}\frac{\partial u}{\partial r}+\frac{\theta_1}{2D}\frac{1}{r}\frac{\partial u}{\partial \varphi}+\frac{\theta_0}{D}\frac{1}{r}\frac{\partial v}{\partial \varphi}=0,\\ \frac{\partial v}{\partial r}-\frac{1}{D}\frac{1}{r}\frac{\partial u}{\partial \varphi}-\frac{\theta_1}{2D}\frac{1}{r}\frac{\partial v}{\partial \varphi}=0.\end{cases} \quad (10)$$

Particularly, for $\theta_0=1, \theta_1=0$ we have $\frac{\partial u}{\partial r}=\frac{1}{r}\frac{\partial v}{\partial \varphi}, \frac{\partial v}{\partial r}=-\frac{1}{r}\frac{\partial u}{\partial \varphi}$.

Further, the system (10) can be rewritten in compact form. To do so we solve the system (10) with respect to $\frac{\partial u}{\partial r}, \frac{\partial u}{\partial \varphi}$

$$\Delta=\begin{vmatrix}1 & \frac{\theta_1}{2Dr}\\ 0 & \frac{1}{Dr}\end{vmatrix}=\frac{1}{Dr}, \quad \Delta_{\frac{\partial u}{\partial r}}=\begin{vmatrix}-\frac{\theta_0}{Dr}\frac{\partial v}{\partial \varphi} & \frac{\theta_1}{2Dr}\\ \frac{\partial v}{\partial r}-\frac{\theta_1}{2Dr}\frac{\partial v}{\partial \varphi} & \frac{1}{Dr}\end{vmatrix}=-\frac{\theta_1}{2Dr}\frac{\partial v}{\partial r}+\frac{1}{Dr^2}\frac{\partial v}{\partial \varphi},$$

$$\Delta_{\frac{\partial u}{\partial \varphi}}=\begin{vmatrix}1 & -\frac{\theta_0}{Dr}\frac{\partial v}{\partial \varphi}\\ 0 & \frac{\partial v}{\partial r}-\frac{\theta_1}{2Dr}\frac{\partial v}{\partial \varphi}\end{vmatrix}=\frac{\partial v}{\partial r}-\frac{\theta_1}{2Dr}\frac{\partial v}{\partial \varphi},$$

$$\frac{\partial u}{\partial r}=\frac{\Delta_{\frac{\partial u}{\partial r}}}{\Delta}=-\frac{\theta_1}{2}\frac{\partial v}{\partial r}+\frac{1}{r}\frac{\partial v}{\partial \varphi}, \quad \frac{\partial u}{\partial \varphi}=\frac{\Delta_{\frac{\partial u}{\partial \varphi}}}{\Delta}=Dr\frac{\partial v}{\partial r}-\frac{\theta_1}{2}\frac{\partial v}{\partial \varphi}.$$

In a similar manner, solving the system (10) with regard to $\frac{\partial v}{\partial r}, \frac{\partial v}{\partial \varphi}$ we have

$$\frac{\partial v}{\partial r}=-\frac{\theta_1}{2\theta_0}\frac{\partial u}{\partial r}-\frac{1}{\theta_0 r}\frac{\partial u}{\partial \varphi}, \quad \frac{\partial v}{\partial \varphi}=-\frac{Dr}{\theta_0}\frac{\partial u}{\partial r}-\frac{\theta_1}{2\theta_0}\frac{\partial u}{\partial \varphi}.$$

Essentially, these equations are equivalent to Laplace equation

$$\begin{aligned}\frac{\partial^2 u}{\partial r^2}+\frac{1}{r}\frac{\partial u}{\partial r}-\frac{1}{Dr^2}\frac{\partial^2 u}{\partial \varphi^2}=0,\\ \frac{\partial^2 v}{\partial r^2}+\frac{1}{r}\frac{\partial v}{\partial r}-\frac{1}{Dr^2}\frac{\partial^2 v}{\partial \varphi^2}=0,\end{aligned} \quad (11)$$

where $D=\frac{\theta_1^2}{4}-\theta_0$. Let consider some examples

EXAMPLES

Example 1. Show that $u(x,y;x_0,y_0)=\ln\frac{1}{r}$ function (where r is the distance between (x,y) and (x_0,y_0) points in a generalized plane R^2, i.e $r=\sqrt{(x-x_0)^2+\theta_1(x-x_0)(y-y_0)+\theta_0 y^2)}$) is harmonic in any domain of the generalized plane R^2, not containing (x_0,y_0) point.

Solution. For the convenience of computing the distance between points we represent the distance in the next form

$$r^2=(x-x_0)^2+\theta_1(x-x_0)(y-y_0)+\theta_0 y^2.$$

From here

$$r_x=\frac{2(x-x_0)+\theta_1(y-y_0)}{2r}, \; r_y=\frac{\theta_1(x-x_0)+2\theta_0(y-y_0)}{2r}, \; r_{xx}=-\frac{D(y-y_0)^2}{r^3}, \; r_{xy}=\frac{D(x-x_0)(y-y_0)}{r^3}, \; r_{yy}=-\frac{D(x-x_0)^2}{r^3}.$$

Then for $u(x,y;x_0,y_0)=\ln\frac{1}{r}$, function we have

$$u_x=-\frac{r_x}{r}, u_y=-\frac{r_y}{r}, u_{xx}=-\frac{r_{xx}r-r_x^2}{r^2}, \; u_{xy}=-\frac{r_{xy}r-r_x r_y}{r^2} \text{ and } u_{yy}=-\frac{r_{yy}r-r_y^2}{r^2}.$$

Substituting found values of the derivatives u_{xx}, u_{xy} and u_{yy} in Laplace equation we obtain

$$\Delta u=-\frac{1}{4D}\left(\theta_0\frac{\partial^2 u}{\partial x^2}-\theta_1\frac{\partial^2 u}{\partial x\partial y}+\frac{\partial^2 u}{\partial y^2}\right)=\frac{1}{4D}\left(\theta_0 r_{xx}-\theta_1 r_{xy}+r_{yy}\right)\frac{1}{r}-$$

$$-\frac{1}{4D}\left(\theta_0 r_x^2-\theta_1 r_x r_y+r_y^2\right)\frac{1}{r^2}=\frac{1}{4D}\frac{1}{r}\frac{-D}{r}-\frac{1}{4D}\frac{1}{r^2}(-D)=\frac{1}{4r^2}-\frac{1}{4r^2}\equiv 0,$$

at all points (x,y) of the generalized plane R^2, except one point (x_0,y_0), since

$$\theta_0 r_{xx}-\theta_1 r_{xy}+r_{yy}=-\frac{D}{r}, \; \theta_0 r_x^2-\theta_1 r_x r_y+r_y^2=-D.$$

By this means, $u=\ln\frac{1}{r}$ function is the solution to Laplace equation in the generalized plane R^2, except (x_0,y_0) point where the function turns to $+\infty$.

Example 2. A solution to Dirichlet problem for the generalized Laplace equation.

Solution. Let consider the interior boundary value problem for Laplace equation with Dirichlet boundary condition

$$\Delta u = \frac{1}{-4D}\left(\theta_0 \frac{\partial^2 u}{\partial x^2} - \theta_1 \frac{\partial^2 u}{\partial x \partial y} + \frac{\partial^2 u}{\partial y^2}\right) = 0$$

in domain $\Omega = \{(x,y)|x^2 + \theta_1 xy + \theta_0 y^2 < 1\}$ with borders $\Gamma: x^2 + \theta_1 xy + \theta_0 y^2 = 1$; where θ_0, θ_1 are real control parameters. Here $D = \frac{\theta_1^2}{4} - \theta_0 < 0, p^2 = -\theta_0 + p\theta_1$.

Dirichlet problem. Find a function $u(x,y)$ in Ω domain satisfying the following conditions:

$$u(x,y) \in C(\overline{\Omega}) \cap C^2(\Omega), \qquad (12)$$

$$\Delta u = \frac{1}{-4D}\left(\theta_0 \frac{\partial^2 u}{\partial x^2} - \theta_1 \frac{\partial^2 u}{\partial x \partial y} + \frac{\partial^2 u}{\partial y^2}\right) = 0,\ (x,y) \in \Omega \qquad (13)$$

$$u(x,y)\,|_\Gamma = f(\varphi), 0 \le \varphi \le 2\pi \qquad (14)$$

where $f(\varphi)$ is a prescribed function; Assume that $f(\varphi) \in C^1(\Gamma)$,

$$f(0) = f(2\pi).$$

In Ω domain we turn to the generalized polar coordinates

$$x = rT(\theta_0, \theta_1, \varphi),\ y = rS(\theta_0, \theta_1, \varphi).$$

Then the equation (13) in the generalized polar coordinates is as follows (see eq. (11))

$$\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} - \frac{1}{Dr^2}\frac{\partial^2 u}{\partial \varphi^2} = 0. \qquad (15)$$

The solution $u(x,y) = u\big(rT(\varphi), rS(\varphi)\big) = \tilde{u}(r,\varphi) = u(r,\varphi)$ of the equation (15) we will search in the form of two functions multiplication

$$u(r,\varphi) = R(r) \cdot \Phi\big(\sqrt{-D}\varphi\big) \ne 0 \text{ в } \Omega \qquad (16)$$

Substituting the expected form of the solution (16) in the equation (15), and separating the variables, we obtain

$$r^2 \frac{R''(r)}{R(r)} + r\frac{R'(r)}{R(r)} = -\frac{\Phi''\big(\sqrt{-D}\varphi\big)}{\Phi\big(\sqrt{-D}\varphi\big)} = \lambda.$$

It follows that the function $R(r)$ must be found by solving the equation below

$$r^2 R''(r) + rR'(r) - \lambda R(r) = 0, \qquad (17)$$

and we obtain an eigenproblem for $\Phi\big(\sqrt{-D}\varphi\big)$ function

$$\begin{aligned} &\Phi''\big(\sqrt{-D}\varphi\big) + \lambda\Phi\big(\sqrt{-D}\varphi\big) = 0, \\ &\Phi\big(\sqrt{-D}\varphi\big) = \Phi\big(\sqrt{-D}\varphi + 2\pi\big). \end{aligned} \qquad (18)$$

Here the periodicity condition of the function $\Phi\big(\sqrt{-D}\varphi\big)$ is a consequence of the frequency of the desired solution $u(r,\varphi)$ in the angular variable with period of 2π. This is only possible if $\lambda = -Dn^2$ and $n\sqrt{-D}$ is an integer. Then a general solution of the differential equation (18) is determined as

$$\Phi\big(\sqrt{-D}\varphi\big) = a_n \cos\big(n\sqrt{-D}\varphi\big) + b_n \sin\big(n\sqrt{-D}\varphi\big),$$

where a_n and b_n are arbitrary constants.

Equation (17) has two linearly independent solutions at $\lambda = -Dn^2$ (where $D < 0$)

$$R_1(r) = r^{\sqrt{-D}n}, R_2(r) = r^{-\sqrt{-D}n}$$

Since we seek particular solutions of this equation (17) at $\lambda = -Dn^2$ in the form of a power function $R(r) = r^k, k = \text{const}$. Substituting this function in equation (17) we can establish that the exponent k is determined from the equation

$$k^2 = -Dn^2, \text{ i.e } k = \pm\sqrt{-D}n.$$

Solution of the inner Dirichlet problem should be limited in the considered domain at $r = 0$. Therefore, from found two solutions should be taken only one

$$R_n(r) = r^{\sqrt{-D}n}.$$

Thus, according to (16) the partial solutions of the equation (15) can be written as

$$u(r,\varphi) = r^{\sqrt{-D}n}\big[a_n \cos\big(n\sqrt{-D}\varphi\big) + b_n \sin\big(n\sqrt{-D}\varphi\big)\big].$$

Because of linearity and uniformity of the equation (15) a composition of particular solutions

$$u(r,\varphi) = \frac{a_0}{2} + \sum_{n=1}^{\infty} r^{\sqrt{-D}n}\big[a_n \cos\big(n\sqrt{-D}\varphi\big) + b_n \sin\big(n\sqrt{-D}\varphi\big)\big], \qquad (19)$$

will also satisfy this equation.

Hence, the series (19) inside Ω domain is a harmonic function. It is known from a general course that the series (19) converges uniformly on $\bar{\Omega}$. Then, satisfying the series (19) with the boundary condition (14) we have

$$u(r,\varphi)|_{r=1} = f(\varphi) \text{ or } f(\varphi) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left[a_n \cos\left(n\sqrt{-D}\varphi\right) + b_n \sin\left(n\sqrt{-D}\varphi\right)\right]. \quad (20)$$

The series (20) represents a transformation into Fourier series of $f(\varphi)$ function in $[0,2\pi]$ interval. Then a_n and b_n coefficients are determined by the following formulas

$$a_n = \frac{1}{\pi}\int_0^{2\pi} f(\varphi)\cos\left(n\sqrt{-D}\varphi\right)d\varphi,\ n = 0,1,2,\dots, \quad (21)$$

$$b_n = \frac{1}{\pi}\int_0^{2\pi} f(\varphi)\sin\left(n\sqrt{-D}\varphi\right)d\varphi,\ n = 1,2,\dots. \quad (22)$$

Theorem. If a function $f(\varphi) \in C^1[0,2\pi]$, and $f(0) = f(2\pi)$, then there exists an ambiguous solution to the Dirichlet problem in Ω domain, which is defined by the series (19).

Poisson formula. Let transform the series (19) with regard to expressions (21) and (22):

$$u(r,\varphi) = \frac{1}{2\pi}\int_0^{2\pi} f(t)dt +$$

$$+\frac{1}{\pi}\sum_{n=1}^{\infty} r^{\sqrt{-D}n}\left[\int_0^{2\pi} f(t)\cos\left(n\sqrt{-D}t\right)dt\cos\left(n\sqrt{-D}\varphi\right) + + \int_0^{2\pi} f(t)\sin\left(n\sqrt{-D}t\right)dt\sin\left(n\sqrt{-D}\varphi\right)\right] = \frac{1}{2\pi}\int_0^{2\pi} f(t)dt +$$

$$+\frac{1}{\pi}\int_0^{2\pi} f(t)\sum_{n=1}^{\infty} r^{\sqrt{-D}n}\left[\cos\left(n\sqrt{-D}t\right)\cos\left(n\sqrt{-D}\varphi\right) + \sin\left(n\sqrt{-D}t\right)\sin\left(n\sqrt{-D}\varphi\right)\right]dt =$$

$$= \frac{1}{2\pi}\int_0^{2\pi}\left[1 + 2\sum_{n=1}^{\infty} r^{\sqrt{-D}n}\cos n\sqrt{-D}(t-\varphi)\right]dt \quad (23)$$

Providing generalized Euler formula (9), we have

$$z^{\sqrt{-D}n} = r^{\sqrt{-D}n}e^{i\sqrt{-D}n\omega} = r^{\sqrt{-D}n}\left[\cos\left(n\sqrt{-D}\omega\right) + i\sin\left(n\sqrt{-D}\omega\right)\right], \omega = t - \varphi.$$

Find series' sum

$$1 + 2\sum_{n=1}^{\infty} r^{\sqrt{-D}n}\cos\left(n\sqrt{-D}(t-\varphi)\right) = 1 + 2\sum_{n=1}^{\infty} r^{\sqrt{-D}n}\cos n\omega =$$

$$= -1 + 2\mathrm{Re}\sum_{n=0}^{\infty} z^{\sqrt{-D}n} = -1 + 2\mathrm{Re}\frac{1}{1 - z^{\sqrt{-D}}} =$$

$$= -1 + \frac{2 - 2r^{\sqrt{-D}}\cos\sqrt{-D}\omega}{1 - 2r^{\sqrt{-D}}\cos\sqrt{-D}\omega + r^{2\sqrt{-D}}} = \frac{1 - r^{2\sqrt{-D}}}{1 - 2r^{\sqrt{-D}}\cos\sqrt{-D}\omega + r^{2\sqrt{-D}}} \quad (24)$$

Then substituting (24) in (23) we find a formula

$$u(r,\varphi) = \frac{1}{2\pi}\int_0^{2\pi} f(t)\frac{1 - r^{2\sqrt{-D}}}{1 - 2r^{\sqrt{-D}}\cos\sqrt{-D}(t-\varphi) + r^{2\sqrt{-D}}}dt, \quad (25)$$

which is called Poisson formula.

Example 3. Evaluate an integral $\int P_n(x)e^{ax}\sin bx\,dx$.

Solution. In case that $D = \frac{\theta_1^2}{4} - \theta_0 < 0$ formula (9) can be rewritten in the next form

$$e^{px} = J(\theta_0,\theta_1,x) + pK(\theta_0,\theta_1,x) = e^{\frac{\theta_1}{2}x}\left[\left(\cos\sqrt{-D}x - \frac{\theta_1}{2\sqrt{-D}}\sin\sqrt{-D}x\right) + p\frac{1}{\sqrt{-D}}\sin\sqrt{-D}x\right]. \quad (26)$$

Then $\int J(x)\,dx + p\int K(x)\,dx = \frac{e^{px}}{p} + C_1 + pC_2$. From here

$$\int J(x)\,dx = \frac{\theta_1}{\theta_0}J(x) + K(x) + C_1,\ \int K(x)\,dx = -\frac{1}{\theta_0}J(x) + C_2;$$

since $\bar{p} = \theta_1 - p$ and $p \cdot \bar{p} = \theta_0$. We can rewrite last integral as

$$\int e^{\frac{\theta_1}{2}x}\sin\sqrt{-D}x\,dx = -\frac{\sqrt{-D}}{\theta_0}e^{\frac{\theta_1}{2}x}\left(\cos\sqrt{-D}x - \frac{\theta_1}{2\sqrt{-D}}\sin\sqrt{-D}x\right) + C.$$

Here $\frac{\theta_1}{2} = a, \sqrt{-D} = b$ и $\theta_0 = a^2 + b^2$. Hence

$$\int e^{ax}\sin bx\,dx = \frac{e^{ax}(a\sin bx - b\cos bx)}{a^2 + b^2} + C.$$

We have following from the exponent decomposition (26):

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$$J'(x) = -\theta_0 K(x),\ K'(x) = J(x) + \theta_1 K(x).$$

We apply these relations:

$$\theta_0 \int K(x)\,dx = -J(x) + C,$$

$$\theta_0 \int xK(x)\,dx = K(x) + \left(-x + \frac{\theta_1}{\theta_0}\right) J(x) + C.$$

Next we replace parameters θ_0, θ_1 by parameters a and b, and we obtain:

$$\int xe^{ax} \sin bx\,dx = \frac{e^{ax}}{a^2+b^2}\left[\sin bx - \left(x - \frac{2a}{a^2+b^2}\right)(b \cos bx - a \sin bx)\right] + C.$$

Now it is very easy to evaluate the given integral by using obtained results. Integration by parts leads to the depression of n degree under the integral. Indeed, we have

$$\int x^n e^{ax} \sin bx\,dx = \int x^n\,d\left[\frac{e^{ax}(a \sin bx - b \cos bx)}{a^2+b^2}\right] = x^n \frac{e^{ax}(a \sin bx - b \cos bx)}{a^2+b^2} - n \int x^{n-1}\left[\frac{e^{ax}(a \sin bx - b \cos bx)}{a^2+b^2}\right] dx.$$

REFERENCES

[1] Lavrentiev M. A., Shabat B. V. 1973. Problems of Hydrodynamics and Mathematical Models, Nauka Publishing House, Moscow (in Russian)

[2] Sagindykov B. Zh., Bimurat Zh. 2013. The generalized complex exponent and its application for finding sums, Natural and mathematical sciences: issues and trends, Novosibirsk (in Russian)