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Sensors and Boundary Reconstruction of Hyperbolic Systems

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ABSTRACT

The aim of this paper is to study regional gradient observability for hyperbolic system in the case where the subregion of interest is a part of the boundary, and the reconstruction of the state gradient without the knowledge of the state. First, we give definitions and characterizations of this new concept and establish necessary conditions for the sensor structure in order to obtain regional boundary gradient observability. The developed approach, based on the Hilbert uniqueness method [7], leads to a reconstruction algorithm. The obtained results are illustrated with numerical examples and simulations.

Indexing terms/Keywords

Distributed hyperbolic systems, regional gradient observability, gradient strategic sensor, gradient reconstruction.

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1 INTRODUCTION

For a distributed parameter system evolving on domain $\Omega \subset IR^n$, the concept of regional observability was introduced by El Jai and Zerrik [3, 5], and refers to problems in which the observed state of interest is not fully specified as a state, but concerns the observability and the reconstruction of the state only on a subregion ω, a portion of the spatial domain Ω. It was developed for parabolic and hyperbolic systems [10, 11, 13], where authors developed approaches for the reconstruction of the state on a subregion ω interior or in the boundary of Ω.

Since in many real problems the need may be only to reconstruct the state gradient, and since there exists a state that cannot be reconstructed but possible for its gradient, Zerrik and Bourray [12] introduced the notion of regional gradient observability and concerns the extension of the state observability to the observability of the gradient in a subregion ω. Results have been developed for parabolic and hyperbolic systems that characterize this notion and give approaches for the reconstruction of the gradient in a subregion interior of the system domain [12, 13, 1].

It is also plausible in real problems that the subregion of interest may be a portion of the boundary $\Gamma \subset \partial\Omega$.

Here, we are interested in the regional gradient observability of distributed hyperbolic systems on Γ. This is the purpose of this paper, which is organized as follows:

Section 2 is devoted to the definitions and characterizations of regional gradient observability. In the third section, we establish the relation between regional boundary gradient observability and sensors structure. Section 4 is devoted to applications for two-dimensional hyperbolic system. In section 5 a reconstruction method is developed using extension of Hilbert uniqueness method. Finally a numerical approach is established and illustrations through numerical simulations are given.

2 PREGIONAL BOUNDARY GRADIENT OBSERVABILITY

2.1 Preliminaries

Let Ω be an open bounded subset of IR^n with a regular boundary $\partial\Omega$ and $T > 0$. Denoted by $Q = \Omega \times \left]0,T\right[$, $\Sigma = \partial\Omega \times \left]0,T\right[$ and consider a system described by the hyperbolic equation

$$\begin{cases} \dfrac{\partial^2 y(x,t)}{\partial t^2} = A\,y(x,t) & on \quad Q \\ y(x,0) = y^0, \dfrac{\partial y(x,0)}{\partial t} = y^1 & on \quad \Omega. \\ y(\zeta,t) = 0 & on \quad \Sigma \end{cases} \tag{2-1}$$

where A is a second order elliptic linear operator with regular coefficients.

Equation (2-1) has a unique solution $y \in \left(0,T;H_0^1(\Omega)\right) \cap C^1\left(0,T;\, L^2(\Omega)\right)$ (see [6]).

Suppose that measurements on system (2-1) are given by means of the output function:

$$z(t) = C\,y(t) \tag{2-2}$$

where $C: H^2(\Omega) \cap H_0^1(\Omega) \to IR^q$ is a linear operator depending on the structure of q sensors.

Let us recall that a sensor is defined by a couple (D, f) where $D \subset \Omega$ is the location of the sensor and $f \in L^2(D)$ is the spatial distribution of the measurement on D. In the case of a pointwise sensor, $D = b \in \Omega$ and $f = \delta_b$ is the Dirac mass concentrated in b, (see [2]) for more details.

Let $\overline{y} = (y, \partial y/\partial t)^T$ and $\overline{C}\,\overline{y} = (C y, 0)$ then the system (2-1) may be written in the form

$$\begin{cases} \dfrac{\partial \overline{y}}{\partial t}(t) = \overline{A}\,\overline{y}(t) \quad 0 < t < T \\ \overline{y}^0 = \left(y^0, y^1\right) \end{cases} \tag{2-3}$$

with $\overline{A} = \begin{pmatrix} 0 & I \\ A & 0 \end{pmatrix}$.

The operator $\overline{A}$ has a compact resolvent and generates a strongly continuous semi-group $(S(t))_{t\geq 0}$ on a subspace of the Hilbert state space $L^2(\Omega)\times L^2(\Omega)$ given by

$$S(t)\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} \displaystyle\sum_{m\geq 1}\sum_{j=1}^{r_m}\left(\langle y_1, w_{mj}\rangle_{L^2(\Omega)}\cos\sqrt{-\lambda_m}\,t + \frac{1}{\sqrt{-\lambda_m}}\langle y_2, w_{mj}\rangle_{L^2(\Omega)}\sin\sqrt{-\lambda_m}\,t\right)w_{mj} \\ \displaystyle\sum_{m\geq 1}\sum_{j=1}^{r_m}\left(-\sqrt{-\lambda_m}\langle y_1, w_{mj}\rangle_{L^2(\Omega)}\sin\sqrt{-\lambda_m}\,t + \langle y_2, w_{mj}\rangle_{L^2(\Omega)}\cos\sqrt{-\lambda_m}\,t\right)w_{mj} \end{pmatrix}$$

(w_{mj}) is a basis in $Z = H^2(\Omega)\cap H_0^1(\Omega)$ of eigenfunctions of A, orthonormal in $L^2(\Omega)$ and $\lambda_m < 0$ the associated eigenvalues with multiplicity r_m. Then (2-3) admits a unique solution $\overline{y} = S(t)\overline{y}^0$ (see [6]).

Let us define the observability operator

$$\begin{aligned} K : Z\times Z &\to L^2(0,T;IR^q) \\ h &\to \overline{C}\,S(.)h \end{aligned}$$

which is linear and bounded, its adjoint is denoted by K^* and let the operator

$$\begin{aligned} \overline{\nabla} : Z\times Z &\to (H^1(\Omega))^n \times (H^1(\Omega))^n \\ (y_1, y_2) &\to (\nabla y_1, \nabla y_2) \end{aligned}$$

where

$$\begin{aligned} \nabla : Z &\to (H^1(\Omega))^n \\ y &\to \nabla y = \left(\frac{\partial y}{\partial x_1}, \frac{\partial y}{\partial x_2}, \ldots, \frac{\partial y}{\partial x_n}\right) \end{aligned}$$

while their adjoints are denoted by $\overline{\nabla}^*$ and ∇^* respectively. Consider a regular boundary subregion Γ of $\partial\Omega$ of positive measure and let ω be a open subregion of Ω with regular boundary $\partial\omega$ such that $\Gamma\subset\partial\Omega\cap\partial\omega$.

We consider the restriction operators

$$\begin{aligned} \overline{\chi}_\Gamma : \left(H^{\frac{1}{2}}(\partial\Omega)\right)^n \times \left(H^{\frac{1}{2}}(\partial\Omega)\right)^n &\to \left(H^{\frac{1}{2}}(\Gamma)\right)^n \times \left(H^{\frac{1}{2}}(\Gamma)\right)^n \\ (y_1, y_2) &\to (\chi_\Gamma y_1, \chi_\Gamma y_2) \end{aligned}$$

with

$$\begin{aligned} \chi_\Gamma : \left(H^{\frac{1}{2}}(\partial\Omega)\right)^n &\to \left(H^{\frac{1}{2}}(\Gamma)\right)^n \\ y &\to y|_\Gamma \end{aligned} \qquad \text{and} \qquad \begin{aligned} \tilde{\chi}_\Gamma : H^{\frac{1}{2}}(\partial\Omega) &\to H^{\frac{1}{2}}(\Gamma) \\ y &\to y|_\Gamma \end{aligned}$$

while their adjoints are denoted by $\overline{\chi}_\Gamma^*$, χ_Γ^* and $\tilde{\chi}_\Gamma^*$ respectively.

We also consider

$$\begin{aligned} \overline{\chi}_\omega : (H^1(\Omega))^n \times (H^1(\Omega))^n &\to (H^1(\omega))^n \times (H^1(\omega))^n \\ (y_1, y_2) &\to (\chi_\omega y_1, \chi_\omega y_2) \end{aligned}$$

where

$$\chi_\omega : \left(H^1(\Omega)\right)^n \to \left(H^1(\omega)\right)^n \qquad \text{and} \qquad \tilde{\chi}_\omega : H^1(\Omega) \to H^1(\omega)$$
$$y \to y|_\omega \qquad\qquad\qquad y \to y|_\omega$$

their adjoints are denoted by $\bar{\chi}_\omega^*$, χ_ω^* et $\tilde{\chi}_\omega^*$ respectively.

And the trace operator

$$\bar{\gamma} : \left(H^1(\Omega)\right)^n \times \left(H^1(\Omega)\right)^n \to \left(H^{\frac{1}{2}}(\partial\Omega)\right)^n \times \left(H^{\frac{1}{2}}(\partial\Omega)\right)^n$$
$$(y_1, y_2) \to (\gamma y_1, \gamma y_2)$$

where

$$\gamma : \left(H^1(\Omega)\right)^n \to \left(H^{\frac{1}{2}}(\partial\Omega)\right)^n$$
$$y \to \gamma y = (\gamma_0 y_1, \gamma_0 y_2, \ldots, \gamma_0 y_n)$$

with $\gamma_0 : H^1(\Omega) \to H^{\frac{1}{2}}(\partial\Omega)$ is the trace operator of order zero which is linear, surjective and continuous, γ_0^* denotes its adjoint, γ^* and $\bar{\gamma}^*$ denotes the adjoints of operators γ and $\bar{\gamma}$.

We finally introduce the operator $\mathrm{H} = \bar{\chi}_\Gamma \bar{\gamma}\, \overline{\nabla} K^*$ from $L^2\left(0, T; IR^q\right)$ into $\left(H^{\frac{1}{2}}(\Gamma)\right)^n \times \left(H^{\frac{1}{2}}(\Gamma)\right)^n$.

2.2 Definitions and counterexample

In this paragraph we will give definitions and characterization of regional gradient observability of system (2-1). Let recall that system (2-1) together with the output (2-2) is exactly (resp. approximately) gradient observable on ω if

$$\mathrm{Im}\left(\bar{\chi}_\omega \overline{\nabla} K^*\right) = \left(H^1(\omega)\right)^n \times \left(H^1(\omega)\right)^n \text{ (resp. } \overline{\mathrm{Im}\left(\bar{\chi}_\omega \overline{\nabla} K^*\right)} = \left(H^1(\omega)\right)^n \times \left(H^1(\omega)\right)^n \text{)}$$

Definition .1

The system (2-1) together with the output equation (2-2) is said to be exactly regionally gradient observable (resp. approximately G-observable) on Γ if

$$\mathrm{Im}(\mathrm{H}) = \left(H^{\frac{1}{2}}(\Gamma)\right)^n \times \left(H^{\frac{1}{2}}(\Gamma)\right)^n \text{ (resp. } \overline{\mathrm{Im}(\mathrm{H})} = \left(H^{\frac{1}{2}}(\Gamma)\right)^n \times \left(H^{\frac{1}{2}}(\Gamma)\right)^n \text{)}$$

Remark .2

1. *This is a natural definition of regional gradient observability extending those given in [1] to the case where we restrict the observability of the gradient to a boundary subregion Γ.*
2. $\left(\overline{\mathrm{Im}(\mathrm{H})} = \left(H^{\frac{1}{2}}(\Gamma)\right)^n \times \left(H^{\frac{1}{2}}(\Gamma)\right)^n\right) \Leftrightarrow \left(\ker\left(\mathrm{H}^*\right) = \{0\}\right).$
3. $\overline{\mathrm{Im}\left(\overline{\nabla} K^*\right)} = \left(H^1(\Omega)\right)^n \times \left(H^1(\Omega)\right)^n \Leftrightarrow \ker\left(K \overline{\nabla}^*\right) = \{0\}.$
4. *There exist systems which are not G-observable on the whole domain Ω but G-observable on Γ. This is shown through the following counter example.*

Example .3

Let $\Omega =]0,1[\times]0,1[$, we consider the two dimensional hyperbolic system described by the equation

$$\begin{cases} \dfrac{\partial^2 y(x_1,x_2,t)}{\partial t^2} = \dfrac{\partial^2 y(x_1,x_2,t)}{\partial x_1^2} + \dfrac{\partial^2 y(x_1,x_2,t)}{\partial x_2^2} & on\ Q \\ y(x_1,x_2,0) = y^0(x_1,x_2) \quad , \quad \dfrac{\partial y(x_1,x_2,0)}{\partial t} = y^1(x_1,x_2) & on\ \Omega \\ y(\zeta_1,\varsigma_2,t) = 0 & on\ \Sigma \end{cases} \tag{2-4}$$

Measurements are given by the output function

$$z(t) = \int_D y(x_1,x_2,t)\, f(x_1,x_2)\, dx_1\, dx_2 \tag{2-5}$$

where $D =]0,1[\times\left\{\frac{1}{2}\right\}$ is the sensor support and $f(x_1,x_2) = \sin(3\pi x_1)$ is the function measure.

Let $\Gamma = \{0\}\times[0,1]$ be the target subregion, and $g = (g^0, g^1)$ the initial gradient to be observed on Γ with $g^0(x_1,x_2) = (\pi\cos(\pi x_1)\sin(\pi x_2),\ \pi\sin(\pi x_1)\cos(\pi x_2))$ and $g^1(x_1,x_2) = (-\pi\sin(\pi x_1)\cos(\pi x_2),\ -\pi\cos(\pi x_1)\sin(\pi x_2))$. Then we have the result.

Proposition .4

The gradient g is not approximately G-observable on the whole domain Ω, however it may be approximately G-observable on Γ.

Proof

To prove that g is not approximately G-observable on Ω, we show that $g \in Ker\left(K\overline{\nabla}^*\right)$.

We have

$$K\overline{\nabla}^*(g^0,g^1) = \sum_{ij=1}^{\infty}\left[\langle\nabla^* g^0, w_{ij}\rangle_{L^2(\Omega)}\cos\sqrt{-\lambda_{ij}}\,t + \frac{1}{\sqrt{-\lambda_{ij}}}\langle\nabla^* g^1, w_{ij}\rangle_{L^2(\Omega)}\sin\sqrt{-\lambda_{ij}}\,t\right]\langle w_{ij}, f\rangle_{L^2(D)}$$

where $\lambda_{ij} = -(i^2+j^2)\pi^2$ associated to the eigenfunctions $w_{ij}(x_1,x_2) = 2\sin(i\pi x_1)\sin(j\pi x_2)$.

$\forall i, j \in IN^*$,

$$\langle w_{ij}, f\rangle_{L^2(D)} = \langle 2\sin(i\pi x_1)\sin(j\pi x_2), \sin(3\pi x_1)\rangle_{L^2(D)}$$

$$= 2\sin\left(j\frac{\pi}{2}\right)\int_0^1 \sin(i\pi x_1)\sin(3\pi x_1)\,dx_1$$

$$= \sin\left(j\frac{\pi}{2}\right)\int_0^1 \left[\cos(i-3)\pi x_1 - \cos(i+3)\pi x_1\right]\,dx_1$$

This gives

$$\langle w_{ij}, f\rangle_{L^2(D)} = \begin{cases} 0 & if \quad (i \neq 3 \text{ or } j \in 2IN) \\ \sin\left(j\dfrac{\pi}{2}\right) & if \quad (i = 3 \text{ or } j \in 2IN+1) \end{cases}$$

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Therefore consider only the case where $i=3$ and $j \in 2IN+1$

$$\left\langle \nabla^* g^0, w_{3j} \right\rangle_{L^2(\Omega)} = \left\langle g^0, \nabla w_{3j} \right\rangle_{(L^2(\Omega))^2}$$

$$= \int_0^1 6\pi^2 \cos(\pi x_1)\cos(3\pi x_1)\, dx_1 \int_0^1 \sin(\pi x_2)\sin(j\pi x_2)\, dx_2 +$$

$$2j\pi^2 \int_0^1 \sin(\pi x_1)\sin(3\pi x_1)\, dx_1 \int_0^1 \cos(\pi x_2)\cos(j\pi x_2)\, dx_2$$

since $\int_0^1 \cos(\pi x_1)\cos(3\pi x_1)\, dx_1 = 0$ and $\int_0^1 \sin(\pi x_1)\sin(3\pi x_1)\, dx_1 = 0$, we have

$$\left\langle \nabla^* g^0, w_{3j} \right\rangle_{L^2(\Omega)} = 0 \qquad , \forall j \in 2IN+1$$

$$\left\langle \nabla^* g^1, w_{3j} \right\rangle_{L^2(\Omega)} = \left\langle g^1, \nabla w_{3j} \right\rangle_{(L^2(\Omega))^2}$$

$$= \int_0^1 -6\pi^2 \sin(\pi x_1)\cos(3\pi x_1)\, dx_1 \int_0^1 \cos(\pi x_2)\sin(j\pi x_2)\, dx_2 -$$

$$2j\pi^2 \int_0^1 \cos(\pi x_1)\sin(3\pi x_1)\, dx_1 \int_0^1 \sin(\pi x_2)\cos(j\pi x_2)\, dx_2$$

We also have $\int_0^1 \sin(\pi x_1)\cos(3\pi x_1)\, dx_1 = 0$ and $\int_0^1 \cos(\pi x_1)\sin(3\pi x_1)\, dx_1 = 0$

Then $\left\langle \nabla^* g^1, w_{3j} \right\rangle_{L^2(\Omega)} = 0 \qquad , \forall j \in 2IN+1$

This gives $K\overline{\nabla}^*(g^0, g^1) = 0$, and then the system is not approximately G-observable on Ω.

On the other hand g may be approximately G-observable on Γ.

Indeed, suppose that $K\overline{\nabla}^*\overline{\gamma}^*\overline{\chi}_\Gamma^*\overline{\chi}_\Gamma\overline{\gamma}(g^0, g^1) = 0$, then

$$\sum_{ij=1}^{\infty}\left[\left\langle \chi_\Gamma\gamma g^0, \chi_\Gamma\gamma\nabla w_{ij}\right\rangle_{(L^2(\Gamma))^2} \cos\sqrt{-\lambda_{ij}}\,t + \frac{1}{\sqrt{-\lambda_{ij}}}\left\langle \chi_\Gamma\gamma g^1, \chi_\Gamma\gamma\nabla w_{ij}\right\rangle_{(L^2(\Gamma))^2} \sin\sqrt{-\lambda_{ij}}\,t\right]\left\langle w_{ij}, f\right\rangle_{L^2(D)} = 0$$

Since for T large enough, the set $\left\{\sin\sqrt{-\lambda_{ij}}\,t, \cos\sqrt{-\lambda_{ij}}\right\}_{ij\geq 1}$ forms a complete orthonormal set of $L^2(0,T)$, we have

$$\left\langle \chi_\Gamma\gamma g^0, \chi_\Gamma\gamma\nabla w_{ij}\right\rangle_{(L^2(\Gamma))^2}\left\langle w_{ij}, f\right\rangle_{L^2(D)} = \left\langle \chi_\Gamma\gamma g^1, \chi_\Gamma\gamma\nabla w_{ij}\right\rangle_{(L^2(\Gamma))^2}\left\langle w_{ij}, f\right\rangle_{L^2(D)} = 0 \qquad \forall i,j \geq 1$$

But for $i=3$ and $j \in 2IN+1$ we have $\left\langle w_{3j}, f\right\rangle_{L^2(D)} = \sin\left(j\frac{\pi}{2}\right) \neq 0$

This gives

$$\left\langle \chi_\Gamma\gamma g^0, \chi_\Gamma\gamma\nabla w_{3j}\right\rangle_{(L^2(\Gamma))^2} = \left\langle \chi_\Gamma\gamma g^1, \chi_\Gamma\gamma\nabla w_{3j}\right\rangle_{(L^2(\Gamma))^2} = 0 \quad \forall j \in 2IN+1$$

But for $j=1$

$$\left\langle \chi_\Gamma \gamma g^0, \chi_\Gamma \gamma \nabla w_{31} \right\rangle_{(L^2(\Gamma))^2} = \int_0^1 6\pi^2 \sin^2(\pi x_2)\, dx_2$$

$$= 3\pi^2 \neq 0$$

thus $K \overline{\nabla}^* \overline{\gamma}^* \overline{\chi}_\Gamma^* \overline{\chi}_\Gamma \overline{\gamma} \left(g^0, g^1\right) \neq 0$.

The following results characterize the G-observability on Γ.

Proposition .5

The system (2-1) together with the output equation (2-2) is

1. *exactly G-observable on* Γ *if and only if there exists* $c > 0$*, such that for all* $z^* \in \left(H^{\frac{1}{2}}(\Gamma)\right)^n \times \left(H^{\frac{1}{2}}(\Gamma)\right)^n$,

$$\left\| z^* \right\|_{\left(H^{\frac{1}{2}}(\Gamma)\right)^n \times \left(H^{\frac{1}{2}}(\Gamma)\right)^n} \leq c \left\| K \overline{\nabla}^* \overline{\gamma}^* \overline{\chi}_\Gamma^* z^* \right\|_{L^2\left(0,T;IR^q\right)}$$

2. *approximately G-observable on* Γ *if and only if the operator* $N_\Gamma = \mathrm{H}\mathrm{H}^*$ *is positive defined. i.e.*

$$\forall z^* \in \left(H^{\frac{1}{2}}(\Gamma)\right)^n \times \left(H^{\frac{1}{2}}(\Gamma)\right)^n ; \left\langle N_\Gamma z^*, z^* \right\rangle_{\left(H^{\frac{1}{2}}(\Gamma)\right)^n \times \left(H^{\frac{1}{2}}(\Gamma)\right)^n} = 0 \Rightarrow \; z^* = 0 .$$

Proof

1. Let us consider the operator $h = Id_{\left(H^{\frac{1}{2}}(\Gamma)\right)^n \times \left(H^{\frac{1}{2}}(\Gamma)\right)^n}$. Since the system is exactly G-observable on Γ, we have $\operatorname{Im} h \subset \operatorname{Im} H$, and by the general result given in [9], this is equivalent to there exist $c > 0$ such that

$$\forall z^* \in \left(H^{\frac{1}{2}}(\Gamma)\right)^n \times \left(H^{\frac{1}{2}}(\Gamma)\right)^n ; \left\| h^* z^* \right\|_{\left(H^{\frac{1}{2}}(\Gamma)\right)^n \times \left(H^{\frac{1}{2}}(\Gamma)\right)^n} \leq c \left\| H^* z^* \right\|_{L^2\left(0,T;IR^q\right)}$$

2. Let $z^* \in \left(H^{\frac{1}{2}}(\Gamma)\right)^n \times \left(H^{\frac{1}{2}}(\Gamma)\right)^n$ such that $\left\langle N_\Gamma z^*, z^* \right\rangle_{\left(H^{\frac{1}{2}}(\Gamma)\right)^n \times \left(H^{\frac{1}{2}}(\Gamma)\right)^n} = 0$.

   So $\left\langle \mathrm{H}^* z^*, \mathrm{H}^* z^* \right\rangle_{L^2\left(0,T;IR^q\right)} = 0$ which means that $z^* \in Ker\, \mathrm{H}^*$ and since (2-1) is approximately G-observable then $z^* = 0$, that is N_Γ is positive defined.

Conversely, let $z^* \in \left(H^{\frac{1}{2}}(\Gamma)\right)^n \times \left(H^{\frac{1}{2}}(\Gamma)\right)^n$ such that $\mathrm{H}^* z^* = 0$, then $\left\langle N_\Gamma z^*, z^* \right\rangle_{\left(H^{\frac{1}{2}}(\Gamma)\right)^n \times \left(H^{\frac{1}{2}}(\Gamma)\right)^n} = 0$, that is the system is approximately G-observable on Γ.

3 BOUNDARY GRADIENT STRATEGIC SENSORS

The aim of this section is to give a characterization of sensors structure (number and location) in order that a system be approximately gradient observable on Γ.

Consider the system (2-1) observed by q sensors $(D_i, f_i)_{1 \leq i \leq q}$, which may be pointwise or zone, where $D_i \subset \Omega$ is the location of the sensor and $f_i \in L^2(D_i)$ is the spatial distribution of the measurements on D_i. The output function is then given by

$$z(t)=\left(z_1(t),\ldots,z_q(t)\right),\quad t\in\left]0,T\right[\tag{3-6}$$

such that

$$z_i(t)=\begin{cases}\sum_{k=1}^{n}\frac{\partial y(b_i,t)}{\partial x_k} & \textit{in the point wise case}\\ \sum_{k=1}^{n}\left\langle\frac{\partial y(.,t)}{\partial x_k},f_i\right\rangle_{L^2(D_i)} & \textit{in the zone case}\end{cases}$$

Definition .6

1. *A sensor (D,f) is said to be gradient strategic on Γ if the observed system is G-observable on Γ.*
2. *A sequence of sensors $(D_i,f_i)_{1\le i\le q}$ is said to be gradient strategic on Γ if there exists at least a sensor (D_{i_0},f_{i_0}) which is G-strategic on Γ.*

We assume that the operator A is of constant coefficients and has a complete set of eigenfunctions (w_{mj}) in $H^2(\Omega)\cap H_0^1(\Omega)$ orthonormal in $L^2(\Omega)$ associated to the eigenvalues λ_m of multiplicity r_m. Assume also that $r=\sup_{m\in I} r_m$ is finite, then we have the following result.

Proposition .7

If the sequence of sensors $(D_i,f_i)_{1\le i\le q}$ is G-strategic on Γ, then $q\ge r$ and $rang\ M_m=r_m$, $\forall m\ge 1$, where for $1\le i\le q$ and $1\le j\le r_m$

$$(M_m)_{i,j}=\begin{cases}\sum_{k=1}^{n}\frac{\partial w_{mj}}{\partial x_k}(b_i) & \textit{in the point wise case}\\ \sum_{k=1}^{n}\left\langle\frac{\partial w_{mj}}{\partial x_k},f_i\right\rangle_{L^2(D_i)} & \textit{in the zone case}\end{cases}$$

Proof

The proof is developed in the case zone sensors located inside Ω. We show that if the system (2-1),(3-6) is G-observable on Γ, then $rank\ M_m=r_m$, $\forall m\ge 1$.

Suppose that there exists $m_0\ge 1$ such that $rank\ M_{m_0}\ne r_{m_0}$, that is, there exists

$$z_{m_0}=\begin{pmatrix}z_{m_0 1}\\ \vdots\\ z_{m_0 r_{m_0}}\end{pmatrix}\ne 0 \quad\text{and}\quad M_{m_0}z_{m_0}=0.$$

Let $z_1^*=\begin{pmatrix}z_{11}^*\\ \vdots\\ z_{1n}^*\end{pmatrix}\in\left(H^{\frac{1}{2}}(\Gamma)\right)^n$ verifying

$$\begin{cases}\left\langle z_1^*,\chi_\Gamma\gamma\nabla w_{m_0 j}\right\rangle_{\left(H^{\frac{1}{2}}(\Gamma)\right)^n}=z_{m_0 j} & ,\forall\, j=1,\ldots,r_{m_0}\\ \left\langle z_1^*,\chi_\Gamma\gamma\nabla w_{mj}\right\rangle_{\left(H^{\frac{1}{2}}(\Gamma)\right)^n}=0 & ,\forall\, m\ne m_0\ ,\forall\, j=1,\ldots,r_m\end{cases}$$

and $z_2^* = \begin{pmatrix} z_{21}^* \\ \vdots \\ z_{2n}^* \end{pmatrix} \in \left(H^{\frac{1}{2}}(\Gamma) \right)^n$ verifying

$$\begin{cases} \left\langle z_2^*, \chi_\Gamma \gamma \nabla w_{m_0 j} \right\rangle_{\left(H^{\frac{1}{2}}(\Gamma)\right)^n} = z_{m_0 j} & , \forall\, j = 1, \ldots, r_{m_0} \\ \left\langle z_2^*, \chi_\Gamma \gamma \nabla w_{m j} \right\rangle_{\left(H^{\frac{1}{2}}(\Gamma)\right)^n} = 0 \quad , \forall\, m \neq m_0 & , \forall\, j = 1, \ldots, r_m \end{cases}$$

Then we have

$$\begin{cases} \sum_{j=1}^{r_{m_0}} \left\langle z_1^*, \chi_\Gamma \gamma \nabla w_{m_0 j} \right\rangle_{\left(H^{\frac{1}{2}}(\Gamma)\right)^n} \sum_{k=1}^{n} \left\langle \frac{\partial w_{m_0 j}}{\partial x_k}, f_i \right\rangle_{L^2(D_i)} = 0, \quad \forall i = 1, \ldots, q. \\ \sum_{j=1}^{r_m} \left\langle z_2^*, \chi_\Gamma \gamma \nabla w_{m j} \right\rangle_{\left(H^{\frac{1}{2}}(\Gamma)\right)^n} \sum_{k=1}^{n} \left\langle \frac{\partial w_{mj}}{\partial x_k}, f_i \right\rangle_{L^2(D_i)} = 0, \quad \forall m \neq m_0, \quad \forall i = 1, \ldots, q. \end{cases}$$

Thus there exists $z^* = \left(z_1^*, z_2^*\right) \in \left(H^{\frac{1}{2}}(\Gamma) \right)^n \times \left(H^{\frac{1}{2}}(\Gamma) \right)^n$ such that $z^* \neq 0$ and $\forall i = 1, \ldots, q$

$$K \overline{\nabla}^* \overline{\gamma}^* \overline{\chi}_\Gamma^* z^* = \sum_{m=1}^{\infty} \sum_{j=1}^{r_m} \left[\left\langle \gamma^* \chi_\Gamma^* z_1^*, \nabla w_{m j} \right\rangle \cos \sqrt{-\lambda_m}\, t + \frac{1}{\sqrt{-\lambda_m}} \left\langle \gamma^* \chi_\Gamma^* z_2^*, \nabla w_{m j} \right\rangle \sin \sqrt{-\lambda_m}\, t \right] \sum_{k=1}^{n} \left\langle \frac{\partial w_{mj}}{\partial x_k}, f_i \right\rangle = 0$$

i.e. $\ker\left(K \overline{\nabla}^* \overline{\gamma}^* \overline{\chi}_\Gamma^* \right) \neq \{0\}$ and the system (2-1)-(3-6) is not G-observable on Γ.

Remark .8

1. *The above proposition implies that the required number of sensors is greater than or equal to the largest multiplicity of eigenvalues.*
2. *By infinitesimally deforming of the domain, the multiplicity of the eigenvalues can be reduced to one [4,8]. Consequently, the regional G-observability on the subregion ω may be possible only by one sensor.*

4 APPLICATION TO SENSOR LOCATION

In this section, we give applications of the above results to a two-dimensional system defined on $\Omega =]0, a[\times]0, d[$ with $\frac{a^2}{d^2} \notin IQ$ by

$$\begin{cases} \frac{\partial^2 y}{\partial t^2}(x_1, x_2, t) = \frac{\partial^2 y}{\partial x_1^2}(x_1, x_2, t) + \frac{\partial^2 y}{\partial x_2^2}(x_1, x_2, t) & \text{on } Q \\ y(x_1, x_2, 0) = y^0(x_1, x_2) \quad , \quad \frac{\partial y}{\partial t}(x_1, x_2, 0) = y^1(x_1, x_2) & \text{on } \Omega \\ y(\zeta_1, \varsigma_2, t) = 0 & \text{on } \Sigma \end{cases} \qquad (4\text{-}7)$$

The eigenfunctions of the operator $A = \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2}$ are $w_{i j}(x_1, x_2) = \frac{2}{\sqrt{a d}} \sin\left(i\pi \frac{x_1}{a} \right) \sin\left(j\pi \frac{x_2}{d} \right)$ associated to the eigenvalues $\lambda_{i j} = -\left(\frac{i^2}{a^2} + \frac{j^2}{d^2} \right) \pi^2$ of multiplicity $r_{i j}$. Since $\frac{a^2}{d^2} \notin IQ$, $r_{i j} = 1$ (see [4, 8]) and system (4-7) may be G-observable only by one sensor.

4.1 Zone case

Here we consider the system (4-7) with sensor (D, f) where $D \subset \Omega$ (or $D \subset \partial\Omega$) is the support of the sensor and $f \in L^2(D)$ is the spatial distribution of the sensing measurements on D.

for $0 < \alpha_1 < \alpha_2 < a$ and $0 < \beta_1 < \beta_2 < d$, denote by $\eta_1 = \dfrac{\alpha_1 + \alpha_2}{2}$, $\eta_2 = \dfrac{\beta_1 + \beta_2}{2}$, $\mu_1 = \dfrac{\alpha_2 - \alpha_1}{2}$ and $\mu_2 = \dfrac{\beta_2 - \beta_1}{2}$.

4.1.1 Internal zone sensor

Here we consider the system (4-7) augmented by the output function

$$z(t) = \int_D \frac{\partial y}{\partial x_1}(x_1, x_2, t)\, f(x_1, x_2)\, dx_1\, dx_2 + \int_D \frac{\partial y}{\partial x_2}(x_1, x_2, t)\, f(x_1, x_2)\, dx_1\, dx_2$$

and assume that the sensor is located inside the domain Ω over $D =]\alpha_1, \alpha_2[\times]\beta_1, \beta_2[$.

Proposition .9

If f is uniformly distributed on $]\alpha_1, \alpha_2[\times]\beta_1, \beta_2[$, then the sensor (D, f) is not G-strategic on Γ if $\dfrac{\mu_1}{a} \in IQ$ or $\dfrac{\mu_2}{d} \in IQ$ or $\left(\dfrac{\eta_1}{a} \in IQ \text{ and } \dfrac{\eta_2}{d} \in IQ\right)$ or if there exists $k, l \in IN^*$ such that $2k\dfrac{\eta_1}{a}$ and $2l\dfrac{\eta_2}{d}$ are odds.

Proof

We have

$$\sum_{k=1}^{2} \left\langle \frac{\partial w_{ij}}{\partial x_k}, f \right\rangle = \frac{2\pi}{\sqrt{a\,d}} \left(\frac{i}{a} \int_{\alpha_1}^{\alpha_2} \int_{\beta_1}^{\beta_2} f(x_1, x_2) \cos i\pi \frac{x_1}{a} \sin j\pi \frac{x_2}{d}\, dx_1\, dx_2 \right.$$
$$\left. + \frac{j}{d} \int_{\alpha_1}^{\alpha_2} \int_{\beta_1}^{\beta_2} f(x_1, x_2) \sin i\pi \frac{x_1}{a} \cos j\pi \frac{x_2}{d}\, dx_1\, dx_2 \right)$$

If $f(x_1, x_2) = c$, we obtain

$$\sum_{k=1}^{2} \left\langle \frac{\partial w_{ij}}{\partial x_k}, f \right\rangle = \frac{2\pi c}{\sqrt{a\,d}} \left(\frac{i}{a} \int_{\alpha_1}^{\alpha_2} \int_{\beta_1}^{\beta_2} \cos i\pi \frac{x_1}{a} \sin j\pi \frac{x_2}{d}\, dx_1 dx_2 + \frac{j}{d} \int_{\alpha_1}^{\alpha_2} \int_{\beta_1}^{\beta_2} \sin i\pi \frac{x_1}{a} \cos j\pi \frac{x_2}{d}\, dx_1 dx_2 \right)$$

$$= \frac{2i\pi c}{a\sqrt{a\,d}} \left(\int_0^{\mu_1} \cos i\pi \frac{(\eta_1 - s)}{a} ds + \int_0^{\mu_1} \cos i\pi \frac{(\eta_1 + s)}{a} ds \right) \left(\int_0^{\mu_2} \sin j\pi \frac{(\eta_2 - s)}{d} ds + \int_0^{\mu_2} \sin j\pi \frac{(\eta_2 + s)}{d} ds \right)$$
$$+ \frac{2j\pi c}{d\sqrt{a\,d}} \left(\int_0^{\mu_1} \sin i\pi \frac{(\eta_1 - s)}{a} ds + \int_0^{\mu_1} \sin i\pi \frac{(\eta_1 + s)}{a} ds \right) \left(\int_0^{\mu_2} \cos j\pi \frac{(\eta_2 - s)}{d} ds + \int_0^{\mu_2} \cos j\pi \frac{(\eta_2 + s)}{d} ds \right)$$
$$= \frac{8c}{\pi\sqrt{a\,d}} \sin i\pi \frac{\mu_1}{a} \sin j\pi \frac{\mu_2}{d} \left(\frac{d}{j} \cos i\pi \frac{\eta_1}{a} \sin j\pi \frac{\eta_2}{d} + \frac{a}{i} \sin i\pi \frac{\eta_1}{a} \cos j\pi \frac{\eta_2}{d} \right)$$

from where one deduce the result.

4.1.2 Boundary zone sensor

Consider the system (4-7) with the output function

$$z(t) = \int_D \frac{\partial y}{\partial x_1}(\xi_1, \xi_2, t)\, f(\xi_1, \xi_2)\, d\xi_1\, d\xi_2 + \int_D \frac{\partial y}{\partial x_2}(\xi_1, \xi_2, t)\, f(\xi_1, \xi_2)\, d\xi_1\, d\xi_2$$

then we have

Proposition .10

1. *If f is uniformly distributed on $D =]\alpha_1, \alpha_2[\times \{0\}$ (or on $D =]\alpha_1, \alpha_2[\times \{d\}$), then the sensor (D, f) is not G-strategic on Γ if $\frac{\mu_1}{a} \in IQ$ or $\frac{\eta_1}{a} \in IQ$.*

2. *If f is symmetric with respect to the point $(\eta_1, 0)$ or with respect to the point (η_1, d), then the sensor (D, f) is not G-strategic on Γ if there exists $k \in IN^*$ such that $2k\frac{\eta_1}{a}$ is odd.*

3. *If f is symmetric with respect to the axis $x_1 = \eta_1$, then the sensor (D, f) is not G-strategic on Γ if $\frac{\eta_1}{a} \in IQ$.*

Proof

We have $\sum_{k=1}^{2} \left\langle \frac{\partial w_{ij}}{\partial x_k}, f \right\rangle = \frac{2j\pi}{d\sqrt{ad}} \int_{\alpha_1}^{\alpha_2} f(x_1, 0) \sin i\pi \frac{x_1}{a} dx_1$

1. if $f(x_1, 0) = c$, we obtain

$$\sum_{k=1}^{2} \left\langle \frac{\partial w_{ij}}{\partial x_k}, f \right\rangle = \frac{2j\pi c}{d\sqrt{ad}} \left(\int_0^{\mu_1} \sin i\pi \frac{(\eta_1 - s)}{a} ds + \int_0^{\mu_1} \sin i\pi \frac{(\eta_1 + s)}{a} ds \right)$$

$$= \frac{2ajc}{id\sqrt{ad}} \left(\cos i\pi \frac{(\eta_1 - \mu_1)}{a} - \cos i\pi \frac{(\eta_1 + \mu_1)}{a} \right)$$

$$= \frac{4ajc}{id\sqrt{ad}} \sin\left(i\pi \frac{\eta_1}{a} \right) \sin\left(i\pi \frac{\mu_1}{a} \right)$$

Then we deduce the result.

In the next, we set $g(x_1) = f(x_1, 0)$, then we have

$$\sum_{k=1}^{2} \left\langle \frac{\partial w_{ij}}{\partial x_k}, f \right\rangle = \frac{2j\pi}{d\sqrt{ad}} \int_{\alpha_1}^{\alpha_2} g(x_1) \sin i\pi \frac{x_1}{a} dx_1$$

$$= \frac{2j\pi}{d\sqrt{ad}} \left(\int_0^{\mu_1} g(\eta_1 - s) \sin i\pi \frac{(\eta_1 - s)}{a} ds + \int_0^{\mu_1} g(\eta_1 + s) \sin i\pi \frac{(\eta_1 + s)}{a} ds \right)$$

2. if f is symmetric with respect to the point $(\eta_1, 0)$, then we have $g(\eta_1 - s) = -g(\eta_1 + s)$ in this case we have

$$\sum_{k=1}^{2} \left\langle \frac{\partial w_{ij}}{\partial x_k}, f \right\rangle = \frac{2j\pi}{d\sqrt{ad}} \left(\int_0^{\mu_1} g(\eta_1 + s) \left(\sin i\pi \frac{(\eta_1 + s)}{a} - \sin i\pi \frac{(\eta_1 - s)}{a} \right) ds \right)$$

$$= \frac{4j\pi}{d\sqrt{ad}} \cos i\pi \frac{\eta_1}{a} \left(\int_0^{\mu_1} g(\eta_1 + s) \sin i\pi \frac{s}{a} ds \right)$$

and the result is proved.

3. if f is symmetric with respect to the axis $x_1 = \eta_1$, we have $g(\eta_1 - s) = g(\eta_1 + s)$, then we have

$$\sum_{k=1}^{2}\left\langle \frac{\partial w_{ij}}{\partial x_k}, f\right\rangle = \frac{2j\pi}{d\sqrt{ad}}\left(\int_0^{\mu_1} g(\eta_1+s)\left(\sin i\pi\frac{(\eta_1+s)}{a}+\sin i\pi\frac{(\eta_1-s)}{a}\right)ds\right)$$

$$= \frac{4j\pi}{d\sqrt{ad}}\sin i\pi\frac{\eta_1}{a}\left(\int_0^{\mu_1} g(\eta_1+s)\cos i\pi\frac{s}{a}ds\right)$$

Then the result is proved.

Proposition .11

1. *If f is uniformly distributed on $D=\{0\}\times]\beta_1,\beta_2[$ (or on $D=\{a\}\times]\beta_1,\beta_2[$), then the sensor (D,f) is not G-strategic on Γ if $\frac{\mu_2}{d}\in IQ$ or $\frac{\eta_2}{d}\in IQ$.*
2. *If f is symmetric with respect to the point $(0,\eta_2)$ or with respect to the point (a,η_2), then the sensor (D,f) is not G-strategic on Γ if there exists $l\in IN^*$ such that $2l\frac{\eta_2}{d}$ is odd.*
3. *If f is symmetric with respect to the axis $x_2=\eta_2$, then the sensor (D,f) is not G-strategic on Γ if $\frac{\eta_2}{d}\in IQ$.*

The proof is similar to that of the previous proposition.

5 REGIONAL BOUNDARY GRADIENT RECONSTRUCTION

In this section, we develop an approach for the reconstruction of the gradient of the initial state of system (2-1) on a boundary subregion Γ of $\partial\Omega$. The considered approach consists in the reconstruction of the gradient on a subregion $\omega\subset\Omega$ such that $\Gamma\subset\partial\Omega\cap\partial\omega$ by extension of the Hilbert uniqueness method.

Proposition .12

If the system (2-1) together with the output equation (2-2) is exactly (resp. approximately) G-observable on ω then, it is exactly (resp. approximately) G-observable on Γ.

Proof

1. We prove that If the system (2-1) together with the output equation (2-2) is exactly (resp. approximately) G-observable in ω then, it is exactly (resp. approximately) G-observable on Γ. For this purpose, it is sufficient to show that

$$\left(H^{\frac{1}{2}}(\Gamma)\right)^n\times\left(H^{\frac{1}{2}}(\Gamma)\right)^n\subset \operatorname{Im}\overline{\chi}_\Gamma\overline{\gamma}\,\overline{\nabla}K^*$$

Let $y=(y^1,y^2)\in\left(H^{\frac{1}{2}}(\Gamma)\right)^n\times\left(H^{\frac{1}{2}}(\Gamma)\right)^n$, and let $\tilde{y}=(\tilde{y}^1,\tilde{y}^2)$ be its extension to $\left(H^{\frac{1}{2}}(\partial\Omega)\right)^n\times\left(H^{\frac{1}{2}}(\partial\Omega)\right)^n$. Applying the trace theorem, there exists a continuous operator

$$\overline{\Re}:\left(H^{\frac{1}{2}}(\partial\Omega)\right)^n\times\left(H^{\frac{1}{2}}(\partial\Omega)\right)^n\to\left(H^1(\Omega)\right)^n\times\left(H^1(\Omega)\right)^n$$
$$(y^1,y^2)\to(\Re y^1,\Re y^2)$$

with

$$\mathfrak{R}:\left(H^{\frac{1}{2}}(\partial\Omega)\right)^n \to \left(H^1(\Omega)\right)^n$$
$$(y_1,y_2,\ldots,y_n)\to\left(\tilde{\mathfrak{R}}y_1,\tilde{\mathfrak{R}}y_2,\ldots,\tilde{\mathfrak{R}}y_n\right)$$

such that $\bar{\gamma}\,\overline{\mathfrak{R}}\,\tilde{y}=\tilde{y}$ where $\tilde{\mathfrak{R}}:H^{\frac{1}{2}}(\partial\Omega)\to H^1(\Omega)$

We have $\bar{\chi}_\omega\overline{\mathfrak{R}}\,\tilde{y}\in\left(H^1(\omega)\right)^n\times\left(H^1(\omega)\right)^n$, since the system (2-1)-(2-2) is exactly G-observable on ω, there exists $z\in L^2\left(0,T;IR^q\right)$ such that $\bar{\chi}_\omega\overline{\mathfrak{R}}\,\tilde{y}=\bar{\chi}_\omega\overline{\nabla}K^*z$ and then $\bar{\gamma}\left(\bar{\chi}_\omega^*\bar{\chi}_\omega\overline{\mathfrak{R}}\,\tilde{y}\right)=\bar{\gamma}\left(\bar{\chi}_\omega^*\bar{\chi}_\omega\overline{\nabla}K^*z\right)$.

Thus $\bar{\chi}_\Gamma\bar{\gamma}\left(\bar{\chi}_\omega^*\bar{\chi}_\omega\overline{\mathfrak{R}}\,\tilde{y}\right)=\bar{\chi}_\Gamma\bar{\gamma}\left(\bar{\chi}_\omega^*\bar{\chi}_\omega\overline{\nabla}K^*z\right)$.

But $\bar{\chi}_\Gamma\bar{\gamma}\left(\bar{\chi}_\omega^*\bar{\chi}_\omega\overline{\nabla}K^*z\right)=\bar{\chi}_\Gamma\bar{\gamma}\left(\overline{\nabla}K^*z\right)$ and $\bar{\chi}_\Gamma\bar{\gamma}\left(\bar{\chi}_\omega^*\bar{\chi}_\omega\overline{\mathfrak{R}}\,\tilde{y}\right)=y$ then $\bar{\chi}_\Gamma\bar{\gamma}\,\overline{\nabla}K^*z=y$.

Consequently the system (2-1)-(2-2) is G-observable on Γ.

2. We show that $\forall\varepsilon>0,\forall y\in\left(H^{\frac{1}{2}}(\Gamma)\right)^n\times\left(H^{\frac{1}{2}}(\Gamma)\right)^n$, $\exists\, z\in L^2\left(0,T;IR^q\right)$ such that

$$\left\|\bar{\chi}_\Gamma\bar{\gamma}\,\overline{\nabla}K^*z-y\right\|_{\left(H^{\frac{1}{2}}(\Gamma)\right)^n\times\left(H^{\frac{1}{2}}(\Gamma)\right)^n}\le\varepsilon$$

Let $y=\left(y^1,y^2\right)\in\left(H^{\frac{1}{2}}(\Gamma)\right)^n\times\left(H^{\frac{1}{2}}(\Gamma)\right)^n$ and $\tilde{y}=\left(\tilde{y}^1,\tilde{y}^2\right)$ be its extension to $\left(H^{\frac{1}{2}}(\partial\Omega)\right)^n\times\left(H^{\frac{1}{2}}(\partial\Omega)\right)^n$.

By the trace theorem, there exists $\overline{\mathfrak{R}}\tilde{y}\in\left(H^1(\Omega)\right)^n\times\left(H^1(\Omega)\right)^n$ such that $\bar{\gamma}\overline{\mathfrak{R}}\tilde{y}=\tilde{y}$.

Since $\bar{\chi}_\omega\overline{\mathfrak{R}}\tilde{y}\in\left(H^1(\omega)\right)^n\times\left(H^1(\omega)\right)^n$ the system (2-1)-(2-2) is approximately G-observable on ω, then

$$\forall\varepsilon>0,\exists\, z\in L^2\left(0,T;IR^q\right)\text{ such that }\left\|\bar{\chi}_\omega\overline{\nabla}K^*z-\bar{\chi}_\omega\overline{\mathfrak{R}}\tilde{y}\right\|_{\left(H^1(\omega)\right)^n\times\left(H^1(\omega)\right)^n}\le\varepsilon \qquad (5\text{-}8)$$

By the continuity of the trace $\bar{\gamma}$, we have

$$\left\|\bar{\gamma}\left(\bar{\chi}_\omega^*\bar{\chi}_\omega\overline{\nabla}K^*z\right)-\bar{\gamma}\left(\bar{\chi}_\omega^*\bar{\chi}_\omega\overline{\mathfrak{R}}\tilde{y}\right)\right\|_{\left(H^{\frac{1}{2}}(\partial\Omega)\right)^n\times\left(H^{\frac{1}{2}}(\partial\Omega)\right)^n}\le\left\|\bar{\chi}_\omega^*\bar{\chi}_\omega\overline{\nabla}K^*z-\bar{\chi}_\omega^*\bar{\chi}_\omega\overline{\mathfrak{R}}\tilde{y}\right\|_{\left(H^1(\Omega)\right)^n\times\left(H^1(\Omega)\right)^n}$$

By the continuity of the operator $\bar{\chi}_\omega^*$, we have

$$\left\|\bar{\chi}_\omega^*\bar{\chi}_\omega\overline{\nabla}K^*z-\bar{\chi}_\omega^*\bar{\chi}_\omega\overline{\mathfrak{R}}\tilde{y}\right\|_{\left(H^1(\Omega)\right)^n\times\left(H^1(\Omega)\right)^n}\le\left\|\bar{\chi}_\omega\overline{\nabla}K^*z-\bar{\chi}_\omega\overline{\mathfrak{R}}\tilde{y}\right\|_{\left(H^1(\omega)\right)^n\times\left(H^1(\omega)\right)^n}$$

From (5-8), we obtain $\left\|\bar{\gamma}\left(\bar{\chi}_\omega^*\bar{\chi}_\omega\overline{\nabla}K^*z\right)-\bar{\gamma}\left(\bar{\chi}_\omega^*\bar{\chi}_\omega\overline{\mathfrak{R}}\tilde{y}\right)\right\|_{\left(H^{\frac{1}{2}}(\partial\Omega)\right)^n\times\left(H^{\frac{1}{2}}(\partial\Omega)\right)^n}\le\varepsilon$

The operator $\bar{\chi}_\Gamma$ is continuous then

$$\left\|\bar{\chi}_\Gamma\bar{\gamma}\left(\bar{\chi}_\omega^*\bar{\chi}_\omega\overline{\nabla}K^*z-\bar{\chi}_\omega^*\bar{\chi}_\omega\overline{\mathfrak{R}}\tilde{y}\right)\right\|_{\left(H^{\frac{1}{2}}(\Gamma)\right)^n\times\left(H^{\frac{1}{2}}(\Gamma)\right)^n}\le\left\|\bar{\gamma}\left(\bar{\chi}_\omega^*\bar{\chi}_\omega\overline{\nabla}K^*z\right)-\bar{\gamma}\left(\bar{\chi}_\omega^*\bar{\chi}_\omega\overline{\mathfrak{R}}\tilde{y}\right)\right\|_{\left(H^{\frac{1}{2}}(\partial\Omega)\right)^n\times\left(H^{\frac{1}{2}}(\partial\Omega)\right)^n}$$

Since, $\bar{\chi}_\Gamma\bar{\gamma}\left(\bar{\chi}_\omega^*\bar{\chi}_\omega\overline{\nabla}K^*z-\bar{\chi}_\omega^*\bar{\chi}_\omega\overline{\mathfrak{R}}\tilde{y}\right)=\bar{\chi}_\Gamma\bar{\gamma}\left(\overline{\nabla}K^*z-\overline{\mathfrak{R}}\tilde{y}\right)=\bar{\chi}_\Gamma\bar{\gamma}\left(\overline{\nabla}K^*z\right)-y$, this gives

$$\left\| \overline{\chi}_{\Gamma}\overline{\gamma}\,\overline{\nabla}K^{*}z-y\right\|_{\left(H^{\frac{1}{2}}(\Gamma)\right)^{n}\times\left(H^{\frac{1}{2}}(\Gamma)\right)^{n}}\leq\varepsilon$$

Then the system (2-1)-(2-2) is approximately G-observable on Γ.

By the above proposition, we are going to reconstruct the components of the gradient of the initial state on ω, and deduce its trace on Γ.

Consider the system (2-1) with the output (2-2), and the set

$$F=\left\{\left(\varphi^{0},\varphi^{1}\right)\in\left(H^{1}(\Omega)\right)^{n}\times\left(H^{1}(\Omega)\right)^{n}/\varphi^{0}=\varphi^{1}=0\ sur\ \Omega\setminus\omega\right\}\cap$$
$$\left\{\overline{\nabla}\left(\phi^{0},\phi^{1}\right)/\left(\phi^{0},\phi^{1}\right)\in\left(H^{2}(\Omega)\cap H_{0}^{1}(\Omega)\right)\times\left(H^{2}(\Omega)\cap H_{0}^{1}(\Omega)\right)\right\}$$

For $\left(\phi^{0},\phi^{1}\right)\in\left(H^{2}(\Omega)\cap H_{0}^{1}(\Omega)\right)\times\left(H^{2}(\Omega)\cap H_{0}^{1}(\Omega)\right)$, consider the system

$$\begin{cases}\dfrac{\partial^{2}\phi(x,t)}{\partial t^{2}}=A\phi(x,t) & \text{on } Q\\ \phi(x,0)=\phi^{0}(x)\quad,\quad\dfrac{\partial\phi(x,t)}{\partial t}=\phi^{1}(x) & \text{on } \Omega\\ \phi(\zeta,t)=0 & \text{on } \Sigma\end{cases}\tag{5-9}$$

has a unique solution $\phi\in C\left(0,T;H_{0}^{1}(\Omega)\cap H^{2}(\Omega)\right)\cap C^{1}\left(0,T;L^{2}(\Omega)\right)$ (see [6]).

Considering a zone sensor, the output equation is then

$$z(t)=\sum_{k=1}^{n}\left\langle\frac{\partial y(.,t)}{\partial x_{k}},f\right\rangle_{L^{2}(D)}\tag{5-10}$$

where D is the sensor support, f the function measures and we consider a semi-norm on F defined by

$$\left(\varphi^{0},\varphi^{1}\right)\in F\mapsto\left\|\left(\varphi^{0},\varphi^{1}\right)\right\|_{F}^{2}=\int_{0}^{T}\left(\sum_{k=1}^{n}\left\langle\frac{\partial\phi(.,t)}{\partial x_{k}},f\right\rangle_{L^{2}(D)}\right)^{2}dt\tag{5-11}$$

where $\phi(x,t)$ is the solution of (5-9).

The reverse system given by

$$\begin{cases}\dfrac{\partial^{2}\psi(x,t)}{\partial t^{2}}=A^{*}\psi(x,t)+\displaystyle\sum_{k=1}^{n}\left\langle\frac{\partial\phi}{\partial x_{k}},f\right\rangle_{L^{2}(D)}\chi_{D}(x)f(x) & \text{on } Q\\ \psi(x,T)=0,\quad\dfrac{\partial\psi(x,T)}{\partial t}=0 & \text{on } \Omega\\ \psi(\zeta,t)=0 & \text{on } \Sigma\end{cases}\tag{5-12}$$

has a unique solution $\psi\in C\left(0,T;H_{0}^{1}(\Omega)\cap H^{2}(\Omega)\right)\cap C^{1}\left(0,T;L^{2}(\Omega)\right)$ [6].

We denote $\psi^{0}(x)=\psi(x,0)$ and $\psi^{1}(x)=\dfrac{\partial\psi(x,0)}{\partial t}$ and consider the operator

$$\Lambda\left(\varphi^{0},\varphi^{1}\right)=P\left(-\Psi^{1},\Psi^{0}\right)$$

where $P=\overline{\chi}_{\omega}^{*}\overline{\chi}_{\omega}$, $\Psi^{1}=\left(\psi^{1},\psi^{1},\ldots,\psi^{1}\right),\Psi^{0}=\left(\psi^{0},\psi^{0},\ldots,\psi^{0}\right)$.

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The retrograde system

$$\begin{cases} \dfrac{\partial^2 Z(x,t)}{\partial t^2} = A^* Z(x,t) + \sum_{k=1}^{n} \left\langle \dfrac{\partial y(.,t)}{\partial x_k}, f \right\rangle_{L^2(D)} \chi_D(x) f(x) & \text{on } Q \\ Z(x,T) = 0 \quad , \quad \dfrac{\partial Z(x,T)}{\partial t} = 0 & \text{on } \Omega \\ Z(\zeta,t) = 0 & \text{on } \Sigma \end{cases} \tag{5-13}$$

has a unique solution $Z \in C\left(0,T; H_0^1(\Omega) \cap H^2(\Omega)\right) \cap C^1\left(0,T; L^2(\Omega)\right)$ (see [6]).

We denote $Z^0(x) = Z(x,0)$ and $Z^1(x) = \dfrac{\partial Z(x,0)}{\partial t}$.

If $\left(\varphi^0, \varphi^1\right)$ is chosen such that $\psi^0 = Z^0$ et $\psi^1 = Z^1$ on ω, then the regional gradient observability turns up to solve the equation

$$\Lambda\left(\varphi^0, \varphi^1\right) = P\left(-\overline{Z}^1, \overline{Z}^0\right) \tag{5-14}$$

where $\overline{Z}^1 = \left(Z^1, Z^1, \ldots, Z^1\right)$ and $\overline{Z}^0 = \left(Z^0, Z^0, \ldots, Z^0\right)$.

Proposition .13

If the sensor (D, f) is G-strategic on ω, then the equation (5-14) has a unique solution $\left(\varphi^0, \varphi^1\right)$ which coincides with the gradient of the initial state $\left(\nabla y^0, \nabla y^1\right)$ in the subregion ω, and the gradient on the subregion Γ is given by $\left(\nabla y_\Gamma^0, \nabla y_\Gamma^1\right) = \overline{\chi}_\Gamma \overline{\gamma}\left(\varphi^0, \varphi^1\right)$.

Proof

1. Let us show first that if the system (2-1) is G-observable, then (5-11) defines a norm on F. Indeed Consider a basis $\left(w_j\right)_{j \geq 1}$ of the eigenfunctions of A, without loss of generality we suppose that the multiplicities of the eigenvalues are simples, then

$\left\| \left(\varphi^0, \varphi^1\right) \right\|_F = 0 \Leftrightarrow \sum_{k=1}^{n} \left\langle \dfrac{\partial \phi}{\partial x_k}, f \right\rangle_{L^2(D)} = 0$ on $\left]0, T\right[$ which is equivalent to

$$\sum_{j=1}^{\infty} \left[\left\langle \phi^0, w_j \right\rangle_{L^2(\Omega)} \cos \sqrt{-\lambda_j}\, t + \frac{1}{\sqrt{-\lambda_j}} \left\langle \phi^1, w_j \right\rangle_{L^2(\Omega)} \sin \sqrt{-\lambda_j}\, t \right] \sum_{k=1}^{n} \left\langle \frac{\partial w_j}{\partial x_k}, f \right\rangle_{L^2(D)} = 0$$

The set $\left\{ \sin \sqrt{-\lambda_j}\, t, \cos \sqrt{-\lambda_j}\, t,\ j \geq 1 \right\}$ forms a complete orthogonal set of $L^2(0,T)$, then we obtain

$$\begin{cases} \left\langle \phi^0, w_j \right\rangle_{L^2(\Omega)} \sum_{k=1}^{n} \left\langle \dfrac{\partial w_j}{\partial x_k}, f \right\rangle_{L^2(D)} = 0, \forall j \geq 1 \\ \left\langle \phi^1, w_j \right\rangle_{L^2(\Omega)} \sum_{k=1}^{n} \left\langle \dfrac{\partial w_j}{\partial x_k}, f \right\rangle_{L^2(D)} = 0, \forall j \geq 1 \end{cases}$$

and since the sensor (D, f) is G-strategic on ω, we have $\sum_{k=1}^{n} \left\langle \dfrac{\partial w_j}{\partial x_k}, f \right\rangle_{L^2(D)} \neq 0, \forall j \geq 1$ (see [1]) then

$\left\langle \phi^0, w_j \right\rangle_{L^2(\Omega)} = \left\langle \phi^1, w_j \right\rangle_{L^2(\Omega)} = 0,\ \forall j \geq 1$.

Consequently $\phi^0 = \phi^1 = 0$ and thus $\varphi^0 = \varphi^1 = 0$.

Conversely, $\varphi^0 = \varphi^1 = 0 \Rightarrow \phi^0 = c_1$ and $\phi^1 = c_2$ (constants),since $\phi \in C\left(0,T; H_0^1(\Omega) \cap H^2(\Omega)\right) \cap C^1\left(0,T; L^2(\Omega)\right)$ and from $\phi = 0$ on Σ, (5-11) is a norm.

2. Let denote by F completion of F by the norm (5-11) and F^* its dual. We show that Λ is an isomorphism from F into F^*. Indeed, let $\left(\hat{\varphi}^0, \hat{\varphi}^1\right) \in F$ and $\hat{\phi}$ the corresponding solution of system (5-9), we multiply system (5-12) by $\dfrac{\partial \hat{\phi}(x,t)}{\partial x_k}$, and integrate on Q, we obtain

$$\left\langle \frac{\partial \hat{\phi}}{\partial x_k}, \frac{\partial^2 \psi}{\partial t^2} \right\rangle_{L^2(Q)} = \left\langle \frac{\partial \hat{\phi}}{\partial x_k}, A^* \psi \right\rangle_{L^2(Q)} + \left\langle \frac{\partial \hat{\phi}}{\partial x_k}, \sum_{l=1}^{n} \left\langle \frac{\partial \phi}{\partial x_l}, f \right\rangle_{L^2(D)} \chi_D f \right\rangle_{L^2(Q)}$$

The first term gives

$$\left\langle \frac{\partial \hat{\phi}}{\partial x_k}, \frac{\partial^2 \psi}{\partial t^2} \right\rangle_{L^2(Q)} = -\left\langle \frac{\partial \hat{\phi}(.,0)}{\partial x_k}, \frac{\partial \psi(.,0)}{\partial t} \right\rangle_{L^2(\Omega)} + \left\langle \frac{\partial}{\partial x_k}\left(\frac{\partial \hat{\phi}(.,0)}{\partial t}\right), \psi(.,0) \right\rangle_{L^2(\Omega)} + \left\langle \frac{\partial}{\partial x_k}\left(A\hat{\phi}\right), \psi \right\rangle_{L^2(Q)}$$

Using Green formula for the second term, we obtain

$$\left\langle \frac{\partial \hat{\phi}}{\partial x_k}, A^* \psi \right\rangle_{L^2(Q)} + \left\langle \frac{\partial \hat{\phi}}{\partial x_k}, \sum_{l=1}^{n} \left\langle \frac{\partial \phi}{\partial x_l}, f \right\rangle_{L^2(D)} \chi_D f \right\rangle_{L^2(Q)} = \left\langle A\frac{\partial \hat{\phi}}{\partial x_k}, \psi \right\rangle_{L^2(Q)}$$
$$+ \int_{\Sigma} \left(\psi(\xi,t) \frac{\partial^2 \hat{\phi}(\xi,t)}{\partial \eta_A \partial x_k} - \frac{\partial \hat{\phi}(\xi,t)}{\partial x_k} \frac{\partial \psi(\xi,t)}{\partial \eta_{A^*}} \right) d\Sigma + \int_0^T \int_{\Omega} \frac{\partial \hat{\phi}(x,t)}{\partial x_k} \sum_{l=1}^{n} \left\langle \frac{\partial \phi}{\partial x_l}, f \right\rangle_{L^2(D)} \chi_D(x) f(x)\, dx\, dt$$

The boundaries conditions gives

$$\left\langle \left(\hat{\varphi}^0, \hat{\varphi}^1\right), \Lambda\left(\varphi^0, \varphi^1\right) \right\rangle = \int_0^T \left(\sum_{k=1}^{n} \left\langle \frac{\partial \hat{\phi}(t)}{\partial x_k}, f \right\rangle_{L^2(D)} \right) \left(\sum_{l=1}^{n} \left\langle \frac{\partial \phi(t)}{\partial x_l}, f \right\rangle_{L^2(D)} \right) dt$$

Using Cauchy-Schwartz inequality, we have

$$\left\langle \left(\hat{\varphi}^0, \hat{\varphi}^1\right), \Lambda\left(\varphi^0, \varphi^1\right) \right\rangle \le \left\| \left(\hat{\varphi}^0, \hat{\varphi}^1\right) \right\|_F \left\| \left(\varphi^0, \varphi^1\right) \right\|_F, \forall \left(\varphi^0, \varphi^1\right), \left(\hat{\varphi}^0, \hat{\varphi}^1\right) \in F$$

Hence,

$$\left\langle \left(\varphi^0, \varphi^1\right), \Lambda\left(\varphi^0, \varphi^1\right) \right\rangle = \left\| \left(\varphi^0, \varphi^1\right) \right\|_{\hat{F}}^2, \ \forall \left(\varphi^0, \varphi^1\right) \in F$$

This proves that Λ is an isomorphism and consequently the equation (5-14) has a unique solution $\left(\varphi^0, \phi^1\right)$ which corresponds to initial state gradient to be observed on the subregion ω. Thus the initial state gradient to be observed on Γ is given by $\left(\nabla y_\Gamma^0, \nabla y_\Gamma^1\right) = \bar{\chi}_\Gamma \bar{\gamma}\left(\varphi^0, \varphi^1\right)$.

6 NUMERICAL APPROACH AND SIMULATIONS

In this section we develop a numerical approach which leads to explicit formulas for the gradient on the subregion ω, and deduce the gradient on $\Gamma \subset \partial\Omega \cap \partial\omega$.

We consider the case where system (2-1) is observed by the output equation

$$z(t) = \sum_{k=1}^{n} \left\langle \frac{\partial y(.,t)}{\partial x_k}, f \right\rangle_{L^2(D)}, t \in \left]0,T\right[$$

6.1 Numerical approach

Proposition .14

If the sensor (D, f) is G-strategic on ω, then the components of the initial gradient on ω may be approached by

$$\hat{\nabla} y^0 \approx \sum_{j=1}^{M} \left[\frac{2}{T\left(\sum_{k=1}^{n}\left\langle \frac{\partial w_j}{\partial x_k}, f\right\rangle\right)^2} \sum_{l=1}^{n}\sum_{m=1}^{M}\langle w_m, f\rangle \int_0^T \sum_{i=1}^{n}\left\langle \frac{\partial y}{\partial x_i}, f\right\rangle \cos\sqrt{-\lambda_m}\,t\, dt \left\langle w_m, \frac{\partial w_j}{\partial x_l}\right\rangle \right] \nabla w_j \qquad (6\text{-}15)$$

$$\hat{\nabla} y^1 \approx \sum_{j=1}^{M} \left[\frac{-2\lambda_j}{T\left(\sum_{k=1}^{n}\left\langle \frac{\partial w_j}{\partial x_k}, f\right\rangle\right)^2} \sum_{l=1}^{n}\sum_{m=1}^{M}\langle w_m, f\rangle \int_0^T \sum_{i=1}^{n}\left\langle \frac{\partial y}{\partial x_i}, f\right\rangle \frac{\sin\sqrt{-\lambda_m}\,t}{\sqrt{-\lambda_m}}\, dt \left\langle w_m, \frac{\partial w_j}{\partial x_l}\right\rangle \right] \nabla w_j \qquad (6\text{-}16)$$

where M *a truncation order.*

Proof

In the previous section, it has been seen that the regional reconstruction of the initial state gradient on ω turns up to solve the equation (5-14). For that consider the functional

$$\Phi\left(\varphi^0,\varphi^1\right)=\frac{1}{2}\left\langle \Lambda\left(\varphi^0,\varphi^1\right),\left(\varphi^0,\varphi^1\right)\right\rangle-\left\langle P\left(-\bar{Z}^1,\bar{Z}^0\right),\left(\varphi^0,\varphi^1\right)\right\rangle$$
$$=\frac{1}{2}\int_0^T\left(\sum_{k=1}^{n}\left\langle \frac{\partial\phi(.,t)}{\partial x_k}, f\right\rangle_{L^2(D)}\right)^2 dt-\left\langle -\bar{Z}^1,\varphi^0\right\rangle-\left\langle \bar{Z}^0,\varphi^1\right\rangle$$

And solving equation (5-14) turns up to minimize Φ with respect to $\left(\varphi^0,\varphi^1\right)$

After development and when $T \longrightarrow +\infty$, we obtain

$$\lim_{T\to+\infty}\frac{1}{2T}\int_0^T\left(\sum_{k=1}^{n}\left\langle \frac{\partial\phi(.,t)}{\partial x_k}, f\right\rangle_{L^2(D)}\right)^2 dt=\frac{1}{4}\sum_{j=1}^{+\infty}\left[\left\langle \phi^0,w_j\right\rangle^2-\frac{1}{\lambda_j}\left\langle \phi^1,w_j\right\rangle^2\right]\left(\sum_{k=1}^{n}\left\langle \frac{\partial w_j}{\partial x_k}, f\right\rangle_{L^2(D)}\right)^2$$

For T large enough, we have

$$\frac{1}{2}\int_0^T\left(\sum_{k=1}^{n}\left\langle \frac{\partial\phi(.,t)}{\partial x_k}, f\right\rangle_{L^2(D)}\right)^2 dt\approx\frac{T}{4}\sum_{j=1}^{+\infty}\left[\left\langle \phi^0,w_j\right\rangle^2-\frac{1}{\lambda_j}\left\langle \phi^1,w_j\right\rangle^2\right]\left(\sum_{k=1}^{n}\left\langle \frac{\partial w_j}{\partial x_k}, f\right\rangle_{L^2(D)}\right)^2$$

On the other hand, we have

$$\phi^0(x)=\sum_{j=1}^{+\infty}\left\langle \phi^0,w_j\right\rangle w_j(x) \quad\text{and}\quad \phi^1(x)=\sum_{j=1}^{+\infty}\left\langle \phi^1,w_j\right\rangle w_j(x)$$

since $\left(\varphi^0,\varphi^1\right)=\bar{\nabla}\left(\phi^0,\phi^1\right)$, then

$$\varphi^0(x)=\sum_{j=1}^{+\infty}\left\langle \phi^0,w_j\right\rangle\left(\frac{\partial w_j}{\partial x_1},\frac{\partial w_j}{\partial x_2},\ldots,\frac{\partial w_j}{\partial x_n}\right) \quad \text{on } \omega \qquad (6\text{-}17)$$

and

$$\varphi^1(x)=\sum_{j=1}^{+\infty}\left\langle \phi^1,w_j\right\rangle\left(\frac{\partial w_j}{\partial x_1},\frac{\partial w_j}{\partial x_2},\ldots,\frac{\partial w_j}{\partial x_n}\right) \quad \text{on } \omega \qquad (6\text{-}18)$$

we obtain

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$$\left\langle -\overline{Z}^1,\varphi^0\right\rangle_{\left(L^2(\omega)\right)^n}=\sum_{k=1}^{n}\sum_{j=1}^{+\infty}\left\langle \phi^0,w_j\right\rangle_{L^2(\Omega)}\left\langle -Z^1,\frac{\partial w_j}{\partial x_k}\right\rangle_{L^2(\omega)}$$

and

$$\left\langle \overline{Z}^0,\varphi^1\right\rangle_{\left(L^2(\omega)\right)^n}=\sum_{k=1}^{n}\sum_{j=1}^{+\infty}\left\langle \phi^1,w_j\right\rangle_{L^2(\Omega)}\left\langle Z^0,\frac{\partial w_j}{\partial x_k}\right\rangle_{L^2(\omega)}$$

The minimization of Φ is equivalent to solve the two following problems

$$\underset{\phi^0}{Inf}\sum_{j=1}^{+\infty}\left\{\frac{T}{4}\left\langle \phi^0,w_j\right\rangle_{L^2(\Omega)}^2\left(\sum_{k=1}^{n}\left\langle\frac{\partial w_j}{\partial x_k},f\right\rangle_{L^2(D)}\right)^2-\left\langle \phi^0,w_j\right\rangle_{L^2(\Omega)}\sum_{k=1}^{n}\left\langle -Z^1,\frac{\partial w_j}{\partial x_k}\right\rangle_{L^2(\omega)}\right\}$$

$$\underset{\phi^1}{Inf}\sum_{j=1}^{+\infty}\left\{\frac{-T}{4\lambda_j}\left\langle \phi^1,w_j\right\rangle_{L^2(\Omega)}^2\left(\sum_{k=1}^{n}\left\langle\frac{\partial w_j}{\partial x_k},f\right\rangle_{L^2(D)}\right)^2-\left\langle \phi^1,w_j\right\rangle_{L^2(\Omega)}\sum_{k=1}^{n}\left\langle Z^0,\frac{\partial w_j}{\partial x_k}\right\rangle_{L^2(\omega)}\right\}$$

which solutions are

$$\left\langle \phi^0,w_j\right\rangle_{L^2(\Omega)}=-\frac{2}{T}\frac{\left\langle \overline{Z}^1,\nabla w_j\right\rangle_{\left(L^2(\omega)\right)^n}}{\left(\sum_{k=1}^{n}\left\langle\frac{\partial w_j}{\partial x_k},f\right\rangle_{L^2(D)}\right)^2}\quad,\quad\forall j\geq 1 \tag{6-19}$$

$$\left\langle \phi^1,w_j\right\rangle_{L^2(\Omega)}=-\frac{2\lambda_j}{T}\frac{\left\langle \overline{Z}^0,\nabla w_j\right\rangle_{\left(L^2(\omega)\right)^n}}{\left(\sum_{k=1}^{n}\left\langle\frac{\partial w_j}{\partial x_k},f\right\rangle_{L^2(D)}\right)^2}\quad,\quad\forall j\geq 1 \tag{6-20}$$

Now, let $Z(x,t)=\sum_{m\geq 1}Z_m(t)w_m(x)$ be the solution of the system (5-13) with

$$Z_m(t)=\frac{\left\langle w_m,f\right\rangle_{L^2(D)}}{\sqrt{-\lambda_m}}\int_0^T\sum_{i=1}^{n}\left\langle\frac{\partial y(.,s)}{\partial x_i},f\right\rangle_{L^2(D)}\sin\sqrt{-\lambda_m}(s-t)\,ds$$

Thus

$$Z^0=Z(x,0)=\sum_{m\geq 1}\frac{\left\langle w_m,f\right\rangle_{L^2(D)}}{\sqrt{-\lambda_m}}\int_0^T\sum_{i=1}^{n}\left\langle\frac{\partial y(.,s)}{\partial x_i},f\right\rangle_{L^2(D)}\sin\sqrt{-\lambda_m}\,s\,ds\ \ w_m(x)$$

and

$$Z^1=\frac{\partial Z}{\partial t}(x,0)=-\sum_{m\geq 1}\left\langle w_m,f\right\rangle_{L^2(D)}\int_0^T\sum_{i=1}^{n}\left\langle\frac{\partial y(.,s)}{\partial x_i},f\right\rangle_{L^2(D)}\cos\sqrt{-\lambda_m}\,s\,ds\ \ w_m(x)$$

Thus we have

$$\left\langle \overline{Z}^0,\nabla w_j\right\rangle_{L^2(\omega)}=\sum_{l=1}^{n}\sum_{m=1}^{+\infty}\frac{\left\langle w_m,f\right\rangle_{L^2(D)}}{\sqrt{-\lambda_m}}\int_0^T\sum_{i=1}^{n}\left\langle\frac{\partial y(.,s)}{\partial x_i},f\right\rangle_{L^2(D)}\sin\sqrt{-\lambda_m}\,s\,ds\left\langle w_m,\frac{\partial w_j}{\partial x_l}\right\rangle_{L^2(\omega)}$$

and

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$$\left\langle \bar{Z}^1, \nabla w_j \right\rangle_{L^2(\omega)} = -\sum_{l=1}^{n}\sum_{m=1}^{+\infty} \langle w_m, f\rangle_{L^2(D)} \int_0^T \sum_{i=1}^{n} \left\langle \frac{\partial y(.,s)}{\partial x_i}, f \right\rangle_{L^2(D)} \cos\sqrt{-\lambda_m}\, s\, ds \left\langle w_m, \frac{\partial w_j}{\partial x_l} \right\rangle_{L^2(\omega)}$$

With these developments, according to (6-19) and (6-20), we obtain. $\forall\, j \geq 1$

$$\left\langle \phi^0, w_j \right\rangle_{L^2(\Omega)} = \frac{2}{T\left(\sum_{k=1}^{n}\left\langle \frac{\partial w_j}{\partial x_k}, f\right\rangle_{L^2(D)}\right)^2} \sum_{l=1}^{n}\sum_{m=1}^{+\infty} \langle w_m, f\rangle_{L^2(D)} \int_0^T \sum_{i=1}^{n} \left\langle \frac{\partial y(.,s)}{\partial x_i}, f \right\rangle_{L^2(D)} \cos\sqrt{-\lambda_m} sds \left\langle w_m, \frac{\partial w_j}{\partial x_l} \right\rangle_{L^2(\omega)}$$

and

$$\left\langle \phi^1, w_j \right\rangle_{L^2(\Omega)} = \frac{-2\lambda_j}{T\left(\sum_{k=1}^{n}\left\langle \frac{\partial w_j}{\partial x_k}, f\right\rangle_{L^2(D)}\right)^2} \sum_{l=1}^{n}\sum_{m=1}^{+\infty} \frac{\langle w_m, f\rangle_{L^2(D)}}{\sqrt{-\lambda_m}} \int_0^T \sum_{i=1}^{n} \left\langle \frac{\partial y(.,s)}{\partial x_i}, f \right\rangle_{L^2(D)} \sin\sqrt{-\lambda_m} sds \left\langle w_m, \frac{\partial w_j}{\partial x_l} \right\rangle_{L^2(\omega)}$$

replacing in relations (6-17) and (6-18), we obtain

$$\varphi^0(x) = \sum_{j=1}^{+\infty} \frac{2}{T\left(\sum_{k=1}^{n}\left\langle \frac{\partial w_j}{\partial x_k}, f\right\rangle\right)^2} \sum_{l=1}^{n}\sum_{m=1}^{+\infty} \langle w_m, f\rangle \int_0^T \sum_{i=1}^{n} \left\langle \frac{\partial y(.,s)}{\partial x_i}, f \right\rangle \cos\sqrt{-\lambda_m}\, s\, ds \left\langle w_m, \frac{\partial w_j}{\partial x_l} \right\rangle \nabla w_j(x)\, on\, \omega$$

and

$$\varphi^1(x) = \sum_{j=1}^{+\infty} \frac{-2\lambda_j}{T\left(\sum_{k=1}^{n}\left\langle \frac{\partial w_j}{\partial x_k}, f\right\rangle\right)^2} \sum_{l=1}^{n}\sum_{m=1}^{+\infty} \frac{\langle w_m, f\rangle}{\sqrt{-\lambda_m}} \int_0^T \sum_{i=1}^{n} \left\langle \frac{\partial y(.,s)}{\partial x_i}, f \right\rangle \sin\sqrt{-\lambda_m}\, s\, ds \left\langle w_m, \frac{\partial w_j}{\partial x_l} \right\rangle \nabla w_j(x)\, on\, \omega$$

We consider a truncation up to order $M, \left(M \in IN^*\right)$, then we obtain formulae (6-15) and (6-16).

We define a final error

$$\xi^2 = \left\|\nabla y^0 - \hat{\nabla} y^0\right\|^2_{L^2(\Gamma)} + \left\|\nabla y^1 - \hat{\nabla} y^1\right\|^2_{L^2(\Gamma)}$$

The good choice of M will be such that $\xi \leq \varepsilon\ (\varepsilon > 0)$ and we have the following algorithm:

Algorithm:

Step 1: Data: The region ω, the sensor parameters D, f and ε.

Step 2: Choose a truncation order M .

Step 3: Computation of

6.2 Simulations

Here, we consider the two-dimensional system evolving in $\Omega =]0,1[\times]0,1[$ by

$$\begin{cases} \frac{\partial^2 y}{\partial t^2}(x_1, x_2, t) = \frac{\partial^2 y}{\partial x_1^2}(x_1, x_2, t) + \frac{\partial^2 y}{\partial x_2^2}(x_1, x_2, t) & Q \\ y(x_1, x_2, 0) = y^0(x_1, x_2) \quad , \quad \frac{\partial y}{\partial t}(x_1, x_2, 0) = y^1(x_1, x_2) & \Omega \\ y(\zeta_1, \varsigma_2, t) = 0 & \Sigma \end{cases}$$

Measurements are given by one pointwise sensor

$$z(t)=\sum_{k=1}^{n}\frac{\partial y(b_1,b_2.,t)}{\partial x_k},\ t\in\left]0,T\right[$$

where $b=(b_1,b_2)\in\Omega$ denote the pointwise location. Let

$$\nabla y_\Gamma^0(\eta)=\begin{pmatrix}2A_1\pi\sin(\pi\eta)\\0\end{pmatrix}\ \text{and}\ \nabla y_\Gamma^1(\eta)=\begin{pmatrix}2B\pi\eta\sin(\pi\eta)\\0\end{pmatrix},\quad\forall\eta\in[0,1]$$

be the initial gradient to be reconstructed on $\Gamma=\{1\}\times[0,1]$.

Let $\omega=\left]0.9,1\right[\times\left]0,1\right[$ and

$$\begin{cases}\nabla y^0(x_1,x_2)=\begin{pmatrix}2A_1\pi\cos(2\pi x_1)\sin(\pi x_2)\\A_1\pi\sin(2\pi x_1)\cos(\pi x_2)\end{pmatrix}\\\nabla y^1(x_1,x_2)=\begin{pmatrix}2B\pi x_1x_2\cos(2\pi x_1)\sin(\pi x_2)+Bx_2\sin(2\pi x_1)\sin(\pi x_2)\\B\pi x_1x_2\sin(2\pi x_1)\cos(\pi x_2)+Bx_1\sin(2\pi x_1)\sin(\pi x_2)\end{pmatrix}\end{cases}$$

be the extensions of $\nabla y_\Gamma^0,\nabla y_\Gamma^1$ to ω, where A_1 and B are selected to satisfy numerical considerations (in order to obtain reasonable amplitude for ∇y_Γ^0 and ∇y_Γ^1). Consider the following data:

$T=3,\quad A_1=0.015,\quad B=0.99,\quad (b_1,b_2)=(0.66,0.98)$.

Applying the previous algorithm and using the formulae (6-15), (6-16), we obtain the following figures.

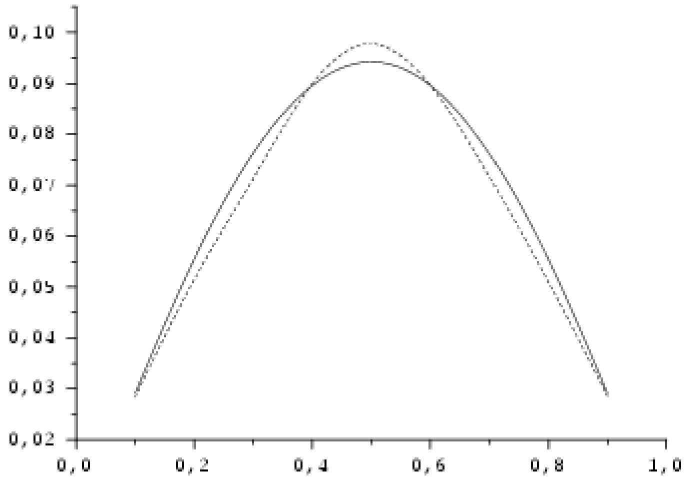

Fig 1: Representes ∇y^0 (continuous line) and $\hat{\nabla} y^0$ (dashed line) on Γ.

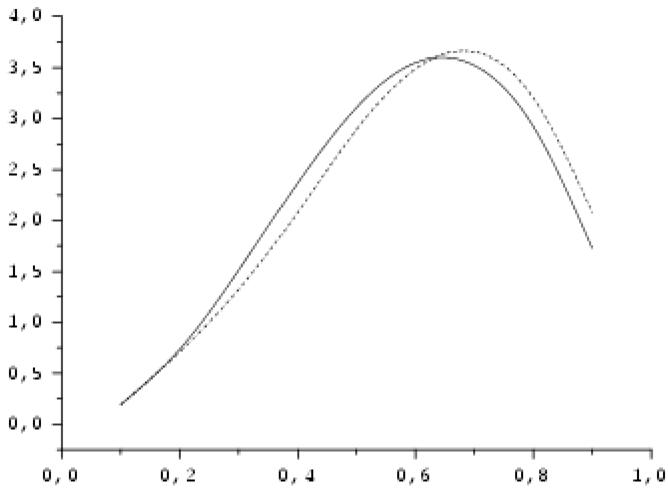

Fig 2: Representes ∇y^1 (continuous line) and $\hat{\nabla} y^1$ (dashed line) on Γ.

Now let study the link between the subregion area and the reconstruction error. Also the evolution of the reconstruction error qith respect to the amplitude A_1 of the initial gradient taking the subregion $\Gamma = \{1\} \times \left]0.9, 1\right[$ .and $B = 0.99$. Numerical simulations leads to the following tables.

Table 1. means that the larger the region is, the greater the error is.

Subregion Γ	Error ξ
$\{1\} \times \left]0.55,1\right[$	3.7835×10^{-2}
$\{1\} \times \left]0.6,1\right[$	3.4298×10^{-2}
$\{1\} \times \left]0.65,1\right[$	2.3829×10^{-2}
$\{1\} \times \left]0.7,1\right[$	1.1944×10^{-2}
$\{1\} \times \left]0.75,1\right[$	3.8238×10^{-3}
$\{1\} \times \left]0.8,1\right[$	5.9483×10^{-4}
$\{1\} \times \left]0.85,1\right[$	1.8861×10^{-5}
$\{1\} \times \left]0.9,1\right[$	6.4113×10^{-6}

Table 2. indicates that the reconstruction error depends on the amplitude of initial gradient, the greater the amplitude is, the greater the error is.

Amplitude A_1	Error ξ
0.05	4.1544×10^{-2}
0.02	5.5612×10^{-3}
0.015	2.8133×10^{-3}
0.01	9.9394×10^{-4}
0.008	5.2611×10^{-4}
0.005	1.0288×10^{-4}
0.004	3.6082×10^{-5}
0.003	6.4113×10^{-6}

CONCLUSION

The question of regional boundary gradient observability of hyperbolic systems was discussed in connection with sensors structure. A reconstruction approach of the gradient on a boundary subregion was developed leading to useful algorithm that successfully illustrated with examples and simulations.

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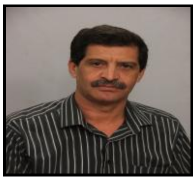

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