Transient Solution of an Batch arrival Queue with Two Types of Service, Multiple Vacation, Random breakdown and Restricted Admissibility

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ABSRACT

In this paper we analyse a batch arrival queue with two types of service subject to random breakdowns having multiple server vacation. We assume that the server provides two types of service, type 1 with probability p_1 and type 2 with probability p_2 with the service times following general distribution and each arriving customer may choose either type of service. The server takes vacation each time the system becomes empty and the vacation period is assumed to be general. On returning from vacation if the server finds no customer waiting in the system, then the server again goes for vacation until he finds at least one customer in the system. The system may breakdown at random and repair time follows exponential distribution. In addition we assume restricted admissibility of arriving batches in which not all batches are allowed to join the system at all times. The time dependent probability generating functions have been obtained in terms of their Laplace transforms and the corresponding steady state results have been obtained explicitly. Average queue lenth and average system size are also computed.

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1 INTRODUCTION

Queueing systems with vacations have been developed for a wide range applications in production, communication systems, computer networks and etc. Vacation queues have been studied by numerous researchers including Doshi [7], Keilson and Servi [9], Chae et al. [4], Madan et al. [13], Madan and Anabosi [14], Madan and Abu-dayyeh [11] and Badamchi Zadeh and Shankar [2]. Borthakur and Choudhury [3] and Hur and Ahn [8] have studied vacation queues with batch arrivals. Queue with multiple vacations has been studied by Tian and Zhang [20], Srinivasan and Maragatha Sundari [17]. Choudhury et al. [6] have studied M/G/1 queue with two phases of service and Bernoulli vacation schedule under multiple vacation policy. Maraghi et al. [16] have studied batch arrival queueing system with random breakdowns and Bernoulli schedule server vacations having general vacation time.

A queueing system might suddenly break down and hence the server will not be able to continue providing service unless the system is repaired. Takine and Sengupta [18], Aissani and Artalejo [1], Vinck and Bruneel [21] have studied different queueing systems subject to random breakdowns. Kulkarni and Choi [10] and Wang et al. [22] have studied retrial queues with system breakdowns and repairs. Thangaraj and Vanitha [19] discussed the single server model with two stages of heterogeneous service with different service time distributions subject to random breakdowns and compulsory service vacations with arbitrary vacation periods.

In some queueing systems with batch arrival there is a restriction such that not all batches are allowed to join the system at all time. This policy is named restricted admissibility. For the first time Madan and Abu-Dayyeh [12], Madan and Choudhury [15] and Choudhury and Madan [5] proposed an $M^{[x]}/G/1$ queueing system with restricted admissibilty of arriving batches and Bernoulli schedule server vacation.

In this paper we consider a batch arrival queue with two types of service where breakdowns may occur at random, and once the system breaks down, it enters a repair process. A single server provides two types of service and each arriving customer has the option of choosing either type of service. If there are no customer waiting in the system then the server goes for vacation with random duration. On returning from vacation, if the server again finds no customer waiting in the system, then the server continues to go for vacation until he finds at least one customer in the system. The service time and the vacation time are generally distributed, while the breakdown and repair times are exponentially distributed. The customers arrive to the system in batches of variable size, but served one by one on a first come - first served basis.

This paper is organized as follows. The mathematical description of our model is given in section 2. Equations governing the system are given in section 3. The time dependent solution have been obtained in section 4 and corresponding steady state results have been derived explicitly in section 5. Average queue lenth and average system size are computed in section 6. Conclusion are given in section 7.

2 MATHEMATICAL DESCRIPTION OF THE MODEL

We assume the following to describe the queueing model of our study.

- Customers arrive at the system in batches of variable size in a compound Poisson process and they are provided

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one by one service on a 'first come - first served basis'. Let $\lambda c_i dt$ (i = 1, 2, . . .) be the first order probability that a batch of i customers arrives at the system during a short interval of time $(t, t+dt]$, where $0 \le c_i \le 1$ and $\sum_{i=1}^{\infty} c_i = 1$ and $\lambda > 0$ is the arrival rate of batches.

- The server provides two types of service, type 1 and type 2, with the service times having general distribution. Let $B_i(v)$ and $b_i(v)$ (i =1, 2) be the distribution and the density function of the type 1 and type 2 service respectively.
- Just before the service of a customer starts he may choose type 1 service with probability p_1 or type 2 service with probability p_2, where $p_1 + p_2 = 1$
- The service time follows a general (arbitrary) distribution with distribution function $B_i(s)$ and density function $b_i(s)$. Let $\mu_i(x)dx$ be the conditional probability density of service completion during the interval $(x, x+dx]$, given that the elapsed time is x, so that

$$\mu_i(x) = \frac{b_i(x)}{1 - B_i(x)}, \quad i = 1,2.$$

and therefore,

$$b_i(s) = \mu_i(s) e^{-\int_0^s \mu_i(x)dx}, \quad i = 1,2.$$

- If there are no customers waiting in the system then the server goes for vacation with random duration. On returning from vacation, if the server again finds no customer waiting in the system, then the server continues to go for vacation until he finds at least one customer in the system. Here the server takes multiple vacation.
- The server's vacation time follows a general (arbitrary) distribution with distribution function V(t) and density function v(t). Let $\gamma(x)dx$ be the conditional probability of a completion of a vacation during the interval $(x, x+dx]$ given that the elapsed vacation time is x, so that

$$\gamma(x) = \frac{v(x)}{1 - V(x)}$$

and therefore,

$$v(t) = \gamma(t) e^{-\int_0^t \gamma(x)dx}.$$

- The system may break down at random, and breakdowns are assumed to occur according to a Poisson stream with mean breakdown rate $\eta > 0$. Further we assume that once the system breaks down, the customer whose service is interrupted comes back to the head of the queue. Once the system breaks down, it enters a repair process immediately. The repair times are exponentially distributed with mean repair rate $\beta > 0$.
- There is a policy restricted admissibility of batches in which not all batches are allowed to join the system at all times. Let α $(0 \le \alpha \le 1)$ and ξ $(0 \le \xi \le 1)$ be the probability that an arriving batch will be allowed to join the system during the period of server's non-vacation period and vacation period respectively.
- Various stochastic processes involved in the system are assumed to be independent of each other.

We define

$P_n^{(1)}(x,t)$: Probability that at time t, the server is active providing first type service and there are n ($n \ge 0$) customers in the queue excluding the one customer in the first type of service being served and the elapsed service time for this customer is x}. $P_n^{(1)}(t) = \int_0^{\infty} P_n^{(1)}(x,t)dx$ denotes the probability that at time t there are n customers in the queue excluding one customer in the first type of service irrespective of the value of x.

$P_n^{(2)}(x,t)$: Probability that at time t, the server is active providing second type service and there are n ($n \ge 0$) customers in the queue excluding the one customer in the second type of service being served and the elapsed service time for this customer is x}. $P_n^{(2)}(t) = \int_0^{\infty} P_n^{(2)}(x,t)dx$ denotes the probability that at time t there are n customers in the queue excluding one customer in the second type of service irrespective of the value of x.

$V_n(x,t)$: Probability that at time t, the server is on vacation with elapsed vacation time x and there are n ($n \geq 0$) customers in the queue}. $V_n(t) = \int_0^\infty V_n(x,t)dx$ denotes the probability that at time t there are n customers in the queue and the server is on vacation irrespective of the value of x.

$R_n(t)$: Probability that at time t, the server is inactive due to system breakdown and the system is under repair, while there are n ($n \geq 0$) customers in the queue.

3 EQUATIONS GOVERNING THE SYSTEM

The model is then, governed by the following set of differential- difference equations:

$$\frac{\partial}{\partial x}P_0^{(1)}(x,t)+\frac{\partial}{\partial t}P_n^{(1)}(x,t)+[\lambda+\mu_1(x)+\eta]P_0^{(1)}(x,t)=\lambda(1-\alpha)P_0^{(1)}(x) \quad (1)$$

$$\frac{\partial}{\partial x}P_n^{(1)}(x,t)+\frac{\partial}{\partial t}P_n^{(2)}(x,t)+[\lambda+\mu_1(x)+\eta]P_n^{(1)}(x,t)=\lambda(1-\alpha)P_{(n)}^{(1)}(x,t)$$
$$+\lambda\alpha\sum_{k=1}^{n}c_kP_{n-k}^{(1)}(x,t),\ n\geq 1 \quad (2)$$

$$\frac{\partial}{\partial x}P_0^{(2)}(x,t)+\frac{\partial}{\partial t}P_n^{(2)}(x,t)+[\lambda+\mu_2(x)+\eta]P_0^{(2)}(x,t)=\lambda(1-\alpha)P_0^{(2)}(x,t) \quad (3)$$

$$\frac{\partial}{\partial x}P_n^{(2)}(x,t)+\frac{\partial}{\partial t}P_n^{(2)}(x,t)+[\lambda+\mu_2(x)+\eta]P_n^{(2)}(x,t)=\lambda(1-\alpha)P_n^{(2)}(x,t)$$
$$+\lambda\alpha\sum_{k=1}^{n}c_kP_{n-k}^{(2)}(x,t),\ n\geq 1 \quad (4)$$

$$\frac{\partial}{\partial x}V_0(x,t)+\frac{\partial}{\partial t}V_0(x,t)+[\lambda+\gamma(x)]V_0(x,t)=\lambda(1-\xi)V_0(x,t) \quad (5)$$

$$\frac{\partial}{\partial x}V_n(x,t)+\frac{\partial}{\partial t}V_n(x,t)+[\lambda+\gamma(x)]V_n(x,t)=\lambda(1-\xi)V_n(x,t)$$
$$+\lambda\xi\sum_{k=1}^{n}c_kV_{n-k}(x,t),\ n\geq 1 \quad (6)$$

$$\frac{d}{dt}R_0(t)+(\lambda+\beta)R_0(t)=0 \quad (7)$$

$$\frac{d}{dt}R_n(t)+(\lambda+\beta)R_n(t)=\lambda\sum_{k=1}^{n}c_kR_{n-k}(t)+\eta\int_0^\infty P_{n-1}^{(1)}(x,t)dx$$
$$+\eta\int_0^\infty P_{n-1}^{(2)}(x,t)dx \quad (8)$$

The above equations are to be solved subject to the following boundary conditions

$$P_n^{(1)}(0,t)=p_1\int_0^\infty\gamma(x)V_{n+1}(x,t)dx+p_1\int_0^\infty\mu_1(x)P_{n+1}^{(1)}(x,t)dx$$
$$+p_1\int_0^\infty\mu_2(x)P_{n+1}^{(2)}(x,t)dx+p_1\beta R_{n+1}(t),\ n\geq 0 \quad (9)$$

$$P_n^{(2)}(0,t)=p_2\int_0^\infty\gamma(x)V_{n+1}(x)dx+p_2\int_0^\infty\mu_1(x)P_{n+1}^{(1)}(x,t)dx$$
$$+p_2\int_0^\infty\mu_2(x)P_{n+1}^{(2)}(x,t)dx+p_2\beta R_{n+1}(t),\ n\geq 0 \quad (10)$$

$$V_0(0,t)=\int_0^\infty \gamma(x)V_0(x,t)dx+\int_0^\infty \mu_1(x)P_0^{(1)}(x,t)dx$$
$$+\int_0^\infty \mu_2(x)P_0^{(2)}(x,t)dx+\beta R_0(t) \quad (11)$$

$$V_n(0,t)=0,\ n\ge 1 \quad (12)$$

we assume that initially there are no customers in the system and the server is idle. so the initial conditions are

$$V_0(0)=V_n(0)=0, P_n^{(i)}(0)=0 \text{ for } n=0,1,2,...,\ i=1,2. \quad (13)$$

4 PROBABILITY GENERATING FUNCTIONS OF THE QUEUE LENGTH:THE TIME-DEPENDENT SOLUTION

We define the probability generating functions,

$$P^{(i)}(x,z,t)=\sum_{n=0}^{\infty} z^n P_n^{(i)}(x,t);\quad P^{(i)}(z,t)=\sum_{n=0}^{\infty} z^n P_n^{(i)}(t), \text{for } i=1,2. \quad (14)$$

$$V(x,z,t)=\sum_{n=0}^{\infty} z^n V_n(x,t);\quad V(z,t)=\sum_{n=0}^{\infty} z^n V_n(t),\ x>0 \quad (15)$$

$$R(z,t)=\sum_{n=0}^{\infty} z^n R_n(t); C(z)=\sum_{n=1}^{\infty} c_n z^n \quad (16)$$

which are convergent inside the circle given by z $\le$ 1 and define the Laplace transform of a function f(t) as

$$\bar{f}(s)=\int_0^\infty e^{-st} f(t)dt,\ \Re(s)>0. \quad (17)$$

We take the Laplace transform of equations (1) to (12) and using equation (13), we get

$$\frac{\partial}{\partial x}\bar{P}_0^{(1)}(x,s)+(s+\lambda\alpha+\mu_1(x)+\eta)\bar{P}_0^{(1)}(x,s)=0 \quad (18)$$

$$\frac{\partial}{\partial x}\bar{P}_n^{(1)}(x,s)+(s+\lambda\alpha+\mu_1(x)+\eta)\bar{P}_n^{(1)}(x,s)=\lambda\alpha\sum_{k=1}^{n} c_k \bar{P}_{n-k}^{(1)}(x,s), n\ge 1 \quad (19)$$

$$\frac{\partial}{\partial x}\bar{P}_0^{(2)}(x,s)+[s+\lambda\alpha+\mu_2(x)+\eta]\bar{P}_0^{(2)}(x,s)=0 \quad (20)$$

$$\frac{\partial}{\partial x}\bar{P}_n^{(2)}(x,s)+[s+\lambda\alpha+\mu_2(x)+\eta]\bar{P}_n^{(2)}(x,s)=\lambda\alpha\sum_{k=1}^{n} c_k \bar{P}_{n-k}^{(2)}(x,s), n\ge 1 \quad (21)$$

$$\frac{\partial}{\partial x}\bar{V}_0(x,s)+[s+\lambda\xi+\gamma(x)]\bar{V}_0(x,s)=0 \quad (22)$$

$$\frac{\partial}{\partial x}\bar{V}_n(x,s)+[s+\lambda\xi+\gamma(x)]\bar{V}_n(x,s)=\lambda\xi\sum_{k=1}^{n} c_k \bar{V}_{n-k}(x,s), n\ge 1 \quad (23)$$

$$(s+\lambda+\beta)\bar{R}_0(s)=0 \quad (24)$$

$$(s+\lambda+\beta)\bar{R}_n(s)=\lambda\sum_{k=1}^{n} c_k \bar{R}_{n-k}(s)+\eta\int_0^\infty \bar{P}_{n-1}^{(1)}(x,s)dx$$
$$+\eta\int_0^\infty \bar{P}_{n-1}^{(2)}(x,s)dx \quad (25)$$

$$\bar{P}_n^{(1)}(0,s)=p_1\int_0^\infty \gamma(x)\bar{V}_{n+1}(x,s)dx+p_1\int_0^\infty \mu_1(x)\bar{P}_{n+1}^{(1)}(x,s)dx$$

$$+ p_1\int_0^\infty \mu_2(x)\overline{P}_{n+1}^{(2)}(x,s)dx + p_1\beta\overline{R}_{n+1}(s),\ n\ge 0 \tag{26}$$

$$\begin{aligned}\overline{P}_n^{(2)}(0,s) &= p_2\int_0^\infty \gamma(x)\overline{V}_{n+1}(x,s)dx + p_2\int_0^\infty \mu_1(x)\overline{P}_{n+1}^{(1)}(x,s)dx\\ &+ p_2\int_0^\infty \mu_2(x)\overline{P}_{n+1}^{(2)}(x,s)dx + p_1\beta\overline{R}_{n+1}(s),\ n\ge 0\end{aligned} \tag{27}$$

$$\begin{aligned}\overline{V}_0(0,s) &= \int_0^\infty \gamma(x)\overline{V}_0(x,s)dx + \int_0^\infty \mu_1(x)\overline{P}_0^{(1)}(x,s)dx\\ &+ \int_0^\infty \mu_2(x)\overline{P}_0^{(2)}(x,s)dx + \beta\overline{R}_0(s)\end{aligned} \tag{28}$$

$$\overline{V}_n(0,s) = 0,\ n\ge 1 \tag{29}$$

Now multiplying equations (19), (21), (23) and (25) by suitable powers of z, adding to equations (18), (20), (22) and (24) and summing over n from 1 to ∞ and using the generating function defined in (14) to (16), we get

$$\frac{\partial}{\partial x}\overline{P}_n^{(1)}(x,z,s) + [s+\lambda\alpha-\lambda\alpha C(z)+\mu_1(x)+\eta]\overline{P}^{(1)}(x,z,s) = 0 \tag{30}$$

$$\frac{\partial}{\partial x}\overline{P}_n^{(2)}(x,z,s) + [s+\lambda\alpha-\lambda\alpha C(z)+\mu_2(x)+\eta]\overline{P}^{(2)}(x,z,s) = 0 \tag{31}$$

$$\frac{\partial}{\partial x}\overline{V}_n(x,z,s) + [s+\lambda\xi-\lambda\xi C(z)+\gamma(x)]\overline{V}(x,z,s) = 0 \tag{32}$$

$$\begin{aligned}(s+\lambda-\lambda C(z)+\beta)\overline{R}(z,s) &= \eta z\int_0^\infty \overline{P}^{(1)}(x,z,s)dx\\ &+ \eta z\int_0^\infty \overline{P}^{(2)}(x,z,s)dx\end{aligned} \tag{33}$$

For the boundary conditions, we multiply both sides of equation (26) by z^n sum over n from 0 to ∞, and use the equations (14) to (16) to get

$$\begin{aligned}zP^{(1)}(0,z,s) &= p_1\int_0^\infty \gamma(x)\overline{V}(x,z,s)dx - p_1\int_0^\infty \gamma(x)\overline{V}_0(x,s)dx\\ &+ p_1\int_0^\infty \mu_1(x)\overline{P}^{(1)}(x,z,s)dx - p_1\int_0^\infty \mu_1(x)\overline{P}_0^{(1)}(x,s)dx\\ &+ p_1\int_0^\infty \mu_2(x)\overline{P}^{(2)}(x,z,s)dx - p_1\int_0^\infty \mu_2(x)\overline{P}_0^{(2)}(x,s)dx\\ &+ p_1\beta\overline{R}(z,s) - p_1\beta\overline{R}_0(s),\ n\ge 0\end{aligned} \tag{34}$$

Performing similar operation on equations (27) to (29), we get

$$\begin{aligned}zP^{(2)}(0,z,s) &= p_2\int_0^\infty \gamma(x)\overline{V}(x,z,s)dx - p_2\int_0^\infty \gamma(x)\overline{V}_0(x,s)dx\\ &+ p_2\int_0^\infty \mu_1(x)\overline{P}^{(1)}(x,z,s)dx - p_2\int_0^\infty \mu_1(x)\overline{P}_0^{(1)}(x,s)dx\\ &+ p_2\int_0^\infty \mu_2(x)\overline{P}^{(2)}(x,z,s)dx - p_2\int_0^\infty \mu_2(x)\overline{P}_0^{(2)}(x,s)dx\\ &+ p_2\beta\overline{R}(z,s) - p_2\beta\overline{R}_0(s),\ n\ge 0\end{aligned} \tag{35}$$

$$\begin{aligned}\overline{V}(0,z,s) &= \int_0^\infty \gamma(x)\overline{V}_0(x,s)dx + \int_0^\infty \mu_1(x)\overline{P}_0^{(1)}(x,s)dx\\ &+ \int_0^\infty \mu_2(x)\overline{P}_0^{(2)}(x,s)dx + \beta\overline{R}_0(s)\end{aligned} \tag{36}$$

Using equation (36) in (34) and (35), we get

$$zP^{(1)}(0,z,s) = p_1\int_0^\infty \gamma(x)\overline{V}(x,z,s)dx + p_1\int_0^\infty \mu_1(x)\overline{P}^{(1)}(x,z,s)dx$$

$$+ p_2\int_0^\infty \mu_2(x)\bar{P}^{(1)}(x,z,s)dx + p_1\beta\bar{R}(z,s) - p_1\bar{V}(0,z,s) \tag{37}$$

$$zP^{(2)}(0,z,s) = p_2\int_0^\infty \gamma(x)\bar{V}(x,z,s)dx + p_2\int_0^\infty \mu_1(x)\bar{P}^{(1)}(x,z,s)dx$$

$$+ p_2\int_0^\infty \mu_2(x)\bar{P}^{(2)}(x,z,s)dx + p_2\beta\bar{R}(z,s) - p_2\bar{V}(0,z,s) \tag{38}$$

Integrating equation (30) between 0 to x , we get

$$\bar{P}^{(1)}(x,z,s) = \bar{P}^{(1)}(0,z,s)e^{-[s+\lambda\alpha-\lambda\alpha C(z)+\eta)]x-\int_0^x \mu_1(t)dt} \tag{39}$$

where $P^{(1)}(0,z,s)$ is given by equation (37).

Again integrating equation (39) by parts with respect to x yields,

$$\bar{P}^{(1)}(z,s) = \bar{P}^{(1)}(0,z,s)\left[\frac{1-\bar{B}_1(s+\lambda\alpha-\lambda\alpha C(z)+\eta)}{s+\lambda\alpha-\lambda\alpha C(z)+\eta}\right] \tag{40}$$

where

$$\bar{B}_1(s+\lambda\alpha-\lambda\alpha C(z)+\eta) = \int_0^\infty e^{-[s+\lambda\alpha-\lambda\alpha C(z)+\eta]x}dB_1(x)$$

is the Laplace-Stieltjes transform of the first stage service time $B_1(x)$. Now multiplying both sides of equation (39) by $\mu_1(x)$ and integrating over x we obtain

$$\int_0^\infty \bar{P}^{(1)}(x,z,s)\mu_1(x)dx = \bar{P}^{(1)}(0,z,s)\bar{B}_1[s+\lambda\alpha-\lambda\alpha C(z)+\eta] \tag{41}$$

Similarly , on integrating equations (31) and (32) from 0 to x , we get

$$\bar{P}^{(2)}(x,z,s) = \bar{P}^{(2)}(0,z,s)e^{-[s+\lambda\alpha-\lambda\alpha C(z)+\eta)]x-\int_0^x \mu_2(t)dt} \tag{42}$$

$$\bar{V}(x,z,s) = \bar{V}(0,z,s)e^{-[s+\lambda\xi-\lambda\xi C(z))]x-\int_0^x \gamma(t)dt} \tag{43}$$

where $P^{(2)}(0,z,s)$ is given by equation (38).

Again integrating equation (42) and (43) by parts with respect to x yields,

$$\bar{P}^{(2)}(z,s) = \bar{P}^{(2)}(0,z,s)\left[\frac{1-\bar{B}_2(s+\lambda\alpha-\lambda\alpha C(z)+\eta)}{s+\lambda\alpha-\lambda\alpha C(z)+\eta}\right] \tag{44}$$

$$\bar{V}(z,s) = \bar{V}(0,z,s)\left[\frac{1-\bar{V}(s+\lambda\xi-\lambda\xi C(z)))}{s+\lambda\xi-\lambda\xi C(z)}\right] \tag{45}$$

where

$$\bar{B}_2(s+\lambda\alpha-\lambda\alpha C(z)+\eta) = \int_0^\infty e^{-[s+\lambda\alpha-\lambda\alpha C(z)+\eta]x}dB_2(x)$$

$$\bar{V}(s+\lambda\xi-\lambda\xi C(z)) = \int_0^\infty e^{-[s+\lambda\xi-\lambda\xi C(z)]x}dV(x)$$

is the Laplace-Stieltjes transform of the second type service time $B_2(x)$ and vacation time V(x). Now multiplying both sides of equation (42) by $\mu_2(x)$, (43) by $\gamma(x)$ and integrating over x we obtain

$$\int_0^\infty \bar{P}^{(2)}(x,z,s)\mu_2(x)dx = \bar{P}^{(2)}(0,z,s)\bar{B}_2[s+\lambda\alpha-\lambda\alpha C(z)+\eta] \tag{46}$$

$$\int_0^\infty \bar{V}(x,z,s)\gamma(x)dx = \bar{V}(0,z,s)\bar{V}[s+\lambda\xi-\lambda\xi C(z)] \tag{47}$$

Using equations (41), (46) and (47) in (37) and (38), we get

$$[z-p_1\bar{B}_1(a)]\bar{P}^{(1)}(0,z,s) = p_1[\bar{V}(c)-1]\bar{V}(0,z,s)+p_1\bar{B}_2(a)\bar{P}^{(2)}(0,z,s) + p_1\beta\bar{R}(z,s) \tag{48}$$

$$[z-p_2\bar{B}_2(a)]\bar{P}^{(2)}(0,z,s) = p_2[\bar{V}(c)-1]\bar{V}(0,z,s)+p_2\bar{B}_1(a)\bar{P}^{(1)}(0,z,s) + p_2\beta\bar{R}(z,s) \tag{49}$$

Using equation (49) in (48) and (48) in (49), we get

$$[z-(p_1\bar{B}_1(a)+p_2\bar{B}_2(a)]\bar{P}^{(1)}(0,z,s) = p_1[\bar{V}(c)-1]\bar{V}(0,z,s)+p_1\beta\bar{R}(z,s) \tag{50}$$

$$[z-(p_1\bar{B}_1(a)+p_2\bar{B}_2(a)]\bar{P}^{(2)}(0,z,s) = p_2[\bar{V}(c)-1]\bar{V}(0,z,s)+p_2\beta\bar{R}(z,s) \tag{51}$$

Substituting equations (39) and (42) in (33), we get

$$\bar{R}(z,s) = \frac{\alpha z}{ab}[\bar{P}^{(1)}(0,z,s)(1-\bar{B}_1(a))+\bar{P}^{(2)}(0,z,s)(1-\bar{B}_2(a))] \tag{52}$$

Using equations (50) and (51) in (52), we get

$$\bar{R}(z,s) = \frac{\eta z[1-(p_1\bar{B}_1(a)+p_2\bar{B}_2(a))][\bar{V}(c)-1]\bar{V}(0,z,s)}{DR} \tag{53}$$

$$DR = ab[z-(p_1\bar{B}_1(a)p_2b\bar{B}_2(a))]-\eta\beta z[1-(p_1\bar{B}_1(a)p_2b\bar{B}_2(a))] \tag{54}$$

By substituting equation (47), (53) in (50) and (51)we get,

$$\bar{P}^{(1)}(0,z,s) = \frac{p_1ab[\bar{V}(c)-1]\bar{V}(0,z,s)}{DR} \tag{55}$$

$$\bar{P}^{(2)}(0,z) = \frac{p_2ab[\bar{V}(c)-1]\bar{V}(0,z,s)}{DR} \tag{56}$$

Using equations (55), (56) in (40) and (44), we have

$$\bar{P}^{(1)}(z,s) = \frac{p_1b[1-\bar{B}_1(a)][\bar{V}(c)-1]\bar{V}(0,z,s)}{DR} \tag{57}$$

$$\bar{P}^{(2)}(z,s) = \frac{p_2b[1-\bar{B}_2(a)][\bar{V}(c)-1]\bar{V}(0,z,s)}{DR} \tag{58}$$

where a= $s+\lambda\alpha-\lambda\alpha C(z)+\eta$, $b=s+\lambda-\lambda C(z)+\beta$ and $c=s+\lambda\xi-\lambda\xi C(z)$

5 The steady state results

In this section, we shall derive the steady state probability distribution for our queueing model. To define the steady probabilities we suppress the argument t wherever it appears in the time-dependent analysis. This can be obtained by applying the well-known Tauberian property,

$$\lim_{s\to 0} s\bar{f}(s) = \lim_{t\to\infty} f(t) \tag{59}$$

In order to determine $\bar{P}^{(1)}(z,s), \bar{P}^{(2)}(z,s), \bar{V}(z,s)$ and $\bar{R}(z,s)$ completely, we have yet to determine the unknown $\bar{V}(0,z,s)$ which appears in the numerators of the right hand sides of equations (45), (53), (57) and (58). For that purpose, we shall use the normalizing condition

$$P^{(1)}(1)+P^{(2)}(1)+V(1)+R(1)=1 \tag{60}$$

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Theorem 5.1 The steady state probabilities for an $M^{[x]}/G/1$ queue with two types of servive subject to random breakdown and multiple vacation with restricted admissibility are given by

$$P^{(1)}(1)=\frac{-\lambda\beta\xi p_1E(I)(1-\bar{B}_1(\eta))\bar{V}'(0)}{dr}V(0,1) \tag{61}$$

$$P^{(2)}(1)=\frac{-\lambda\beta\xi p_2E(I)(1-\bar{B}_2(\eta))\bar{V}'(0)}{dr}V(0,1) \tag{62}$$

$$R(1)=\frac{-\lambda\eta\xi E(I)[1-(p_1\bar{B}_1(\eta)+p_2\bar{B}_2(\eta))]\bar{V}'(0)}{dr} \tag{63}$$

$$V(1)=-V'(0)V(0,1) \tag{64}$$

where $dr=\eta\beta-[1-(p_1\bar{B}_1(\eta)+p_2\bar{B}_2(\eta))][\lambda E(I)(\alpha\beta+\eta)+\eta\beta]$

$P^{(1)}(1),P^{(2)}(1)$, V(1), R(1) denote the steady state probabilities that the server is providing first type of service, second type of service, server on vacation and server under repair without regard to the number of customers in the queue.

Proof: Multiplying both sides of equations (45), (53), (57) and (58) by s, taking limit as $s\to 0$, applying property (59) and simplifying, we obtain

$$P^{(1)}(z)=\frac{p_1(\lambda-\lambda C(z)+\beta)[\bar{V}(T)-1][1-\bar{B}_1(R)]V(0,z)}{D(z)} \tag{65}$$

$$P^{(2)}(z)=\frac{p_2(\lambda-\lambda C(z)+\beta)[\bar{V}(T)-1][1-\bar{B}_2(R)]V(0,z)}{D(z)} \tag{66}$$

$$V(z)=\frac{[1-\bar{V}(T)]}{T}V(0,z) \tag{67}$$

$$R(z)=\frac{\eta z[\bar{V}(T)-1][1-(p_1\bar{B}_1(R)+p_2\bar{B}_2(R))]}{D(z)}\bar{V}(0,z) \tag{68}$$

$$\begin{aligned}D(z)&=R(\lambda-\lambda C(z)+\beta)[z-(p_1\bar{B}_1(R)+p_2\bar{B}_2(R)]\\&\quad-\eta z\beta[1-(p_1\bar{B}_1(R)+p_2\bar{B}_2(R))]\end{aligned} \tag{69}$$

where $R=\lambda\alpha-\lambda\alpha C(z)+\eta$, $T=\lambda\xi-\lambda\xi C(z)$.

Let $P_q(z)$ denote the probability generating function of the queue size irrespective of the server state. Then adding equations (65) to (68), we obtain

$$P_q(z)=P^{(1)}(z)+P^{(2)}(z)+V(z)+R(z)$$

$$P_q(z)=\frac{N(z)}{D(z)}+\left(\frac{1-\bar{V}(T)}{T}\right)V(0,1) \tag{70}$$

$$N(z)=[\bar{V}(T)-1][1-(p_1\bar{B}_1(R)+p_2\bar{B}_2(R))][\lambda(1-C(z))+\beta+\eta z]V(0,1)$$

and D(z) is given in the equation (69).

$$V(0,1)=\frac{\eta\beta-[\lambda E(I)(\alpha\beta+\eta)+\eta\beta][1-p_1\bar{B}_1(\eta)+p_2\bar{B}_2(\eta)]}{dr_1} \tag{71}$$

$dr_1=\bar{V}'(0)[1-(p_1\bar{B}_1(\eta)+p_2\bar{B}_2(\eta))][\lambda E(I)(\beta(\alpha-\xi)+\eta(1-\xi))+\eta\beta]-\eta\beta\bar{V}'(0)$ and hence, the utilization factor ρ of the system is given by

$$\rho=\frac{\lambda\xi(\beta+\eta)E(I)[1-p_1\bar{B}_1(\eta)+p_2\bar{B}_2(\eta)]}{\eta\beta-[1-p_1\bar{B}_1(\eta)+p_2\bar{B}_2(\eta)][\lambda E(I)(\beta(\alpha-\xi)+\eta(1-\xi))+\eta\beta]} \tag{72}$$

where $\rho < 1$ is the stability condition under which the steady states exits.

Substituting for $V(0,1)$ from (71) into (70), we have completely explicitely determined the probability generating function of the queue size.

6 THE AVERAGE QUEUE SIZE AND THE AVERAGE SYSTEM SIZE

Let L_q the denote the mean number of customers in the queue under the steady state. Then we have

$$L_q = \frac{d}{dz} P_q(z) \ \ at\ z = 1$$

$$L_q = \lim_{z \to 1} \frac{D'(1)N''(1) - N'(1)D''(1)}{2D'(1)^2} V(0,1) + \frac{\lambda \xi E(I) E(V)^2}{2} V(0,1) \tag{73}$$

where primes and double primes in (73) denote first and second derivative at z = 1, respectively. Carrying out the derivative at z = 1 we have

$$N'(1) = \lambda \xi E(I)(\beta + \eta) E(V)[1 - (p_1 \overline{B}_1(\eta) + p_2 \overline{B}_2(\eta))] \tag{74}$$

$$\begin{aligned} N''(1) = {} & [1 - (p_1 \overline{B}_1(\eta) + p_2 \overline{B}_2(\eta))][\lambda^2 \xi^2 (E(I))^2 (\beta + \eta) E(V)^2 \\ & + \lambda \xi E(V)[(\beta + \eta) E(I(I-1)) + 2E(I)(\eta - \lambda E(I))] \\ & + 2\lambda^2 \alpha \xi (E(I))^2 (\beta + \eta) E(V)[p_1 \overline{B}_1'(\eta) + p_2 \overline{B}_2'(\eta)] \end{aligned} \tag{75}$$

$$D'(1) = \eta \beta - [\lambda E(I)(\alpha \beta + \eta) + \eta \beta][1 - (p_1 \overline{B}_1(\eta) + p_2 \overline{B}_2(\eta))] \tag{76}$$

$$\begin{aligned} D''(1) = {} & [1 - (p_1 \overline{B}_1(\eta) + p_2 \overline{B}_2(\eta))][-\lambda E(I(I-1))(\alpha \beta + \eta) \\ & + 2\lambda^2 \alpha (E(I))^2] - 2\lambda(\alpha \beta + \eta) E(I)[1 + \lambda \alpha E(I)(p_1 \overline{B}_1(\eta) + p_2 \overline{B}_2(\eta))] \\ & - 2\lambda \alpha \beta \eta E(I)(p_1 \overline{B}_1'(\eta) + p_2 \overline{B}_2'(\eta)) \end{aligned} \tag{77}$$

Then if we substitute the values from (74), (75), (76) and (77) into (73) we obtain L_q in the closed form. Further we find the mean system size L using Little's formula. Thus we have

$$L = L_q + \rho \tag{78}$$

where L_q has been found by equation (73) and ρ is obtained from equation (72).

7 CONCLUSION

In this paper we have studied an $M^{[x]}/G/1$ queue with two types of service subject to breakdown and repair having multiple vacation and restricted admissibility. This model can be utilized in large scale manufacturing industries and communication networks.

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